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EXISTENCE AND MULTIPLICITY FOR ELLIPTIC P-LAPLACIAN PROBLEMS WITH CRITICAL GROWTH IN THE GRADIENT

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ABSTRACT. We consider the boundary value problem

$$(P_\lambda) \quad -\Delta_p u = \lambda c(x)|u|^{p-2}u + \mu(x)|\nabla u|^p + h(x), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with smooth boundary. We assume $c, h \in L^q(\Omega)$ for some $q > \max\{N/p, 1\}$ with $c \geq 0$ and $\mu \in L^\infty(\Omega)$. We prove existence and uniqueness results in the coercive case $\lambda \leq 0$ and existence and multiplicity results in the non-coercive case $\lambda > 0$. Also, considering stronger assumptions on the coefficients, we clarify the structure of the set of solutions in the non-coercive case.

1. INTRODUCTION AND MAIN RESULTS

Let $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ denote the p -Laplacian operator. We consider, for any $1 < p < \infty$, the boundary value problem

$$(P_\lambda) \quad -\Delta_p u = \lambda c(x)|u|^{p-2}u + \mu(x)|\nabla u|^p + h(x), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

under the assumptions

$$(A_0) \quad \begin{cases} \Omega \subset \mathbb{R}^N, N \geq 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^{0,1}, \\ c \text{ and } h \text{ belong to } L^q(\Omega) \text{ for some } q > \max\{N/p, 1\}, \\ c \geq 0 \text{ and } \mu \in L^\infty(\Omega). \end{cases}$$

The study of quasilinear elliptic equations with a gradient dependence up to the critical growth $|\nabla u|^p$ was initiated by L. Boccardo, F. Murat and J.P. Puel in the 80's and it has been an active field of research until now. Under the condition $\lambda c(x) \leq -\alpha_0 < 0$ for some $\alpha_0 > 0$, which is now referred to as the *coercive case*, the existence of solution is a particular case of the results of [9, 11, 16]. The *weakly coercive case* ($\lambda = 0$) was studied in [24] where, for $\|\mu h\|_{N/p}$ small enough, the existence of a unique solution is obtained, see also [1]. The *limit coercive case*, where one just require that $\lambda c(x) \leq 0$ and hence c may vanish only on some parts of Ω , is more complex and was left open until [8]. In that paper, for the case $p = 2$, it was observed that, under the assumption (A_0) , the existence of solutions to (P_λ) is not guaranteed. Sufficient conditions in order to ensure the existence of solution were given.

The case $\lambda c(x) \geq 0$ also remained unexplored until very recently. First, in [31] the authors studied problem (P_λ) with $p = 2$. Assuming $\lambda > 0$ and μh small enough, in an appropriate sense, they proved the existence of at least two solutions. This result has now be complemented in several ways. In [30] the existence of two solutions is obtained, allowing the function c to change sign with $c^+ \not\equiv 0$ but assuming $h \geq 0$. In both [30, 31]

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$\mu > 0$ is assumed constant. In [8] the restriction μ constant was removed but assuming that $h \geq 0$. Finally, in [17], under stronger regularity on the coefficients, cases where μ is non constant and h is non-positive or has no sign were treated. Actually in [17], under different sets of assumptions, the authors clarify the structure of the set of solutions to (P_λ) in the non-coercive case. Now, concerning (P_λ) with $p \neq 2$, the only results in the case $\lambda c \geq 0$ are, up to our knowledge, presented in [1, 27]. In [27] the case c constant and $h \equiv 0$ is covered and in [1], the model equation is $-\Delta_p u = |\nabla u|^p + \lambda f(x)(1+u)^b$, $b \geq p-1$ and $f \geq 0$.

