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Abstract The stabilization of the wave equation in a polygonal domain with cracks is analyzed. Using the multiplier method, we show that a boundary stabilization augmented by an internal one concentrated in a small neighbourhood of the cracks lead to the exponential stability of the problem.

Keywords Stability · Wave equation · Cracks

1 Introduction

We consider a bounded polygonal domain Ω of $\mathbb{R}^2$ with straight cracks and denote by Γ its boundary. We assume that $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ where Γ_D and Γ_N are two open connected parts of Γ. We also assume that $\Gamma_N = \Gamma_N^1 \cup \Gamma_N^2$ where Γ_N^1 is the set of cracks. If $\bar{\Gamma}_D \cap \bar{\Gamma}_N^2$ is not empty, we assume that the interior angle at each corner between Γ_D and Γ_N^2 is $< \pi$ (that means that Ω is convex in a neighbourhood of this corner). We further suppose that the cracks emerge from Γ_N^2 in the following sense. If we denote by $(\sigma_j)_{1 \le j \le n}$ the different cracks of Ω, then for all $j \in \{1 \ldots, n\}$, each σ_j is supposed to have one extremity T_j in common with Γ_N^2, while the other extremity

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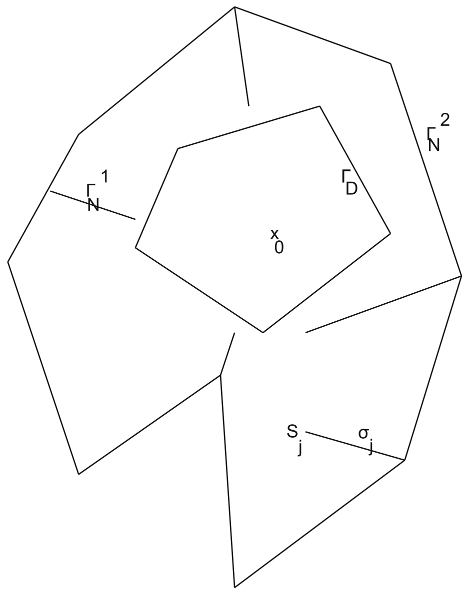

Fig. 1 Polygonal domain with cracks

S_j does not belong to $\bar{\Gamma}_D \cup \bar{\Gamma}_N^2$, see Fig. 1. This assumption implies that the cracks never meet the Dirichlet part of the boundary. In the sequel ν denotes the unit outward normal vector along Γ_N^2 (defined everywhere except at the vertices of Γ_N^2), while on a crack σ_j it is simply a fixed unit normal vector.

On this domain Ω, we consider the following problem

$$\begin{cases} u_{tt} - \triangle u + a u_t = 0 \text{ in } \ \Omega \times (0, +\infty) \\ u = 0 \ \text{ on } \ \Gamma_D \times (0, +\infty) \\ \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \ \Gamma_N^1 \times (0, +\infty) \\ \frac{\partial u}{\partial \nu} + \alpha u + b u_t = 0 \ \text{ on } \ \Gamma_N^2 \times (0, +\infty) \\ u(\cdot, 0) = u_0, \quad u_t(\cdot, 0) = u_1 \ \text{ in } \ \Omega, \end{cases} \tag{1.1}$$

where $a \in L^\infty(\Omega)$, $\alpha, b \in H^s(\Gamma_N^2)$, for some $s > \frac{1}{2}$ are such that

$$a \geq 0 \text{ a. e. in } \Omega, \ \alpha \geq 0 \text{ a. e. in } \Gamma_N^2, \ b > b_0 > 0 \text{ a. e. in } \Gamma_N^2$$

and

$$a > a_0 > 0 \text{ a. e. in } O \cap \Omega,$$

where $O = \cup_{j=1}^n O_j$, O_j being a fixed neighbourhood of σ_j as small as we want and a_0, b_0 are positive real numbers. We further assume that

$$\text{int } \Gamma_D \neq \emptyset \text{ or } \alpha(x) > 0, \forall x \in \Gamma_N^2. \tag{1.2}$$

Note that the boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_N^1 \times (0, +\infty)$$

means that it holds on both sides of the cracks.

We suppose that the domain verifies the following geometric conditions:

$$\exists x_0 \in \mathbb{R}^2 : (x - x_0) \cdot \nu \leq 0 \text{ on } \Gamma_D \text{ and } (x - x_0) \cdot \nu > 0 \text{ on } \bar{\Gamma}_N^2, \tag{1.3}$$

$$(x - S_j) \cdot \nu(x) > 0 \text{ on } (\bar{O}_j \cap \bar{\Gamma}_N^2) \backslash \{T_j\}, \quad \forall j = 1, \ldots, n. \tag{1.4}$$

Problem (1.1) is a wave equation with a standard damping term on Γ_N^2 but set in a domain with cracks. Such a problem was largely studied in the literature but always without the presence of cracks, see [4,10,12,13,16]. We also refer to [2,3] for stability results concerning the elastodynamic system with mixed boundary conditions. Let us notice that some controllability results in domain with cracks are obtained in [6] with some geometrical assumptions and in [17–19] with weaker assumptions by adding internal control near the crack tips. In the spirit of these later papers, due to the expected singular behavior of the solution near the crack tips, we add an additional internal damping in a small neighbourhood of the cracks. Under the previous assumptions, using the multiplier method we show the exponential stability of our problem.

Note that all the above assumptions are used in the proof of our stability result (this will be specified through the paper), and are not easy to relax except the one concerning the convexity near the corner of Γ_D and Γ_N^2, indeed if this is not the case it suffices to add an internal damping in a neighbourhood of these corners to get an exponential decay.

The paper is organized as follows. Section 2 is devoted to the well posedness of problem (1.1) obtained by using standard semigroup theory. In Sect. 3, we state the exponential stability result and recall a useful integral inequality from [10]. Some technical results concerning regularity results for elements in the domain of the Laplace operator and the use of some Green's formulas are proved in Sect. 4. Finally by the multiplier method with an appropriated combination of different multipliers we prove our stability result.

2 Well posedness of the problem

The well posedness of problem (1.1) follows from standard semigroup theory. By setting $\mathcal{U} = (u, u_t)^T$, we have $\mathcal{U}_t = (u_t, u_{tt})^T = (u_t, \Delta u - a(x)v)^T$.

Then problem (1.1) can be formally written in the form

$$\begin{cases} \mathcal{U}_t + \mathcal{A}\mathcal{U} = 0 \\ \mathcal{U}(0) = (u_0, u_1)^T, \end{cases} \tag{2.1}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}(u, v)^T = (-v, -\triangle u + av)^T ,$$

with domain

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{l} (u, v)^T \in (E(\triangle; L^2(\Omega)) \cap H^1_{\Gamma_D}(\Omega)) \times H^1_{\Gamma_D}(\Omega) \\ \quad \text{such that } \dfrac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma^1_N \text{ and } \dfrac{\partial u}{\partial \nu} = -(\alpha u + bv) \ \text{ on } \Gamma^2_N \end{array} \right\},$$

where

$$H^1_{\Gamma_D}(\Omega) = \left\{ u \in H^1(\Omega), u = 0 \,\text{on}\, \Gamma_D \right\},$$

and

$$E(\triangle; L^2(\Omega)) = \left\{ u \in H^1(\Omega), -\triangle u \in L^2(\Omega) \right\}.$$

From these definitions, we see that for $(u, v)^T$ in $\mathcal{D}(\mathcal{A})$ then u belongs to the domain $D(\triangle)$ of $\triangle$ defined by

$$D(\triangle) = \left\{ u \in E(\triangle; L^2(\Omega)) \cap H^1_{\Gamma_D}(\Omega) : \frac{\partial u}{\partial \nu} = 0 \ \text{ on } \Gamma^1_N \text{ and } \frac{\partial u}{\partial \nu} \in H^{\frac{1}{2}}(\Gamma^2_N) \right\}.$$

Indeed as $\alpha, b \in H^s(\Gamma^2_N)$, by using Theorem 1.4.4.2 of [9], for $u, v \in H^1(\Omega)$, we have $\gamma_0(\alpha u + bv) \in H^{\frac{1}{2}}(\Gamma^2_N)$ (where γ_0 is the trace operator).

Let us now introduce the Hilbert space

$$\mathcal{H} = H^1_{\Gamma_D}(\Omega) \times L^2(\Omega)$$

with the norm

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2_{\mathcal{H}} = ||\nabla u||^2_{L^2(\Omega)} + \| \sqrt{\alpha} u \|^2_{L^2(\Gamma^2_N)} + ||v||^2_{L^2(\Omega)}$$

and the natural associated inner product

$$\left(\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u^* \\ v^* \end{pmatrix} \right) = \int_\Omega (\nabla u \cdot \nabla u^* + v v^*) \, dx + \int_{\Gamma^2_N} \alpha u u^* \, d\sigma.$$

By Green's formula we see that

$$\left(\mathcal{A}\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix}\right) = -\int_{\Omega} \nabla v \cdot \nabla u \, dx + \int_{\Omega} (-\Delta u + av) v \, dx - \int_{\Gamma_N^2} \alpha v u \, d\sigma$$

$$= \int_{\Gamma_N^2} b|v|^2 d\Gamma(x) + \int_{\Omega} a|v|^2 dx \geq 0,$$

for all $(u, v)^T \in \mathcal{D}(\mathcal{A})$. Hence $\mathcal{A}$ is monotone.