To state our first main result let us define

$$m_{p,\lambda}^+ := \begin{cases} \inf_{u \in W_\lambda} \int_{\Omega} \left(|\nabla u|^p - \left(\frac{\|\mu^+\|_\infty}{p-1} \right)^{p-1} h(x) |u|^p \right) dx, & \text{if } W_\lambda \neq \emptyset, \\ +\infty, & \text{if } W_\lambda = \emptyset, \end{cases}$$

and

$$m_{p,\lambda}^- := \begin{cases} \inf_{u \in W_\lambda} \int_{\Omega} \left(|\nabla u|^p - \left(\frac{\|\mu^-\|_\infty}{p-1} \right)^{p-1} h(x) |u|^p \right) dx, & \text{if } W_\lambda \neq \emptyset, \\ +\infty, & \text{if } W_\lambda = \emptyset. \end{cases}$$

where

$$W_\lambda := \{w \in W_0^{1,p}(\Omega) : \lambda c(x)w(x) = 0 \text{ a.e. } x \in \Omega, \|w\| = 1\}.$$

Note that $W_0 = W_0^{1,p}(\Omega)$ and W_λ is independent of λ when $\lambda \neq 0$. Using these notations, we state the following result which generalizes the results obtained in [8, Section 3]. In fact, if h is either non-negative or non-positive our hypothesis corresponds to the ones introduced in [8] for $p = 2$. However, if h does not have a sign, our hypothesis are weaker even for $p = 2$.

Theorem 1.1. *Assume that (A_0) holds and that $\lambda \leq 0$. Then if $m_{p,\lambda}^+ > 0$ and $m_{p,\lambda}^- > 0$, the problem (P_λ) has at least one solution.*

In the rest of the paper we assume that μ is constant. Namely, we replace (A_0) by

$$(A_1) \quad \begin{cases} \Omega \subset \mathbb{R}^N, N \geq 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^{0,1}, \\ c \text{ and } h \text{ belong to } L^q(\Omega) \text{ for some } q > \max\{N/p, 1\}, \\ c \geq 0 \text{ and } \mu > 0. \end{cases}$$

Observe that there is no loss of generality in assuming $\mu > 0$ since, if u is a solution to (P_λ) with $\mu < 0$, then $w = -u$ satisfies

$$-\Delta_p w = \lambda c(x) |w|^{p-2} w - \mu |\nabla w|^p - h(x).$$

In [7], for $p = 2$ but assuming only (A_0) , the uniqueness of solution when $\lambda \leq 0$ was obtained as a direct consequence of a comparison principle, see [7, Corollary 3.1]. As we show in Remark 3.3, such kind of principle does not hold in general when $p \neq 2$. Actually the issue of uniqueness for equations of the form of (P_λ) appears widely open. If partial results, assuming for example $1 < p \leq 2$ or $\lambda c(x) \leq -\alpha_0 < 0$, seem reachable adapting existing techniques, see in particular [33, 38, 39], a result covering the full generality of (P_λ) seems, so far, out of reach. Theorem 1.2 below, whose proof makes use of some ideas from [3], crucially relies on the assumption that μ is constant. It permits however to treat the limit case (P_0) which plays an important role in our paper.

Theorem 1.2. *Assume that (A_1) holds and suppose $\lambda \leq 0$. Then (P_λ) has at most one solution.*

Let us now introduce

$$(1.1) \quad m_p := \inf \left\{ \int_{\Omega} \left(|\nabla w|^p - \left(\frac{\mu}{p-1} \right)^{p-1} h(x) |w|^p \right) dx : w \in W_0^{1,p}(\Omega), \|w\| = 1 \right\}.$$

We can state the following result.

Theorem 1.3. *Assume that (A_1) holds. Then (P_0) has a solution if, and only if, $m_p > 0$.*

Theorem 1.3 provides, so to say, a characterization in term of a first eigenvalue of the existence of solution to (P_0) . This result again improves, for μ constant, [8] and it allows to observe that, in case $h \not\geq 0$, (P_0) has always a solution while the case $h \geq 0$ is the “worse” case for the existence of a solution. In case h changes sign, the negative part of h “helps” in order to have a solution to (P_0) . We give in Appendix A, sufficient conditions on h^+ in order to ensure $m_p > 0$.

Remark 1.1. Observe that the sufficient part of Theorem 1.3 is direct. Indeed, if $m_p > 0$ then $m_{p,0}^+ > 0$ and $m_{p,0}^- > 0$ and Theorem 1.1 implies that (P_0) has a solution.

Remark 1.2. We see, combining Theorems 1.1 and 1.3, that if (P_0) has a solution then (P_λ) has a solution for any $\lambda \leq 0$. Moreover this solution is unique by Theorem 1.2.

Now, we turn to the study the non-coercive case, namely when $\lambda > 0$. First, using mainly variational techniques we prove the following result.