Let us prove that the operator $\mathcal{A} + \lambda I$ is surjective for at least one $\lambda > 0$. For $(f, g)^T \in \mathcal{H}$, we look for $(u, v)^T \in \mathcal{D}(\mathcal{A})$ solution of

$$\begin{cases} -v + \lambda u = f & \text{in } \Omega, \\ \lambda v - \Delta u + av = g & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_N^1, \\ \frac{\partial u}{\partial \nu} + \alpha u + bv = 0 & \text{on } \Gamma_N^2. \end{cases} \tag{2.2}$$

Eliminating $v = \lambda u - f$, it remains to find $u \in H^1_{\Gamma_D}(\Omega) \cap E(\Delta; L^2(\Omega))$ which verifies

$$\begin{cases} -\Delta u + (\lambda a + \lambda^2) u = g + (a + \lambda) f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_N^1, \\ \frac{\partial u}{\partial \nu} + (\alpha + b\lambda) u = bf & \text{on } \Gamma_N^2. \end{cases} \tag{2.3}$$

The problem (2.3) admits a unique weak solution by using Lax-Milgram's Lemma. Indeed multiplying the first equation by $v \in H^1_{\Gamma_D}(\Omega)$ and by integrating formally by parts we get

$$c(u, v) = F(v), \forall v \in H^1_{\Gamma_D}(\Omega), \tag{2.4}$$

where the bilinear and continuous form c is given by

$$c(u, v) = \int_{\Omega} \left(\nabla u \cdot \nabla v + (\lambda a + \lambda^2) u v\right) dx + \int_{\Gamma_N^2} (\alpha + b\lambda) u v \, d\sigma, \forall u, v \in H^1_{\Gamma_D}(\Omega),$$

while the linear form F is

$$F(v) = \int_{\Omega} (g + (a + \lambda) f) v \, dx + \int_{\Gamma_N^2} b f v \, d\sigma, \forall v \in H^1_{\Gamma_D}(\Omega).$$

Since c is clearly strongly coercive on $H^1_{\Gamma_D}(\Omega)$ and F is continuous on $H^1_{\Gamma_D}(\Omega)$ (because $g + (a+\lambda) f$ belongs to $L^2(\Omega)$), by Lax-Milgram's Lemma, problem (2.4) admits a unique solution $u \in H^1_{\Gamma_D}(\Omega)$. By taking first test functions $v \in \mathcal{D}(\Omega)$, we recover the first identity of (2.3). This garantees that u belongs to $E(\triangle; L^2(\Omega)$. Then using Green's formula (see Theorem 1.5.3.11 of [9]), we see that u satisfies the third and fourth identities of (2.3). Setting $v = \lambda u - f$, we have found a pair $(u, v)^T \in \mathcal{D}(\mathcal{A})$ solution of (2.2). This shows that the operator $\mathcal{A}$ is maximal monotone and therefore $-\mathcal{A}$ generates a $\mathcal{C}_0$ semi-group of contractions in $\mathcal{H}$. Consequently, we can state the following existence results.

Theorem 2.1 *If (u_0, u_1) belongs to $H^1_{\Gamma_D}(\Omega) \times L^2(\Omega)$, then problem (1.1) has one and only one weak solution u which satisfies $u \in C([0,\infty), H^1_{\Gamma_D}(\Omega)) \cap C^1([0,\infty), L^2(\Omega))$. Furthermore, if (u_0, u_1) belongs to $\mathcal{D}(\mathcal{A})$, then problem (1.1) has one and only one strong solution u which satisfies $(u, u_t) \in C([0,\infty), \mathcal{D}(\mathcal{A}))$.*

3 Stabilization result

We define the energy of problem (1.1) by

$$E(t) := \frac{1}{2}\left(\int_{\Omega} u_t^2(x,t) + |\nabla u(x,t)|^2\, dx + \int_{\Gamma_N^2} \alpha |u(x,t)|^2 d\sigma\, dt\right). \tag{3.1}$$

Now, we give the following exponential stability of problem (1.1).

Theorem 3.1 *Let Ω be a polygonal domain of $\mathbb{R}^2$ which satisfies (1.2), (1.3) and (1.4). Then there exist $\gamma > 0$ and $\beta > 0$ such that the energy of the solution of problem (1.1) satisfies*

$$E(t) \leq \gamma e^{-\beta t} E(0), \quad \forall\ t \geq 0$$

Remark 3.2 To our best knowledge, no stability result is known if an internal damping is not added near the cracks. It is mainly added in order to be able to use the multiplier method (see below). If the domain has no crack, then the exponential stability result holds only by a boundary feedback on Γ_N^2 under the assumptions (1.2) and (1.3), we refer to [4, 10, 13, 16].

For our future purposes we recall the following fundamental result which is proved in [10]:

Lemma 3.3 [10] *Let $E : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a non-increasing function such that there exists a constant $T > 0$ independent of t which verifies*

$$\int_t^{+\infty} E(s) ds \leq T E(t), \quad \forall\, t \geq 0. \tag{3.2}$$

Then

$$E(t) \leq E(0)e^{(1-\frac{t}{T})}, \quad \forall\, t \geq T. \tag{3.3}$$

4 Some technical results

Our first goal is to state some regularity results for any element of $D(\triangle)$. For that purpose, let us recall that the standard crack singularity $\mathcal{S}_j$ associated with the crack σ_j with crack tip S_j is defined by

$$\mathcal{S}_j(r_j, \theta_j) = r_j^{\frac{1}{2}} \cos\left(\frac{\theta_j}{2}\right),$$

where (r_j, θ_j) are polar coordinates centred at S_j such that $\theta_j = 0$ or 2π on the crack σ_j. For further purposes we also fix a radial cut-off function Φ_j equal to 1 near S_j and such that Φ_j is equal to zero outside a ball $B(S_j, \varepsilon_j)$ with a small enough ε_j in order that Φ_j is equal to zero near $\partial\Omega \backslash \sigma_j$.

Lemma 4.1 *There exists* $\alpha_0 > 0$ (*small enough and depending on* Ω) *such that any* $u \in D(\triangle)$ *admits the splitting*

$$u = u_R + \sum_{j=1}^{n} c_j(u)\Phi_j \mathcal{S}_j,$$

where $c_j(u) \in \mathbb{R}$ *and* u_R *belongs to* $W^{2,p}(\Omega)$*, with* $p = \frac{4}{3} + \alpha^*$ *for all* $\alpha^* \in (0, \alpha_0)$.

Proof It suffices to apply Corollary 4.4.3.8 of [9] (see Remarks 4.4.4.15 and 4.4.4.15 of [9]) with $p \leq 2$ such that

$$\frac{1}{2} < 2 - \frac{2}{p} < \frac{\pi}{\omega_m},$$

where ω_m is the maximal angle between the Dirichlet edges of Γ_D, the Neumann edges of Γ_N^2 and the edges between Γ_N^1 and Γ_N^2 (that is $< 2\pi$ due to our assumptions). Note that the trivial embedding

$$H^1(\Omega) \hookrightarrow W^{1,p}(\Omega), \forall p \leq 2,$$

implies, via a trace theorem, the embedding

$$H^{\frac{1}{2}}(E) \hookrightarrow W^{1-\frac{1}{p},p}(E), \forall p \leq 2,$$

for any edge E of Γ_N^2. □

Lemma 4.2 *For all $u, z \in D(\triangle)$ we have*

$$\int\limits_{\Omega} \triangle u z\, dx = -\int\limits_{\Omega} \nabla u \cdot \nabla z\, dx + \int\limits_{\Gamma} \frac{\partial u}{\partial \nu} z\, d\sigma. \tag{4.1}$$

Proof According to the previous lemma, we have

$$u = u_R + \sum_{j=1}^{n} c_j(u)\Phi_j \mathcal{S}_j, \tag{4.2}$$

$$z = z_R + \sum_{j=1}^{n} c_j(z)\Phi_j \mathcal{S}_j, \tag{4.3}$$

with $c_j(u), c_j(z) \in \mathbb{R}$, and u_R, z_R in $W^{2,p}(\Omega)$ for any $p = \frac{4}{3} + \alpha^*$ for all $\alpha^* \in (0, \alpha_0)$ where $\alpha_0 > 0$ is small enough. But we notice that for $\alpha^* < \frac{2}{3}$, the Sobolev embedding theorem yields

$$W^{2,p}(\Omega) \hookrightarrow W^{1,q}(\Omega),$$

for all $p \leq q \leq q_0 := \frac{8+6\alpha^*}{2-3\alpha^*}$. We further see that the conjuguate $p' > 1$ of p (i.e. $\frac{1}{p} + \frac{1}{p'} = 1$) is smaller than q_0, consequently we get

$$W^{2,p}(\Omega) \hookrightarrow W^{1,p'}(\Omega).$$

But Theorem 1.4.5.3 of of [9] implies that

$$\Phi_j \mathcal{S}_j \in W^{1,p'}(\Omega), \quad \forall j = 1, \ldots, n.$$

These two properties implies that

$$D(\triangle) \hookrightarrow W^{1,p'}(\Omega).$$

This embedding allows to use the standard Green's formula

$$\int\limits_{\Omega} \triangle u_R z\, dx = -\int\limits_{\Omega} \nabla u_R \cdot \nabla z\, dx + \int\limits_{\Gamma} \frac{\partial u_R}{\partial \nu} z\, d\sigma. \tag{4.4}$$