Theorem 1.4. *Assume that (A_1) holds and suppose that (P_0) has a solution. Then there exists $\Lambda > 0$ such that, for any $0 < \lambda < \Lambda$, (P_λ) has at least two solutions.*

As we shall see in Corollary 9.4, the existence of a solution to (P_0) is, in some sense, necessary for the existence of a solution when $\lambda > 0$.

Next, considering stronger regularity assumptions, we derive informations on the structure of the set of solutions in the non-coercive case. These informations complement Theorem 1.4. We denote by $\gamma_1 > 0$ the first eigenvalue of the problem

$$(1.2) \quad -\Delta_p u = \gamma c(x)|u|^{p-2}u, \quad u \in W_0^{1,p}(\Omega),$$

and, under the assumptions

$$(A_2) \quad \begin{cases} \Omega \subset \mathbb{R}^N, N \geq 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^2, \\ c \text{ and } h \text{ belong to } L^\infty(\Omega), \\ c \not\geq 0 \text{ and } \mu > 0, \end{cases}$$

we state the following theorem.

Theorem 1.5. *Assume that (A_2) holds and suppose that (P_0) has a solution. Then:*

- *If $h \not\geq 0$, for every $\lambda > 0$, (P_λ) has at least two solutions u_1, u_2 with $u_1 \ll 0$.*
- *If $h \geq 0$, then $u_0 \geq 0$ and there exists $\bar{\lambda} \in (0, \gamma_1)$ such that:*
 - *for every $0 < \lambda < \bar{\lambda}$, (P_λ) has at least two solutions satisfying $u_i \geq u_0$;*
 - *for $\lambda = \bar{\lambda}$, (P_λ) has at least one solution satisfying $u \geq u_0$;*
 - *for any $\lambda > \bar{\lambda}$, (P_λ) has no non-negative solution.*

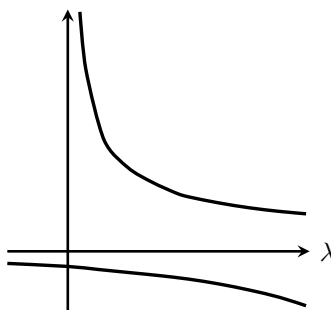
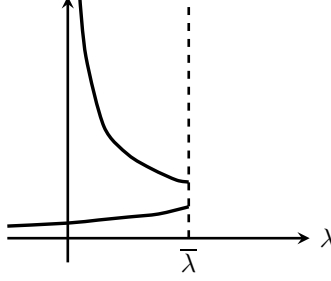


FIGURE 1. Illustration of Theorem 1.5 with $h \not\geq 0$

FIGURE 2. Illustration of Theorem 1.5 with $h \not\geq 0$

Remark 1.3.

- As observed above, in the case $h \leq 0$, the assumption that (P_0) has a solution is automatically satisfied.
- In the case $\mu < 0$, we have the opposite result i.e., two solutions for every $\lambda > 0$ in case $h \geq 0$ and, in case $h \leq 0$, the existence of $\bar{\lambda} > 0$ such that (P_λ) has at least two negative solutions, at least one negative solution or no non-positive solution according to $0 < \lambda < \bar{\lambda}$, $\lambda = \bar{\lambda}$ or $\lambda > \bar{\lambda}$.

In case $h \geq 0$, we know that for $\lambda > \bar{\lambda}$, (P_λ) has no non-negative solution but this does not exclude the possibility of having negative or sign changing solutions. Actually, we are able to prove the following result changing a little the point of view. We consider the boundary value problem

$$(P_{\lambda,k}) \quad -\Delta_p u = \lambda c(x)|u|^{p-2}u + \mu|\nabla u|^p + kh(x), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

with a dependence in the size of h and we obtain the following result.