For the singular part of u, for an arbitrary $j = 1, \ldots, n$ we write

$$\int\limits_{\Omega} \triangle(\Phi_j \mathcal{S}_j) z\, dx = \lim_{\varepsilon \to 0} \int\limits_{\Omega_{j,\varepsilon}} \triangle(\Phi_j \mathcal{S}_j) z\, dx,$$

where

$$\Omega_{j,\varepsilon} = \Omega \backslash B(S_j, \varepsilon).$$

Since $\Phi_j \mathcal{S}_j$ is regular in $\Omega_{j,\varepsilon}$, we can apply Green's formula and obtain

$$\int_{\Omega_{j,\varepsilon}} \triangle(\Phi_j \mathcal{S}_j) z\, dx = -\int_{\Omega_{j,\varepsilon}} \nabla(\Phi_j \mathcal{S}_j) \cdot \nabla z\, dx + \int_{\Gamma \backslash B(S_j,\varepsilon)} \frac{\partial}{\partial \nu}(\Phi_j \mathcal{S}_j) z\, d\sigma - \frac{1}{2}\varepsilon^{\frac{1}{2}} \int_0^{2\pi} \cos\left(\frac{\theta_j}{2}\right) z(\varepsilon, \theta_j)\, d\theta_j. \tag{4.5}$$

Let us show that the last term tends to zero as ε tends to 0. Indeed using (4.3), we may write

$$\varepsilon^{\frac{1}{2}} \int_0^{2\pi} \cos\left(\frac{\theta_j}{2}\right) z(\varepsilon, \theta_j)\, d\theta_j = \varepsilon^{\frac{1}{2}} \int_0^{2\pi} \cos\left(\frac{\theta_j}{2}\right) z_R(\varepsilon, \theta_j)\, d\theta_j + \varepsilon c_j(z) \int_0^{2\pi} \cos^2\left(\frac{\theta_j}{2}\right) d\theta_j.$$

Since $W^{2,p}(\Omega) \hookrightarrow C(\bar{\Omega})$, $z_R(\varepsilon, \theta_j)$ can be uniformly estimated in ε and therefore

$$\lim_{\varepsilon \to 0} \varepsilon^{\frac{1}{2}} \int_0^{2\pi} \cos\left(\frac{\theta_j}{2}\right) z(\varepsilon, \theta_j)\, d\theta_j = 0.$$

Coming back to (4.5) and passing to the limit in $\varepsilon \to 0$, we find that

$$\int_\Omega \triangle(\Phi_j \mathcal{S}_j) z\, dx = -\int_\Omega \nabla(\Phi_j \mathcal{S}_j) \cdot \nabla z\, dx + \int_\Gamma \frac{\partial}{\partial \nu}(\Phi_j \mathcal{S}_j) z\, d\sigma. \tag{4.6}$$

The identities (4.4) and (4.6) lead to (4.1). □

Remark 4.3 The identity (4.1) of the previous Lemma is well-known (see Theorem 1.5.3.11 of [9]) but the difference stays on the assumption on v and on the fact that in the right-hand side an integral can be used instead of a duality bracket.

Now given $u \in D(\triangle)$, we are interested in $z \in H^1(\Omega)$ solution of

$$\begin{cases} \triangle z = 0 & \text{in } \Omega, \\ z = 0 & \text{on } \Gamma_D, \\ \frac{\partial z}{\partial \nu} = 0 & \text{on } \Gamma_N^1, \\ z = u & \text{on } \Gamma_N^2. \end{cases} \tag{4.7}$$

This is equivalent to $z = \omega + u$, where $\omega \in V := \{w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_D \cup \Gamma_N^2\}$ solution of

$$\int_\Omega \nabla \omega \cdot \nabla w \, dx = -\int_\Omega \nabla u \cdot \nabla w \, dx, \forall w \in V.$$

This means that $\omega \in V$ is a weak solution of the problem

$$\begin{cases} \triangle \omega = -\triangle u & \text{in } \Omega \\ \omega = 0 & \text{on } \Gamma_D \cup \Gamma_N^2, \\ \frac{\partial \omega}{\partial \nu} = 0 & \text{on } \Gamma_N^1. \end{cases}$$

In other words, ω belongs to

$$D(\triangle_{Dir}) = \{v \in E(\triangle; L^2(\Omega)) \cap V : \frac{\partial u}{\partial \nu} = 0 \ \text{ sur } \Gamma_N^1\}.$$

Lemma 4.4 *There exists* $\alpha_1 > 0$ (*small enough and depending on* Ω) *such that any* $v \in D(\triangle_{Dir})$ *admits the splitting*

$$v = v_R + \sum_{j=1}^{n} c_j(u) \Phi_j \mathcal{S}_j,$$

where $c_j(v) \in \mathbb{R}$ *and* v_R *belongs to* $W^{2,p}(\Omega)$*, with* $p = \frac{4}{3} + \alpha^*$ *for all* $\alpha^* \in (0, \alpha_1)$.

Proof It suffices to apply Corollary 4.4.3.8 of [9] (see Remarks 4.4.4.15 and 4.4.4.15 of [9]) with $p \leq 2$ such that

$$\frac{1}{2} < 2 - \frac{2}{p} < \min\left\{\frac{\pi}{\omega_m}, \frac{\pi}{2\omega_{mixed}}\right\},$$

where ω_{mixed} is the maximal angle between the edges of Γ_N^1 and Γ_N^2 that is $< \pi$ due to our assumption (1.4). □

Corollary 4.5 *Let z be the unique solution of* (4.7) *with $u \in D(\triangle)$ and let $v \in D(\triangle_{Dir})$. Then we have*

$$\int_{\Omega} \triangle u z \, dx = -\int_{\Omega} \nabla u \cdot \nabla z \, dx + \int_{\Gamma} \frac{\partial u}{\partial \nu} z \, d\sigma, \tag{4.8}$$

$$\int_{\Omega} \triangle v z \, dx = -\int_{\Omega} \nabla v \cdot \nabla z \, dx + \int_{\Gamma} \frac{\partial v}{\partial \nu} z \, d\sigma. \tag{4.9}$$

Proof As $u \in D(\triangle)$ it admits the splitting (4.2), and by the previous Lemma, z admits the splitting (4.3) with

$$z_R = u_R + \omega_R \text{ and } c_j(z) = c_j(u) + c_j(\omega), \quad \forall j = 1, \dots, n,$$

when

$$\omega = \omega_R + \sum_{j=1}^{n} c_j(\omega) \Phi_j \mathcal{S}_j.$$

Hence, the identities (4.8) and (4.9) are proved exactly as in Lemma 4.2.

5 Proof of the main result

For a strong solution u, we can derive (3.1) and by Green's formula (see Lemma 4.2) we obtain

$$E'(t) = -\int_{\Omega} a(x) u_t^2 dx - \int_{\Gamma_N^2} b(x) u_t^2 d\sigma \leq 0. \tag{5.1}$$

Then the energy is nonincreasing and furthermore

$$E(S) - E(T) = \int_S^T \int_{\Omega} a(x) u_t^2 dx dt + \int_S^T \int_{\Gamma_N^2} b(x) u_t^2 d\sigma \, dt, \quad \forall \ 0 \leq S < T. \tag{5.2}$$

Now we use the piecewise multiplier method (see [12]) but adapted to our singular setting. For any extremity S_j of a crack, we consider a cut-off function η_j, such that $\eta_j = 1$ near σ_j and with support in a neighbourhood of σ_j. We also assume that the support of η_j does not meet the support of η_i if $i \neq j$.
Setting

$$\widetilde{m}(x) = \left(1 - \sum_{j=1}^{n} \eta_j(x)\right)(x - x_0),$$

as multiplier we take

$$\widetilde{M}(u)(x) = 2\left(\sum_{j=1}^{n}(x - S_j)\eta_j(x) + \widetilde{m}(x)\right) \cdot \nabla u(x) + u(x).$$

Lemma 5.1 *If u is the strong solution of problem* (1.1), *then u verifies*

$$\begin{aligned}
&-2\int_0^T E(t)dt + \int_0^T\int_\Omega |\nabla u|^2 \sum_{j=1}^{n}(x - S_j) \cdot \nabla\eta_j dxdt \\
&-\int_0^T\int_\Omega |\nabla u|^2 \sum_{j=1}^{n}(x - x_0) \cdot \nabla\eta_j dxdt \\
&-2\int_0^T\int_\Omega (x_k - x_{0k})\sum_{j=1}^{n}\frac{\partial\eta_j}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}dxdt - 2\int_0^T\int_\Omega \sum_{j=1}^{n}(x_k - S_{jk})\frac{\partial u}{\partial x_i}\frac{\partial\eta_j}{\partial x_i}\frac{\partial u}{\partial x_k}dxdt \\
&-\int_\Omega \frac{\partial u}{\partial t}\widetilde{M}(u)\,dx\Big|_0^T - \int_0^T\int_\Omega (\frac{\partial u}{\partial t})^2\sum_{j=1}^{n}(x - S_j)\cdot\nabla\eta_j dxdt \\
&+\int_0^T\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2\sum_{j=1}^{n}(x - x_0)\cdot\nabla\eta_j dxdt - \int_\Omega\int_0^T a(x)\frac{\partial u}{\partial t}\widetilde{M}(u)dtdx \\
&+\int_0^T\int_\Gamma \left(\sum_{j=1}^{n}(x - S_j)\eta_j) + \widetilde{m}\right)\cdot\nu(\frac{\partial u}{\partial t})^2 d\sigma dt \\
&-\int_0^T\int_\Gamma \left[\widetilde{m} + \sum_{j=1}^{n}\eta_j(x - S_j)\right]\cdot\nu|\nabla u|^2 d\sigma dt + \int_0^T\int_{\Gamma_N}\frac{\partial u}{\partial\nu}ud\sigma\ dt \\
&+2\int_0^T\int_\Gamma \frac{\partial u}{\partial\nu}\left[\widetilde{m} + \sum_{j=1}^{n}\eta_j(x - S_j)\right]\cdot\nabla ud\sigma dt \\
&+\int_0^T\int_{\Gamma_N^2}\alpha|u|^2\,d\sigma dt = 0.
\end{aligned} \tag{5.3}$$