Theorem 1.6. *Assume that (A_2) holds and that $h \geq 0$. Let*

$$k_0 = \sup \left\{ k \in [0, +\infty) : \forall w \in W_0^{1,p}(\Omega), \int_\Omega \left(|\nabla w|^p - \left(\frac{\mu}{p-1}\right)^{p-1} k h(x) |w|^p \right) dx > 0 \right\}.$$

Then:

- For all $\lambda \in (0, \gamma_1)$, there exists $\bar{k} = \bar{k}(\lambda) \in (0, k_0)$ such that, for all $k \in (0, \bar{k})$, the problem $(P_{\lambda,k})$ has at least two solutions u_1, u_2 with $u_i \geq 0$ and for all $k > \bar{k}$, the problem $(P_{\lambda,k})$ has no solution. Moreover, the function $\bar{k}(\lambda)$ is non-increasing.
- For $\lambda = \gamma_1$, the problem $(P_{\lambda,k})$ has a solution if and only if $k = 0$. In that case, the solution is unique and it is equal to 0.
- For all $\lambda > \gamma_1$, there exist $0 < \tilde{k}_1 \leq \tilde{k}_2 < +\infty$ such that, for all $k \in (0, \tilde{k}_1)$, the problem $(P_{\lambda,k})$ has at least two solutions with $u_{\lambda,1} \ll 0$ and $\min u_{\lambda,2} < 0$, for all $k > \tilde{k}_2$, the problem $(P_{\lambda,k})$ has no solution and, in case $\tilde{k}_1 < \tilde{k}_2$, for all $k \in (\tilde{k}_1, \tilde{k}_2)$, the problem $(P_{\lambda,k})$ has at least one solution u with $u \ll 0$ and $\min u < 0$. Moreover, the function $\tilde{k}_1(\lambda)$ is non-decreasing.

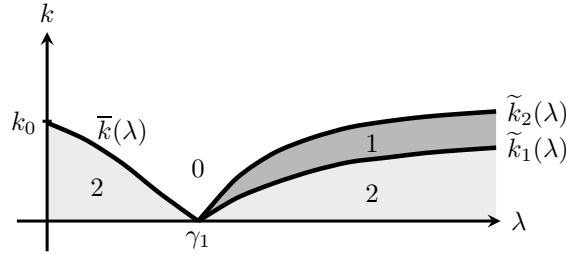


FIGURE 3. Existence regions of Theorem 1.6

Let us now say some words about our proofs. First note that when μ is assumed constant it is possible to perform a Hopf-Cole change of variable. Introducing

$$v = \frac{p-1}{\mu} \left(e^{\frac{\mu}{p-1}u} - 1 \right),$$

we can check that u is a solution of (P_λ) if, and only if, $v > -\frac{p-1}{\mu}$ is a solution of

$$(1.3) \quad -\Delta_p v = \lambda c(x)g(v) + \left(1 + \frac{\mu}{p-1}v\right)^{p-1} h(x), \quad v \in W_0^{1,p}(\Omega),$$

where g is an arbitrary function satisfying

$$g(s) = \left| \frac{p-1}{\mu} \left(1 + \frac{\mu}{p-1}s\right) \ln \left(1 + \frac{\mu}{p-1}s\right) \right|^{p-2} \frac{p-1}{\mu} \left(1 + \frac{\mu}{p-1}s\right) \ln \left(1 + \frac{\mu}{p-1}s\right), \quad \text{if } s > -\frac{p-1}{\mu}.$$

Working with problem (1.3) presents the advantage that one may assume, with a suitable choice of g when $s \leq -\frac{p-1}{\mu}$, that it has a variational structure. Nevertheless from this point we face several difficulties.

First, we need a control from below on the solutions to (1.3), i.e. having found a solution to (1.3) one needs to check that it satisfies $v > -\frac{p-1}{\mu}$, in order to perform the opposite change of variable and obtain a solution to (P_λ) . To that end, in Section 4, we prove the existence of a lower solution \underline{u}_λ to (P_λ) such that every upper solution β of (P_λ) satisfies $\beta \geq \underline{u}_\lambda$. This allows us to transform the problem (1.3) in a new one, which has the advantage of being completely equivalent to (P_λ) . Note that the existence of the lower solution ultimately relies on the existence of an a priori lower bound. See Lemma 4.1 for a more general result.

We denote by I_λ the functional associated to the new problem, see (5.5) for a precise definition. The “geometry” of I_λ crucially depends on the sign of λ . When $\lambda \leq 0$ is it essentially coercive and one may search for a critical point as a global minimum. When $\lambda > 0$ the functional I_λ becomes unbounded from below and presents something like a concave-convex geometry. Then, in trying to obtain a critical point, the fact that g is only slightly superlinear at infinity is a difficulty. It implies that I_λ does not satisfies an Ambrosetti-Rabinowitz-type condition and proving that Palais-Smale or Cerami sequences are bounded may be challenging. In the case of the Laplacian, when $p = 2$, dealing with this issue is now relatively standard but for elliptic problems with a p -Laplacian things are more complex and we refer to [18, 27, 28, 34] in that direction. Note however that in these last works, it is always assumed a kind of homogeneity condition which is not available here. Consequently, some new ideas are required, see Section 8.

Having at hand the Cerami condition for I_λ with $\lambda > 0$, in order to prove Theorems 1.4, 1.5 and 1.6, we shall look for critical points which are either local-minimum or of mountain-pass type. In Theorem 1.4 the geometry of I_λ is “simple” and permits to use only variational arguments. In Theorems 1.5 and 1.6 however it is not so clear, looking directly to I_λ , where to search for critical points. We shall then make uses of lower and upper solutions arguments. In both theorems a first solution is obtained through the existence of well-ordered lower and upper solutions. This solution is further proved to be a local minimum of I_λ and it is then possible to obtain a second solution by a mountain pass argument. Our approach here follows the strategy presented in [12, 13, 20]. See also [6].

Finally, concerning Theorem 1.1, where μ is not assumed to be constant, we obtain our solution through the existence of lower and an upper solution which correspond to solutions to (P_λ) where $\mu = -\|\mu^-\|_\infty$ and $\mu = \|\mu^+\|_\infty$ respectively, see Section 6.

The paper is organized as follows. In Section 2, we recall preliminary general results that are used in the rest of the paper. In Section 3, we give a comparison principle and prove the uniqueness result for $\lambda \leq 0$. Section 4 is devoted to the existence of the lower solution. In Section 5, we construct the modified problem that we use to obtain the existence results. The coercive and limit-coercive cases, corresponding to $\lambda \leq 0$ are studied in Section 6 where we prove Theorem 1.1. Theorem 1.3 which gives a necessary and sufficient condition to the existence of a solution to (P_0) is established in Section 7. In Section 8 we show that I_λ has, for $\lambda > 0$ small, a mountain pass geometry and that the Cerami compactness condition holds. This permits to give the proof of Theorem 1.4. Section 9 contains the proofs of Theorems 1.5 and 1.6. Finally in an Appendix we give conditions on h^+ that ensure that $m_p > 0$.

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Notation.

- 1) For $p \in [1, +\infty[$, the norm $(\int_{\Omega} |u|^p dx)^{1/p}$ in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$. We denote by p' the conjugate exponent of p , namely $p' = p/(p-1)$ and by p^* the Sobolev critical exponent i.e. $p^* = \frac{Np}{N-p}$ if $p < N$ and $p^* = +\infty$ in case $p \geq N$. The norm in $L^\infty(\Omega)$ is $\|u\|_\infty = \text{esssup}_{x \in \Omega} |u(x)|$.
- 2) For $v \in L^1(\Omega)$ we define $v^+ = \max(v, 0)$ and $v^- = \max(-v, 0)$.
- 3) The space $W_0^{1,p}(\Omega)$ is equipped with the norm $\|u\| := (\int_{\Omega} |\nabla u|^p dx)^{1/p}$.
- 4) We denote $\mathbb{R}^+ = (0, +\infty)$ and $\mathbb{R}^- = (-\infty, 0)$.
- 5) For $a, b \in L^1(\Omega)$ we denote $\{a \leq b\} = \{x \in \Omega : a(x) \leq b(x)\}$.

2. PRELIMINARIES

In this section we present some definitions and known results which are going to play an important role throughout all the work. First of all, we present some results on lower and upper solutions adapted to our setting. Let us consider the problem

$$(2.1) \quad -\Delta_p u + H(x, u, \nabla u) = f(x), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

where f belongs to $L^1(\Omega)$ and $H : \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a Carathéodory function.