Proof We use the multiplier method, namely we multiply (1.1) by $\widetilde{M}(u)$ and integrate by parts.

$$\bullet\text{Transformation of } \int_0^T\int_\Omega -\Delta u\,\widetilde{M}(u)\,dxdt.$$

For any extremity S_j of the crack σ_j and u strong solution of problem (1.1), we get

$$\begin{aligned}
&\int_\Omega -\Delta u \ \eta_j(x-S_j)\cdot\nabla u\,dx \\
&= \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}(\eta_j(x_k-S_{jk})\frac{\partial u}{\partial x_k})dx - \int_\Gamma \frac{\partial u}{\partial \nu}\eta_j(x-S_j)\cdot\nabla u d\sigma \\
&= \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial(x_k-S_{jk})}{\partial x_i}\eta_j\frac{\partial u}{\partial x_k}dx + \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial \eta_j}{\partial x_i}(x_k-S_{jk})\frac{\partial u}{\partial x_k}dx \\
&\quad + \int_\Omega \eta_j(x_k-S_{jk})\frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}(\frac{\partial u}{\partial x_k})dx - \int_\Gamma \frac{\partial u}{\partial \nu}\eta_j(x-S_j)\cdot\nabla u d\sigma \\
&= \int_\Omega |\nabla u|^2\eta_j dx + \int_\Omega (x_k-S_{jk})\frac{\partial u}{\partial x_i}\frac{\partial \eta_j}{\partial x_i}\frac{\partial u}{\partial x_k}dx \\
&\quad -\frac{1}{2}\int_\Omega |\nabla u|^2\mathrm{div}((x-S_j)\eta_j)dx \\
&\quad +\frac{1}{2}\int_\Gamma \eta_j(x-S_j)\cdot\nu|\nabla u|^2 d\sigma - \int_\Gamma \eta_j\frac{\partial u}{\partial \nu}(x-S_j)\cdot\nabla u d\sigma.
\end{aligned}\tag{5.4}$$

Furthermore Green's formula yields

$$\begin{aligned}
\int_\Omega -\Delta u\,\widetilde{m}\cdot\nabla u\,dx &= -\frac{1}{2}\int_\Omega |\nabla u|^2\mathrm{div}(\widetilde{m})dx + \int_\Omega \frac{\partial \widetilde{m}_k}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}dx \\
&\quad +\frac{1}{2}\int_\Gamma \widetilde{m}\cdot\nu|\nabla u|^2 d\sigma - \int_\Gamma \frac{\partial u}{\partial \nu}\widetilde{m}\cdot\nabla u d\sigma,
\end{aligned}$$

and hence

$$\begin{aligned}
&\int_0^T\int_\Omega -\Delta u\,\widetilde{M}(u)\,dx \\
&= -\int_0^T\int_\Omega |\nabla u|^2\mathrm{div}\left[\sum_{j=1}^n (x-S_j)\eta_j) + \widetilde{m}\right]dxdt + 2\int_0^T\int_\Omega \frac{\partial \widetilde{m}_k}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}dxdt \\
&\quad +2\int_0^T\int_\Omega |\nabla u|^2\sum_{j=1}^n \eta_j dxdt \\
&\quad +2\int_0^T\int_\Omega \sum_{j=1}^n (x_k-S_{jk})\frac{\partial u}{\partial x_i}\frac{\partial \eta_j}{\partial x_i}\frac{\partial u}{\partial x_k}dxdt + \int_0^T\int_\Omega |\nabla u|^2 dxdt
\end{aligned}$$

$$+\int_0^T\int_\Gamma \left[\widetilde{m}+\sum_{j=1}^n \eta_j(x-S_j)\right]\cdot \nu|\nabla u|^2 d\sigma dt - \int_0^T\int_{\Gamma_N} \frac{\partial u}{\partial \nu} u d\sigma\ dt$$

$$-2\int_0^T\int_\Gamma \frac{\partial u}{\partial \nu}\left[\widetilde{m}+\sum_{j=1}^n \eta_j(x-S_j)\right]\cdot \nabla u d\sigma dt.$$

As $\operatorname{div}\widetilde{m}=(x-x_0)\cdot\nabla(1-\sum_{j=1}^n \eta_j)+2(1-\sum_{j=1}^n \eta_j)$, we may write

$$-\int_0^T\int_\Omega |\nabla u|^2 \operatorname{div}\left[\sum_{j=1}^n (x-S_j)\eta_j)+\widetilde{m}\right]dxdt = -2\int_0^T\int_\Omega |\nabla u|^2 dxdt$$

$$-\int_0^T\int_\Omega |\nabla u|^2\sum_{j=1}^n (x-S_j)\cdot\nabla\eta_j dxdt + \int_0^T\int_\Omega |\nabla u|^2 \sum_{j=1}^n (x-x_0)\cdot\nabla\eta_j dxdt,$$

and

$$2\int_0^T\int_\Omega \frac{\partial \widetilde{m}_k}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k} dxdt$$

$$= 2\int_0^T\int_\Omega (x_k-x_{0k})\sum_{j=1}^n \frac{\partial \eta_j}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}dxdt + 2\int_0^T\int_\Omega\left(1-\sum_{j=1}^n\eta_j\right)|\nabla u|^2 dxdt.$$

These identities lead to

$$\int_0^T\int_\Omega -\Delta u\,\widetilde{M}(u)\,dx$$

$$= -2\int_0^T\int_\Omega |\nabla u|^2 dxdt - \int_0^T\int_\Omega |\nabla u|^2\sum_{j=1}^n (x-S_j)\cdot\nabla\eta_j dxdt$$

$$+\int_0^T\int_\Omega |\nabla u|^2\sum_{j=1}^n (x-x_0)\cdot\nabla\eta_j dxdt$$

$$+2\int_0^T\int_\Omega (x_k-x_{0k})\sum_{j=1}^n \frac{\partial \eta_j}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}dxdt + 2\int_0^T\int_\Omega (1-\sum_{j=1}^n\eta_j)|\nabla u|^2 dxdt$$

$$+2\int_0^T\int_\Omega |\nabla u|^2\sum_{j=1}^n \eta_j dxdt$$

$$+2\int_0^T\int_\Omega \sum_{j=1}^n (x_k - S_{jk})\frac{\partial u}{\partial x_i}\frac{\partial \eta_j}{\partial x_i}\frac{\partial u}{\partial x_k}dxdt + \int_0^T\int_\Omega |\nabla u|^2 dxdt$$
$$+\int_0^T\int_\Gamma \left[\widetilde{m} + \sum_{j=1}^n \eta_j(x - S_j)\right]\cdot \nu|\nabla u|^2 d\sigma dt - \int_0^T\int_{\Gamma_N} \frac{\partial u}{\partial \nu} u d\sigma\ dt$$
$$-2\int_0^T\int_\Gamma \frac{\partial u}{\partial \nu}\left[\widetilde{m} + \sum_{j=1}^n \eta_j(x - S_j)\right]\cdot \nabla u d\sigma dt,$$

and after some elementary calculations, we obtain

$$\int_0^T\int_\Omega -\Delta u\, \widetilde{M}(u)\, dx$$
$$= -\int_0^T\int_\Omega |\nabla u|^2 \sum_{j=1}^n (x - S_j)\cdot \nabla \eta_j dxdt + \int_0^T\int_\Omega |\nabla u|^2 \sum_{j=1}^n (x - x_0)\cdot \nabla \eta_j dxdt$$
$$+2\int_0^T\int_\Omega (x_k - x_{0k})\sum_{j=1}^n \frac{\partial \eta_j}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}dxdt$$
$$+2\int_0^T\int_\Omega \sum_{j=1}^n (x_k - S_{jk})\frac{\partial u}{\partial x_i}\frac{\partial \eta_j}{\partial x_i}\frac{\partial u}{\partial x_k}dxdt + \int_0^T\int_\Omega |\nabla u|^2 dxdt$$
$$+\int_0^T\int_\Gamma \left[\widetilde{m} + \sum_{j=1}^n \eta_j(x - S_j)\right]\cdot \nu|\nabla u|^2 d\sigma dt - \int_0^T\int_{\Gamma_N} \frac{\partial u}{\partial \nu} u d\sigma\ dt$$
$$-2\int_0^T\int_\Gamma \frac{\partial u}{\partial \nu}\left[\widetilde{m} + \sum_{j=1}^n \eta_j(x - S_j)\right]\cdot \nabla u d\sigma dt. \tag{5.5}$$