Definition 2.1. We say that $\alpha \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a *lower solution* of (2.1) if $\alpha^+ \in W_0^{1,p}(\Omega)$ and, for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\varphi \geq 0$, it follows that

$$\int_{\Omega} |\nabla \alpha|^{p-2} \nabla \alpha \nabla \varphi dx + \int_{\Omega} H(x, \alpha, \nabla \alpha) \varphi dx \leq \int_{\Omega} f(x) \varphi dx.$$

Similarly, $\beta \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is an *upper solution* of (2.1) if $\beta^- \in W_0^{1,p}(\Omega)$ and, for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\varphi \geq 0$, it follows that

$$\int_{\Omega} |\nabla \beta|^{p-2} \nabla \beta \nabla \varphi dx + \int_{\Omega} H(x, \beta, \nabla \beta) \varphi dx \geq \int_{\Omega} f(x) \varphi dx.$$

Theorem 2.1. [10, Theorems 3.1 and 4.2] *Assume the existence of a non-decreasing function $b : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and a function $k \in L^1(\Omega)$ such that*

$$|H(x, s, \xi)| \leq b(|s|)[k(x) + |\xi|^p], \quad \text{a.e. } x \in \Omega, \quad \forall (s, \xi) \in \mathbb{R} \times \mathbb{R}^N.$$

If there exist a lower solution α and an upper solution β of (2.1) with $\alpha \leq \beta$, then there exists a solution u of (2.1) with $\alpha \leq u \leq \beta$. Moreover, there exists u_{\min} (resp. u_{\max}) minimum (resp. maximum) solution of (2.1) with $\alpha \leq u_{\min} \leq u_{\max} \leq \beta$ and such that, every solution u of (2.1) with $\alpha \leq u \leq \beta$ satisfies $u_{\min} \leq u \leq u_{\max}$.

Next, we state the strong comparison principle for the p -Laplacian and the following order notions.

Definition 2.2. For $h_1, h_2 \in L^1(\Omega)$ we write

- $h_1 \leq h_2$ if $h_1(x) \leq h_2(x)$ for a.e. $x \in \Omega$,
- $h_1 \not\leq h_2$ if $h_1 \leq h_2$ and $\text{meas}(\{x \in \Omega : h_1(x) < h_2(x)\}) > 0$.

For $u, v \in C^1(\overline{\Omega})$ we write

- $u < v$ if, for all $x \in \Omega$, $u(x) < v(x)$,
- $u \ll v$ if $u < v$ and, for all $x \in \partial\Omega$, either $u(x) < v(x)$, or, $u(x) = v(x)$ and $\frac{\partial u}{\partial \nu}(x) > \frac{\partial v}{\partial \nu}(x)$, where ν denotes the exterior unit normal.

Theorem 2.2. [36, Theorem 1.3] [15, Proposition 2.4] *Assume that $\partial\Omega$ is of class C^2 and let $f_1, f_2 \in L^\infty(\Omega)$ with $f_2 \not\leq f_1 \geq 0$. If $u_1, u_2 \in C_0^{1,\tau}(\overline{\Omega})$, $0 < \tau \leq 1$, are respectively solution of*

$$(P_i) \quad -\Delta_p u_i = f_i, \quad \text{in } \Omega, \quad \text{for } i = 1, 2,$$

such that $u_2 = u_1 = 0$ on $\partial\Omega$. Then $u_2 \gg u_1$.

We need also the following anti-maximum principle.

Proposition 2.3. [26, Theorem 5.1] *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, a bounded domain with $\partial\Omega$ of class $C^{1,1}$, $c, \bar{h} \in L^\infty(\Omega)$, γ_1 the first eigenvalue of (1.2). If $\bar{h} \not\geq 0$, then there exists $\delta_0 > 0$ such that, for all $\lambda \in (\gamma_1, \gamma_1 + \delta_0)$, every solution w of*

$$(2.2) \quad -\Delta_p w = \lambda c(x)|w|^{p-2}w + \bar{h}(x), \quad u \in W_0^{1,p}(\Omega)$$

satisfies $w \ll 0$.

The following result is the well known Picone's inequality for the p -Laplacian. We state it for completeness.