□

Remark 5.2 In the case $\overline{\Gamma_N^2} \cap \overline{\Gamma_D} = \{s_1, s_2\}$ treated in [2] (but excluded here), we would get

$$\int_0^T\int_\Omega -\Delta u\, \widetilde{M}(u)\, dx$$
$$\geq -\int_0^T\int_\Omega |\nabla u|^2 \sum_{j=1}^n (x - S_j)\cdot \nabla \eta_j dxdt + \int_0^T\int_\Omega |\nabla u|^2 \sum_{j=1}^n (x - x_0)\cdot \nabla \eta_j dxdt$$

$$+2\int_0^T\int_\Omega (x_k - x_{0k})\sum_{j=1}^n \frac{\partial \eta_j}{\partial x_i}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_k}dxdt$$

$$+2\int_0^T\int_\Omega \sum_{j=1}^n (x_k - S_{jk})\frac{\partial u}{\partial x_i}\frac{\partial \eta_j}{\partial x_i}\frac{\partial u}{\partial x_k}dxdt + \int_0^T\int_\Omega |\nabla u|^2 dxdt$$

$$+\int_0^T\int_\Gamma \left[\widetilde{m} + \sum_{j=1}^n \eta_j(x - S_j)\right]\cdot \nu |\nabla u|^2 d\sigma dt - \int_0^T\int_{\Gamma_N} \frac{\partial u}{\partial \nu} u d\sigma\ dt$$

$$-2\int_0^T\int_\Gamma \frac{\partial u}{\partial \nu}\left[\widetilde{m} + \sum_{j=1}^n \eta_j(x - S_j)\right]\cdot \nabla u d\sigma dt.$$

$$\bullet\text{Transformation of } \int_\Omega\int_0^T \frac{\partial^2 u}{\partial t^2}\widetilde{M}(u)\, dxdt.$$

Integrating by parts in time, we have

$$\int_\Omega\int_0^T \eta_j \frac{\partial^2 u}{\partial t^2}(x - S_j)\cdot \nabla u dtdx$$

$$= \int_\Omega \eta_j \frac{\partial u}{\partial t}(x - S_j)\cdot \nabla u\, dx\Big|_0^T - \int_0^T\int_\Omega \eta_j (x_k - S_{jk})\frac{\partial u}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x_k}\right) dtdx$$

$$= \int_\Omega \eta_j \frac{\partial u}{\partial t}(x - S_j)\cdot \nabla u\, dx\Big|_0^T + \frac{1}{2}\int_0^T\int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 \mathrm{div}(\eta_j(x - S_j)) dxdt$$

$$-\frac{1}{2}\int_0^T\int_\Gamma \eta_j(x - S_j)\cdot \nu \left(\frac{\partial u}{\partial t}\right)^2 d\sigma dt,$$

and then

$$\int_\Omega\int_0^T \frac{\partial^2 u}{\partial t^2}\widetilde{m}\cdot \nabla u dtdx$$

$$= \int_\Omega \frac{\partial u}{\partial t}\widetilde{m}\cdot \nabla u\, dx\Big|_0^T - \int_0^T\int_\Omega \widetilde{m}_k \frac{\partial u}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x_k}\right) dtdx$$

$$= \int\limits_{\Omega} \frac{\partial u}{\partial t}\widetilde{m}\cdot\nabla u\,dx\Big|_0^T + \frac{1}{2}\int\limits_0^T\int\limits_{\Omega}(\frac{\partial u}{\partial t})^2\mathrm{div}\widetilde{m}dxdt - \frac{1}{2}\int\limits_0^T\int\limits_{\Gamma}\widetilde{m}\cdot\nu\left(\frac{\partial u}{\partial t}\right)^2 d\sigma dt.$$

Similarly

$$\begin{aligned}
&\int\limits_{\Omega}\int\limits_0^T \frac{\partial^2 u}{\partial t^2}\widetilde{M}(u)dtdx \\
&= \int\limits_{\Omega}\frac{\partial u}{\partial t}\widetilde{M}(u)\,dx\Big|_0^T + \int\limits_0^T\int\limits_{\Omega}(\frac{\partial u}{\partial t})^2\mathrm{div}\left[\sum_{j=1}^n (x-S_j)\eta_j) + \widetilde{m}\right]dxdt \\
&\quad -\int\limits_0^T\int\limits_{\Omega}(\frac{\partial u}{\partial t})^2 dxdt - \int\limits_0^T\int\limits_{\Gamma}\left(\sum_{j=1}^n (x-S_j)\eta_j) + \widetilde{m}\right)\cdot\nu(\frac{\partial u}{\partial t})^2 d\sigma dt,
\end{aligned}$$

and

$$\begin{aligned}
&\int\limits_0^T\int\limits_{\Omega}\left(\frac{\partial u}{\partial t}\right)^2\mathrm{div}\left[\sum_{j=1}^n (x-S_j)\eta_j) + \widetilde{m}\right]dxdt \\
&= 2\int\limits_0^T\int\limits_{\Omega}\left(\frac{\partial u}{\partial t}\right)^2 dxdt + \int\limits_0^T\int\limits_{\Omega}(\frac{\partial u}{\partial t})^2\sum_{j=1}^n (x-S_j)\cdot\nabla\eta_j dxdt \\
&\quad -\int\limits_0^T\int\limits_{\Omega}\left(\frac{\partial u}{\partial t}\right)^2\sum_{j=1}^n (x-x_0)\cdot\nabla\eta_j dxdt,
\end{aligned}$$

it then follows that

$$\begin{aligned}
\int\limits_{\Omega}\int\limits_0^T \frac{\partial^2 u}{\partial t^2}\widetilde{M}(u)dtdx &= \int\limits_{\Omega}\frac{\partial u}{\partial t}\widetilde{M}(u)\,dx\Big|_0^T + \int\limits_0^T\int\limits_{\Omega}\left(\frac{\partial u}{\partial t}\right)^2\sum_{j=1}^n (x-S_j)\cdot\nabla\eta_j dxdt \\
&\quad -\int\limits_0^T\int\limits_{\Omega}\left(\frac{\partial u}{\partial t}\right)^2\sum_{j=1}^n (x-x_0)\cdot\nabla\eta_j dxdt + \int\limits_0^T\int\limits_{\Omega}\left(\frac{\partial u}{\partial t}\right)^2 dxdt \\
&\quad -\int\limits_0^T\int\limits_{\Gamma}\left(\sum_{j=1}^n (x-S_j)\eta_j + \widetilde{m}\right)\cdot\nu\left(\frac{\partial u}{\partial t}\right)^2 d\sigma dt. \qquad (5.6)
\end{aligned}$$

We deduce the requested identity from (5.5) and (5.6).

Setting

$$I_1 = \int_0^T \int_\Omega |\nabla u|^2 \sum_{j=1}^n (x - S_j) \cdot \nabla \eta_j dxdt - \int_0^T \int_\Omega |\nabla u|^2 \sum_{j=1}^n (x - x_0) \cdot \nabla \eta_j dxdt$$

$$- 2 \int_0^T \int_\Omega (x_k - x_{0k}) \sum_{j=1}^n \frac{\partial \eta_j}{\partial x_i} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_k} dxdt$$

$$- 2 \int_0^T \int_\Omega \sum_{j=1}^n (x_k - S_{jk}) \frac{\partial u}{\partial x_i} \frac{\partial \eta_j}{\partial x_i} \frac{\partial u}{\partial x_k} dxdt$$

$$- \int_\Omega \frac{\partial u}{\partial t} \widetilde{M}\, dx \Big|_0^T - \int_0^T \int_\Omega \left(\frac{\partial u}{\partial t} \right)^2 \sum_{j=1}^n (x - S_j) \cdot \nabla \eta_j dxdt$$

$$+ \int_0^T \int_\Omega \left(\frac{\partial u}{\partial t} \right)^2 \sum_{j=1}^n (x - x_0) \cdot \nabla \eta_j dxdt - \int_\Omega \int_0^T a(x) \frac{\partial u}{\partial t} \widetilde{M}(u) dtdx$$

and

$$I_2 = \int_0^T \int_\Gamma \left(\sum_{j=1}^n (x - S_j)\eta_j + \widetilde{m} \right) \cdot \nu \left(\frac{\partial u}{\partial t} \right)^2 d\sigma dt$$

$$- \int_0^T \int_\Gamma \left[\widetilde{m} + \sum_{j=1}^n \eta_j (x - S_j) \right] \cdot \nu |\nabla u|^2 d\sigma dt + \int_0^T \int_{\Gamma_N} \frac{\partial u}{\partial \nu} u d\sigma\, dt$$

$$+ 2 \int_0^T \int_\Gamma \frac{\partial u}{\partial \nu} \left[\widetilde{m} + \sum_{j=1}^n \eta_j (x - S_j) \right] \cdot \nabla u d\sigma dt$$

$$+ \int_0^T \int_{\Gamma_N^2} \alpha |u|^2\, d\sigma dt,$$

by Lemma 5.1, we get

$$2 \int_0^T E(t) dt = I_1 + I_2.$$

We now need to estimate appropriately I_1 and I_2.
The values of I_2 on Γ_D, Γ_N^1 and Γ_N^2, are respectively

$$I_2(\Gamma_D) = \int_0^T \int_{\Gamma_D} (x - x_0) \cdot \nu |\nabla u|^2 d\sigma dt$$

$$+\int_0^T\int_{\Gamma_D}\left(\sum_{j=1}^n (x-S_j)\eta_j+\widetilde{m}\right)\cdot\nu\left(\frac{\partial u}{\partial t}\right)^2 d\sigma dt\le 0,$$

$$I_2(\Gamma_N^1)=0,$$

and

$$\begin{aligned} I_2(\Gamma_N^2)=&\int_0^T\int_{\Gamma_N^2}\left(\sum_{j=1}^n (x-S_j)\eta_j+\widetilde{m}\right)\cdot\nu\left(\frac{\partial u}{\partial t}\right)^2 d\sigma dt\\ &-\int_0^T\int_{\Gamma_N^2}\left[\widetilde{m}+\sum_{j=1}^n\eta_j(x-S_j)\right]\cdot\nu|\nabla u|^2 d\sigma dt+\int_0^T\int_{\Gamma_N^2}\frac{\partial u}{\partial\nu}u d\sigma\, dt\\ &+2\int_0^T\int_{\Gamma_N^2}\frac{\partial u}{\partial\nu}\left[\widetilde{m}+\sum_{j=1}^n\eta_j(x-S_j)\right]\cdot\nabla u d\sigma dt\\ &+\int_0^T\int_{\Gamma_N^2}\alpha|u|^2\, d\sigma dt. \end{aligned}$$

Remark 5.3 If the cracks are not straight we cannot garantee that $I_2(\Gamma_N^1)=0$, in the same manner if Γ_D meets one crack, then $I_2(\Gamma_D)$ is no more non positive.

Lemma 5.4 *Let u be a strong solution of problem* (1.1)*. There exist* $C>0$ *and* $\varepsilon_0>0$ *such that for every* $\varepsilon\in]0,\varepsilon_0[$

$$I_2(\Gamma_N^2)\le\varepsilon\int_0^T E(t)dt+\frac{C}{\varepsilon}\left(E(0)+\int_0^T\int_{\Gamma_N^2}\alpha|u|^2\, d\sigma dt\right),\quad\forall T>0. \tag{5.7}$$

Proof Set $m=\widetilde{m}+\sum_{j=1}^n\eta_j(x-S_j)$, then

$$\begin{aligned} I_2(\Gamma_N^2)=&\int_0^T\int_{\Gamma_N^2}\left(-m\cdot\nu|\nabla u|^2+m\cdot\nu\left(\frac{\partial u}{\partial t}\right)^2-\left(\alpha u+b\,\frac{\partial u}{\partial t}\right)\widetilde{M}(u)\right)d\sigma dt\\ &+\int_0^T\int_{\Gamma_N^2}\alpha|u|^2\, d\sigma dt. \end{aligned}$$

On Γ_N^2, we have

$$m \cdot \nu(x) = \left(1 - \sum_{j=1}^{n} \eta_j(x)\right)(x - x_0) \cdot \nu(x) + \sum_{j=1}^{n} \eta_j(x)(x - S_j) \cdot \nu(x),$$

and we deduce by the hypotheses (1.3) and (1.4), that there exists a constant $m_0 > 0$ such that

$$m \cdot \nu(x) \geq m_0, \quad \forall x \in \Gamma_N^2.$$

By using Young's inequality, we obtain

$$I_2(\Gamma_N^2) \leq \int_0^T \int_{\Gamma_N^2} \left(\left(||m||_\infty + \frac{||b||_\infty ||m||_\infty^2 b}{\varepsilon} + \frac{||b||_\infty b}{2\varepsilon}\right)\left(\frac{\partial u}{\partial t}\right)^2 + \varepsilon u^2\right) d\sigma dt$$
$$+ \int_0^T \int_{\Gamma_N^2} \Big(2\varepsilon - m \cdot \nu)\Big)|\nabla u|^2 d\sigma dt$$

where C is a positive constant. The previous estimate for $\varepsilon < \frac{m_0}{2}$, allows to deduce that

$$I_2(\Gamma_N^2) \leq \int_0^T \int_{\Gamma_N^2} \left(\left(||m||_\infty + \frac{||b||_\infty ||m||_\infty^2 b}{\varepsilon} + \frac{||b||_\infty b}{2\varepsilon}\right)\left(\frac{\partial u}{\partial t}\right)^2 + \varepsilon u^2\right) d\sigma dt$$
$$+ \frac{C}{\varepsilon} \int_0^T \int_{\Gamma_N^2} \alpha |u|^2 \, d\sigma dt.$$

As b is uniformly positive definite, there exists a constant $c_1 > 0$ such that

$$\begin{aligned}
\int_0^T \int_{\Gamma_N^2} \left(||m||_\infty + \frac{||m||_\infty b}{\varepsilon} + \frac{b}{2\varepsilon}\right)\left(\frac{\partial u}{\partial t}\right)^2 d\sigma dt &\leq \frac{c_1}{\varepsilon} \int_0^T \int_{\Gamma_N^2} b \left(\frac{\partial u}{\partial t}\right)^2 d\sigma dt \\
&\leq -\frac{c_1}{\varepsilon} \int_0^T E'(t) dt \\
&\leq \frac{c_1}{\varepsilon} E(0).
\end{aligned}$$

By applying Poincaré's inequality, we get

$$\begin{aligned} \varepsilon \int_{\Gamma_N^2} u^2 d\sigma &\leq c_2 \varepsilon \int_{\Omega} |\nabla u|^2 dx \\ &\leq 2 c_2 \varepsilon E(t), \end{aligned}$$

where c_2 is a positive constant independent of t. These three estimates imply (5.7). □

For all j, we consider a cut-off function $\widetilde{\eta_j}$ such that $\widetilde{\eta_j} = 1$ in supp η_j with support in a neighbourhood of supp η_j.
By setting $O = \bigcup_{j=1}^n \operatorname{supp} \eta_j$ and $O' = \bigcup_{j=1}^n \operatorname{supp} \widetilde{\eta_j}$, we have the following result:

Lemma 5.5 *Let u be a strong solution of* (1.1). *Then, there exists $C > 0$ such that for all $T > 0$*

$$\int_0^T \int_O |\nabla u|^2 dx dt \leq C \left[E(0) + \int_0^T \int_{O'} \left(\left(\frac{\partial u}{\partial t} \right)^2 + u^2 \right) dx dt + \int_0^T \int_{O' \cap \Gamma_N^2} \left(\frac{\partial u}{\partial t} \right)^2 d\sigma dt + \int_0^T \int_{O' \cap \Gamma_N^2} u^2 d\sigma dt \right]. \tag{5.8}$$

Proof Multiplying (1.1) by $\widetilde{\eta_j} u$ and integrate by parts, we obtain

$$\begin{aligned} A &= \int_0^T \int_{\Omega} -\Delta u \widetilde{\eta_j} u dx dt \\ &= \int_0^T \int_{\Omega} \nabla u \cdot \nabla (\widetilde{\eta_j} u) dx dt - \int_0^T \int_{\Gamma} \frac{\partial u}{\partial \nu} \widetilde{\eta_j} u d\sigma dt \\ &= \int_0^T \int_{\Omega} u \nabla u \cdot \nabla \widetilde{\eta_j} dx dt + \int_0^T \int_{\Omega} |\nabla u|^2 \widetilde{\eta_j} dx dt - \int_0^T \int_{\Gamma} \frac{\partial u}{\partial \nu} \widetilde{\eta_j} u d\sigma dt \\ &= \int_0^T \int_{\Omega} |\nabla u|^2 \widetilde{\eta_j} dx dt - \frac{1}{2} \int_0^T \int_{\Omega} u^2 \Delta \widetilde{\eta_j} dx dt \\ &\quad - \int_0^T \int_{\Gamma} \frac{\partial u}{\partial \nu} \widetilde{\eta_j} u d\sigma dt + \frac{1}{2} \int_0^T \int_{\Gamma} u^2 \frac{\partial \widetilde{\eta_j}}{\partial \nu} d\sigma dt \end{aligned}$$

$$= \int_0^T \int_\Omega |\nabla u|^2 \widetilde{\eta}_j dxdt - \frac{1}{2} \int_0^T \int_\Omega u^2 \triangle \widetilde{\eta}_j dxdt$$
$$+ \int_0^T \int_{\Gamma_N^2} ((\alpha u + b u_t)\widetilde{\eta}_j u + \frac{1}{2} u^2 \frac{\partial \widetilde{\eta}_j}{\partial \nu}) d\sigma dt.$$

$$B = \int_0^T \int_\Omega \frac{\partial^2 u}{\partial t^2} \widetilde{\eta}_j u dxdt = \int_\Omega \frac{\partial u}{\partial t} \widetilde{\eta}_j u dx \Big|_0^T - \int_0^T \int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 \widetilde{\eta}_j dxdt$$

and

$$C = \int_0^T \int_\Omega a \frac{\partial u}{\partial t} \widetilde{\eta}_j u dxdt.$$

Adding A, B and C, and summing on j, we obtain

$$\int_0^T \int_\Omega |\nabla u|^2 \sum_{j=1}^n \widetilde{\eta}_j dxdt$$
$$= \frac{1}{2} \int_0^T \int_\Omega u^2 \triangle \left(\sum_{j=1}^n \widetilde{\eta}_j \right) dxdt - \int_\Omega \frac{\partial u}{\partial t} \sum_{j=1}^n \widetilde{\eta}_j u dx \Big|_0^T + \int_0^T \int_\Omega \left(\frac{\partial u}{\partial t}\right)^2 \widetilde{\eta}_j dxdt$$
$$- \int_0^T \int_\Omega a \frac{\partial u}{\partial t} \widetilde{\eta}_j u dxdt + \int_0^T \int_{\Gamma_N^2} \left(-(\alpha u + b u_t)\widetilde{\eta}_j u - \frac{1}{2} u^2 \frac{\partial \widetilde{\eta}_j}{\partial \nu} \right) d\sigma dt.$$

Hence by Young's and Cauchy–Schwarz's inequalities, we obtain (5.8). ◻

In order to obtain the requested integral inequality, we need to estimate the terms $\int_0^T \int_{O'} u^2 dxdt$ and $\int_0^T \int_{O' \cap \Gamma_N^2} u^2 d\sigma dt$ in (5.8), as well as the term $\int_0^T \int_{\Gamma_N^2} \alpha |u|^2 \, d\sigma dt$ in (5.7). For that purposes, we prove the following Lemmas.