Proposition 2.4. [5, Theorem 1.1] *Let $u, v \in W^{1,p}(\Omega)$ with $u \geq 0$, $v > 0$ in Ω and $\frac{u}{v} \in L^\infty(\Omega)$. Denote*

$$\begin{aligned} L(u, v) &= |\nabla u|^p + (p-1)\left(\frac{u}{v}\right)^p |\nabla v|^p - p\left(\frac{u}{v}\right)^{p-1} |\nabla v|^{p-2} \nabla v \nabla u, \\ R(u, v) &= |\nabla u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right) |\nabla v|^{p-2} \nabla v. \end{aligned}$$

Then, it follows that

- $L(u, v) = R(u, v) \geq 0$ a.e. in Ω .
- $L(u, v) = 0$ a.e. in Ω if, and only if, $u = kv$ for some constant $k \in \mathbb{R}$.

Now, we consider the boundary value problem

$$(2.3) \quad -\Delta_p v = g(x, v), \quad v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

being $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ a Carathéodory function such that, for all $s_0 > 0$, there exists $A > 0$, with

$$(2.4) \quad |g(x, s)| \leq A, \quad \text{a.e. } x \in \Omega, \quad \forall s \in [-s_0, s_0].$$

This problem can be handled variationally. Let us consider the associated functional $\Phi : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\Phi(v) := \frac{1}{p} \int_{\Omega} |\nabla v|^p dx - \int_{\Omega} G(x, v) dx, \quad \text{where } G(x, s) := \int_0^s g(x, t) dt.$$

We can state the following result.

Proposition 2.5. [19, Proposition 3.1] *Under the assumption (2.4), assume that α and β are respectively a lower and an upper solution of (2.3) with $\alpha \leq \beta$ and consider*

$$M := \{v \in W_0^{1,p}(\Omega) : \alpha \leq v \leq \beta\}.$$

Then the infimum of Φ on M is achieved at some v , and such v is a solution of (2.3).

Definition 2.3. A lower solution $\alpha \in C^1(\bar{\Omega})$ is said to be *strict* if every solution u of (2.1) with $u \geq \alpha$ satisfies $u \gg \alpha$.

Similarly, an upper solution $\beta \in C^1(\bar{\Omega})$ is said to be *strict* if every solution u of (2.1) such that $u \leq \beta$ satisfies $u \ll \beta$.

Corollary 2.6. *Assume that (2.4) is valid and that α and β are strict lower and upper solutions of (2.3) belonging to $C^1(\bar{\Omega})$ and satisfying $\alpha \ll \beta$. Then there exists a local minimizer v of the functional Φ in the C_0^1 -topology. Furthermore, this minimizer is a solution of (2.3) with $\alpha \ll v \ll \beta$.*

Proof. First of all observe that Proposition 2.5 implies the existence of $v \in W_0^{1,p}(\Omega)$ solution of (2.3), which minimizes Φ on $M := \{v \in W_0^{1,p}(\Omega) : \alpha \leq v \leq \beta\}$. Moreover, since g is an L^∞ -Carathéodory function, the classical regularity results (see [21, 35]) imply that $v \in C^{1,\tau}(\bar{\Omega})$ for some $0 < \tau < 1$. Since the lower and the upper solutions are strict, it follows that $\alpha \ll v \ll \beta$ and so, there is a C_0^1 -neighbourhood of v in M . Hence, it follows that v minimizes locally Φ in the C_0^1 -topology. \square

Proposition 2.7. [19, Proposition 3.9] *Assume that g satisfies the following growth condition*

$$|g(x, s)| \leq d(1 + |s|^\sigma), \quad \text{a.e. } x \in \Omega, \quad \text{all } s \in \mathbb{R},$$

for some $\sigma \leq p^* - 1$ and some positive constant d . Let $v \in W_0^{1,p}(\Omega)$ be a local minimizer of Φ for the C_0^1 -topology. Then $v \in C_0^{1,\tau}(\bar{\Omega})$ for some $0 < \tau < 1$ and v is a local minimizer of Φ in the $W_0^{1,p}$ -topology.

We now recall abstract results in order to find critical points of Φ other than local minima.

Definition 2.4. Let $(X, \|\cdot\|)$ be a real Banach space with dual space $(X^*, \|\cdot\|_*)$ and let $\Phi : X \rightarrow \mathbb{R}$ be a \mathcal{C}^1 functional. The functional Φ satisfies the *Cerami condition at level $c \in \mathbb{R}$* if, for any *Cerami sequence* at level $c \in \mathbb{R}$, i.e. for any sequence $\{x_n\} \subset X$ with

$$\Phi(x_n) \rightarrow c \quad \text{and} \quad \|\Phi'(x_n)\|_*(1 + \|x_n\|) \rightarrow 0,$$

there exists a subsequence $\{x_{n_k}\}$ strongly convergent in X .