Lemma 5.6 *Let u be a strong solution of problem* (1.1)*. Then there exists a positive constant C such that for all* $T > 0$ *and* $\varepsilon \in (0, 1)$*, we have*

$$\int_0^T \int_\Omega a u^2 dxdt \le \frac{C}{\varepsilon} \left(E(0) + \int_0^T \int_{\Gamma_N^2} \alpha u^2 \, d\sigma dt \right) + \varepsilon \int_0^T E(t) dt. \tag{5.9}$$

Proof For all $t \geq 0$, consider the problem: find $z = z(t)$ solution of

$$\begin{cases} \triangle z = au & \text{in } \Omega \\ z = 0 & \text{on } \Gamma_D, \\ \frac{\partial z}{\partial \nu} = 0 & \text{on } \Gamma_N. \end{cases}$$

This problem admits the weak formulation: let $z \in H^1_{\Gamma_D}(\Omega)$ be the unique solution of

$$\int_\Omega \nabla z \cdot \nabla w \, dx = -\int_\Omega auw \, dx, \quad \forall w \in H^1_{\Gamma_D}(\Omega). \tag{5.10}$$

Taking $w = z$ in (5.10) we obtain

$$\int_\Omega |\nabla z|^2 dxdt = -\int_\Omega auzdx.$$

By Cauchy–Schwarz's and Poincaré's inequalities, this implies that

$$\int_\Omega |\nabla z|^2 dxdt \leq C_1 \int_\Omega au^2 dx \leq C_2 E(t).$$

In particular, again using Poincaré's inequality and a trace theorem, we get

$$||z||^2_{L^2(\Omega)} \leq C_3 ||\sqrt{a}u||^2_{L^2(\Omega)} \leq C_4 E(t), \tag{5.11}$$

$$||z||^2_{L^2(\Gamma_N^2)} \leq C_5 ||\sqrt{a}u||^2_{L^2(\Omega)} \leq C_6 E(t). \tag{5.12}$$

Differentiating (5.10) with respect to t, z' is solution of the problem

$$\begin{cases} \triangle z' = au' & \text{in } \Omega \\ z' = 0 & \text{on } \Gamma_D, \\ \frac{\partial z'}{\partial \nu} = 0 & \text{on } \Gamma_N, \end{cases} \tag{5.13}$$

then the above considerations lead to

$$||z'||^2_{L^2(\Omega)} \leq c_1 \int_\Omega au'^2 dx \leq -c_1 E'(t). \tag{5.14}$$

Multiplying (1.1) by z, we obtain

$$\int_0^T \int_\Omega \left(\frac{\partial^2 u}{\partial t^2} - \Delta u + au_t \right) zdxdt = 0. \tag{5.15}$$

We now modify the first two terms of this left-hand side. Since u and z belong to the domain $D(\triangle)$ of $\triangle$, by Lemma 4.2 we have

$$-\int_0^T\int_\Omega \triangle uzdxdt = \int_0^T\int_\Omega \nabla z\cdot\nabla udxdt - \int_0^T\int_\Gamma \frac{\partial u}{\partial\nu}zd\sigma dt.$$

Consequently, by (5.10) (since u belongs to $H^1_{\Gamma_D}(\Omega)$) and the boundary conditions satisfied by u, we get

$$-\int_0^T\int_\Omega \triangle uzdxdt = -\int_0^T\int_\Omega au^2dxdt + \int_0^T\int_{\Gamma_N^2} b\frac{\partial u}{\partial t}zd\sigma dt + \int_0^T\int_{\Gamma_N^2}\alpha uzd\sigma dt.$$

For the second term in (5.15) by integrating by parts in time, we directly have

$$\int_0^T\int_\Omega \frac{\partial^2 u}{\partial t^2}zdxdt = \int_\Omega \frac{\partial u}{\partial t}zdx\Big|_0^T - \int_0^T\int_\Omega \frac{\partial u}{\partial t}\frac{\partial z}{\partial t}dxdt.$$

These two identities in (5.15) lead to

$$\begin{aligned}\int_0^T\int_\Omega au^2dxdt = \int_\Omega \frac{\partial u}{\partial t}zdx\Big|_0^T - \int_0^T\int_\Omega \frac{\partial u}{\partial t}\frac{\partial z}{\partial t}dxdt + \int_0^T\int_\Omega a\frac{\partial u}{\partial t}zdxdt\\ + \int_0^T\int_{\Gamma_N^2} b\frac{\partial u}{\partial t}zd\sigma dt + \int_0^T\int_{\Gamma_N^2}\alpha uzd\sigma dt.\end{aligned} \tag{5.16}$$

Using Young's inequality and the previous estimates we get

$$\begin{aligned}\int_0^T\int_\Omega \frac{\partial u}{\partial t}\frac{\partial z}{\partial t}dxdt &\le \varepsilon\int_0^T E(t)dt - \frac{C_1}{2\varepsilon}\int_0^T E'(t)dt\\ &\le \frac{C_1}{2\varepsilon}E(0) + \varepsilon\int_0^T E(t)dt.\end{aligned}$$

$$\int_0^T\int_{\Gamma_N^2} b\frac{\partial u}{\partial t}zd\sigma dt \le \frac{1}{2\varepsilon}\int_0^T\int_{\Gamma_N^2} b(\frac{\partial u}{\partial t})^2dtd\sigma + \varepsilon\|b\|_\infty\int_0^T\int_{\Gamma_N^2} z^2dtd\sigma$$

$$\leq \frac{1}{2\varepsilon}E(0) + C\varepsilon \int_0^T E(t)dt.$$

$$\int_0^T \int_\Omega a\frac{\partial u}{\partial t} z dx dt \leq \frac{\| a \|_\infty}{2\varepsilon} \int_\Omega \int_0^T a\left(\frac{\partial u}{\partial t}\right)^2 dx dt + \varepsilon \int_0^T \int_\Omega z^2 dx dt$$

$$\leq C\left(\frac{1}{\varepsilon}E(0) + \varepsilon \int_0^T E(t)dt\right).$$

$$-\int_\Omega \frac{\partial u}{\partial t} z dx\Big|_0^T = -\int_\Omega \frac{\partial u}{\partial t}(T)z(T)dx + \int_\Omega \frac{\partial u}{\partial t}(0)z(0)dx$$

$$\leq \frac{1}{2\varepsilon}\int_\Omega u_t^2(T)dx + \varepsilon\int_\Omega z^2(T)dx + \frac{1}{2\varepsilon}\int_\Omega u_t^2(0)dx + \varepsilon \int_\Omega z^2(0)dx$$

$$\leq CE(0).$$

$$\int_0^T \int_{\Gamma_N^2} \alpha u z d\sigma dt \leq \frac{\| \alpha \|_\infty}{2\varepsilon} \int_{\Gamma_N^2} \int_0^T \alpha u^2 d\sigma dt + \varepsilon \int_0^T \int_{\Gamma_N^2} z^2 d\sigma dt$$

$$\leq C\left(\frac{1}{\varepsilon}\int_{\Gamma_N^2} \int_0^T \alpha u^2 d\sigma dt + \varepsilon \int_0^T E(t)dt\right).$$

These five estimates in (5.16) lead to (5.9). □

Lemma 5.7 *Let u be the strong solution of problem* (1.1). *Then there exist a positive constant C and $\varepsilon_0 > 0$ small enough such that for all $T > 0$ and $\varepsilon \in]0, \varepsilon_0[$, we have*

$$\int_0^T \int_{O' \cap \Gamma_N^2} u^2 d\sigma dt \leq \frac{C}{\varepsilon^5}\left(E(0)\int_{\Gamma_N^2}\int_0^T \alpha u^2 d\sigma dt\right) + \varepsilon \int_0^T E(t)dt. \tag{5.17}$$

Proof By a trace theorem in O', there exists $C > 0$ such that

$$\int_{O' \cap \Gamma_N^2} u^2 d\sigma \leq C\| u \|^2_{H^{\frac{3}{4}}(0')}.$$

Now using an interpolation inequality (see for instance Theorem 1.4.4.3 of 7), we deduce that there exists $K > 0$ such that

$$\int_{O' \cap \Gamma_N^2} u^2 d\sigma \le C \left(\varepsilon^2 \| u \|^2_{H^1(0')} + K^2 \varepsilon^{-6} \| u \|^2_{L^2(0')} \right),$$

for all $\varepsilon > 0$. Setting $\eta = C\varepsilon^2$, we find that

$$\int_{O' \cap \Gamma_N^2} u^2 d\sigma \le \eta \int_{O'} |\nabla u|^2 \, dx + C\eta^{-3} \| u \|^2_{L^2(0')},$$

for all $\eta \in (0, 1)$.

Integrating this expression in $(0, T)$ and using Lemma 5.6 (estimate (5.9)), we obtain

$$\int_0^T \int_{O' \cap \Gamma_N^2} u^2 d\sigma dt \le \eta \int_0^T E(t) dt + C\eta^{-3} \left(\frac{C}{\varepsilon} \left(E(0) + \int_{\Gamma_N^2} \int_0^T \alpha u^2 d\sigma dt \right) + \varepsilon \int_0^T E(t) dt \right),$$

for all $\eta \in (0, 1)$ and all $\varepsilon \in (0, 1)$. This estimate is trivially equivalent to

$$\int_0^T \int_{O' \cap \Gamma_N^2} u^2 d\sigma dt \le \left(\eta + \frac{C\varepsilon}{\eta^3} \right) \int_0^T E(t) dt + \frac{C}{\varepsilon \eta^3} E(0),$$

for all $\eta \in (0, 1)$ and all $\varepsilon \in (0, 1)$. By chosing ε such that

$$\frac{C\varepsilon}{\eta^3} = \eta,$$

or equivalently

$$\varepsilon = C^{-1} \eta^4,$$

we obtain

$$\int_0^T \int_{O' \cap \Gamma_N^2} u^2 d\sigma dt \le 2\eta \int_0^T E(t) dt + \frac{C}{\eta^5} \left(E(0) + \int_{\Gamma_N^2} \int_0^T \alpha u^2 d\sigma dt \right),$$

for all $\eta \in (0, 1)$ small enough. This proves (5.17) by renaming $2\eta = \varepsilon$. □

Lemma 5.8 *Let u be the strong solution of problem (1.1). Then there exists a positive constant C such that for all $T > 0$ and $\varepsilon \in]0, 1[$, we have*

$$\int_0^T \int_{\Gamma_N^2} \alpha u^2 d\sigma dt \leq \frac{C}{\varepsilon} E(0) + \varepsilon \int_0^T E(t) dt. \tag{5.18}$$

Proof We proceed as in [10] but here by taking into account the presence of cracks. Namely for all $t \geq 0$, we consider $z = z(t) \in H^1(\Omega)$ solution of (see section 4)

$$\begin{cases} \triangle z = 0 & \text{in } \Omega \\ z = 0 & \text{on } \Gamma_D, \\ \frac{\partial z}{\partial \nu} = 0 & \text{on } \Gamma_N^1, \\ z = u & \text{on } \Gamma_N^2. \end{cases} \tag{5.19}$$

This is equivalent to $z = \omega + u$, where $\omega \in V = \{w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_D \cup \Gamma_N^2\}$ solution of

$$\int_\Omega \nabla \omega \cdot \nabla w \, dx = - \int_\Omega \nabla u \cdot \nabla w \, dx, \quad \forall w \in V.$$

This identity is equivalent to

$$\int_\Omega \nabla z \cdot \nabla w \, dx = 0, \quad \forall w \in V. \tag{5.20}$$

Taking $w = z - u$, we find in particular

$$\int_\Omega \nabla z \cdot \nabla (z - u) \, dx = 0,$$

or equivalently

$$\int_\Omega \nabla z \cdot \nabla u \, dx = \int_\Omega \nabla z \cdot \nabla z \, dx \geq 0. \tag{5.21}$$

Let us now show that z also satisfies

$$\int_\Omega f \cdot z dx = - \int_\Gamma z \frac{\partial v_f}{\partial \nu} d\Gamma, \quad \forall f \in L^2(\Omega), \tag{5.22}$$

where $v_f \in V$ is the unique solution of

$$\int_\Omega \nabla v_f \cdot \nabla w dx = \int_\Omega f w dx, \quad \forall w \in V.$$

Indeed from (5.20) we may write

$$\int_\Omega \nabla v_f \cdot \nabla z dx = 0.$$

As z is solution of (5.19) with $u(t) \in D(\triangle)$ and v_f belong to $D(\triangle_{Dir})$ by Green's formula (see Corollary 4.5), we deduce that

$$-\int_\Omega \triangle v_f \, z dx + \int_\Gamma z \frac{\partial v_f}{\partial \nu} d\Gamma = 0.$$

This proves the identity (5.22) since $\triangle v_f = -f$.

In the identity (5.22) taking $f = z$, we may write

$$\int_\Omega |z|^2 dx = -\int_\Gamma z \frac{\partial v_z}{\partial \nu} d\Gamma = -\int_{\Gamma_N^2} u \frac{\partial v_z}{\partial \nu} d\Gamma,$$

since $z = u$ on Γ_N^2, $\frac{\partial v_z}{\partial \nu} = 0$ on Γ_N^1 and $z = 0$ on Γ_D. By Cauchy–Schwarz's inequality we obtain

$$\int_\Omega |z|^2 dx \leq \| u \|_{L^2(\Gamma_N^2)} \left\| \frac{\partial v_z}{\partial \nu} \right\|_{L^2(\Gamma_N^2)}. \tag{5.23}$$

By Lemma 4.1 we have for all $\alpha > 0$ small enough

$$\| v_{zR} \|_{W^{2,p}(\Omega)} \leq K \| z \|_{L^2(\Omega)}, \tag{5.24}$$

for some positive constant K when $p = \frac{4}{3} + \alpha^*$. This estimate, a standard trace theorem and the embedding $W^{1-\frac{1}{p},p}(\Gamma_N^2) \hookrightarrow L^2(\Gamma_N^2)$ lead to

$$\left\| \frac{\partial v_z}{\partial \nu} \right\|_{L^2(\Gamma_N^2)} = \left\| \frac{\partial v_{zR}}{\partial \nu} \right\|_{L^2(\Gamma_N^2)} \leq K_1 \| z \|_{L^2(\Omega)},$$

for some positive constant K_1. Inserting this estimate in (5.23) we arrive at

$$\int_\Omega |z|^2 dx \leq C_0 \int_{\Gamma_N^2} |u|^2 d\Gamma,$$

where C_0 is a positive constant. Since z' is solution of problem (5.19) with u' instead of u, the above arguments yield

$$\int_{\Omega} |z'|^2 dx \le C_0 \int_{\Gamma_N^2} |u'|^2 d\Gamma.$$

Multiplying the first identity of (1.1) by z and integrating on $\Omega \times (0, T)$, we obtain

$$\int_{\Omega\times(0,T)} z(u'' - \triangle u + au')dxdt = 0.$$

Applying Green's formula (allowed by Corollary 4.5) and taking into account the boundary conditions in (1.1), we get

$$\int_{\Omega\times(0,T)} (zu'' + \nabla z \cdot \nabla u + au'z)dxdt + \int_{\Gamma_N^2\times(0,T)} (\alpha u + bu')ud\sigma dt = 0.$$

Integrating by part in t and using (5.21), we obtain

$$\begin{aligned}\int_{\Gamma_N^2\times(0,T)} \alpha|u|^2 d\sigma dt \le &- \int_{\Gamma_N^2\times(0,T)} buu'd\sigma dt + \int_{\Omega\times(0,T)} z'u'\,dxdt \\ &- \int_{\Omega\times(0,T)} au'zdxdt - \int_{\Omega} zu'\Big|_0^T.\end{aligned}$$

The rest of the proof is as in the proof of Lemma 5.6 using several times (5.2), (5.25), (5.25) and Young's inequality. □

Remark 5.9 Lemma 5.7 is only necessary if α is not uniformly bounded from below in $O' \cap \Gamma_N^2$, but according to our assumption this could occur for instance in the case when int $\Gamma_D \neq \emptyset$.

Proof of Theorem 3.1 By Lemmas 5.4 and 5.8 we have

$$\begin{aligned}I_2 &= I_2(\Gamma_N^2) + I_2(\Gamma_D) + I_2(\Gamma_N^1) \\ &\le I_2(\Gamma_N^2) \\ &\le \frac{C}{\varepsilon}E(0) + \varepsilon \int_0^T E(t)dt,\end{aligned}$$

recalling that $I_2(\Gamma_D) \leq 0$ and $I_2(\Gamma_N^1) = 0$.
Applying Young's inequality, we obtain

$$I_1 \leq C \int_0^T \int_O \left(\left(\frac{\partial u}{\partial t} \right)^2 + |\nabla u|^2 \right) dx dt + \frac{C}{\varepsilon} \int_0^T \int_\Omega a(x) \left(\frac{\partial u}{\partial t} \right)^2 dx dt$$
$$+ \varepsilon \int_0^T \int_\Omega u^2 dx dt + C E(0)$$
$$\leq C \int_0^T \int_O \left(\left(\frac{\partial u}{\partial t} \right)^2 + |\nabla u|^2 \right) dx dt + \frac{C}{\varepsilon} E(0) + \varepsilon \int_0^T E(t) dt.$$

We know that

$$2 \int_0^T E(t) dt \leq I_1 + I_2,$$

hence, from Lemmas 5.1, 5.6 and 5.7 we obtain that there exists $C > 0$ such that

$$2 \int_0^T E(t) dt \leq \frac{C}{\varepsilon^5} E(0) + \varepsilon \int_0^T E(t) dt,$$

for all ε small enough. By choosing $\varepsilon < 2$, we get

$$\int_0^T E(t) dt \leq C_1 E(0),$$

where C_1 is a positive constant independent of T. Since our system is invariant by translation we get for all $S \geq 0$,

$$\int_S^{S+T} E(t) dt \leq C_1 E(S)$$

and by letting T tend to infinity we have shown that the energy of our system satisfies (3.2). Hence Lemma 3.3 allows us to conclude the exponential stability of (1.1).

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