Theorem 2.8. [23, Corollary 9, Section 1, Chapter IV] *Let $(X, \|\cdot\|)$ be a real Banach space. Suppose that $\Phi : X \rightarrow \mathbb{R}$ is a \mathcal{C}^1 functional. Take two points $e_1, e_2 \in X$ and define*

$$\Gamma := \{\varphi \in \mathcal{C}([0, 1], X) : \varphi(0) = e_1, \varphi(1) = e_2\},$$

and

$$c := \inf_{\varphi \in \Gamma} \max_{t \in [0, 1]} \Phi(\varphi(t)).$$

Assume that Φ satisfies the *Cerami condition at level c* and that

$$c > \max\{\Phi(e_1), \Phi(e_2)\}.$$

Then, there is a critical point of Φ at level c , i.e. there exists $x_0 \in X$ such that $\Phi(x_0) = c$ and $\Phi'(x_0) = 0$.

Theorem 2.9. [25, Corollary 1.6] *Let $(X, \|\cdot\|)$ be a real Banach space and let $\Phi : X \rightarrow \mathbb{R}$ be a \mathcal{C}^1 functional. Suppose that $u_0 \in X$ is a local minimum, i.e. there exists $\varepsilon > 0$ such that*

$$\Phi(u_0) \leq \Phi(u), \quad \text{for } \|u - u_0\| \leq \varepsilon,$$

and assume that Φ satisfies the *Cerami condition at any level $d \in \mathbb{R}$* . Then, the following alternative holds:

- i) either there exists $0 < \gamma < \varepsilon$ such that $\inf\{\Phi(u) : \|u - u_0\| = \gamma\} > \Phi(u_0)$,
- ii) or, for each $0 < \gamma < \varepsilon$, Φ has a local minimum at a point u_γ with $\|u_\gamma - u_0\| = \gamma$ and $\Phi(u_\gamma) = \Phi(u_0)$.

Remark 2.1. In [25], Theorem 2.9 is proved assuming the Palais-Smale condition which is stronger than our Cerami condition. Nevertheless, modifying slightly the proof, it is possible to obtain the same result with the Cerami condition.

3. COMPARISON PRINCIPLE AND UNIQUENESS RESULTS

In this section, we state a comparison principle and, as a consequence, we obtain uniqueness result for (P_λ) with $\lambda \leq 0$, proving Theorem 1.2. Consider the boundary value problem

$$(3.1) \quad -\Delta_p u = \mu |\nabla u|^p + f(x, u), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

under the assumption

$$(3.2) \quad \begin{cases} \Omega \subset \mathbb{R}^N, N \geq 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } \mathcal{C}^{0,1}, \\ f : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \text{ is a } L^1\text{-Carathéodory function with } f(x, s) \leq f(x, t) \text{ for a.e. } x \in \Omega \text{ and all } t \leq s, \\ \mu > 0. \end{cases}$$

Remark 3.1. As above, the assumption $\mu > 0$ is not a restriction. If $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a solution of (3.1) with $\mu < 0$ then $w = -u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a solution of

$$-\Delta_p w = -\mu |\nabla w|^p - f(x, -w), \quad w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

with $-f(x, -s)$ satisfying the assumption (3.2).

Under a stronger regularity on the solutions, we can prove a comparison principle for (3.1). The proof relies on the Picone's inequality (Proposition 2.4) and is inspired by some ideas of [3].

Theorem 3.1. *Assume that (3.2) holds. If $u_1, u_2 \in W^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ are respectively a lower and an upper solution of (3.1), then $u_1 \leq u_2$.*

Proof. Suppose that u_1, u_2 are respectively a lower and an upper solution of (3.1). For simplicity denote $t = \frac{p\mu}{p-1}$ and consider as test function

$$\varphi = [e^{tu_1} - e^{tu_2}]^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

First of all, observe that

$$\nabla\varphi = t[\nabla u_1 e^{tu_1} - \nabla u_2 e^{tu_2}] \chi_{\{u_1 > u_2\}},$$

with χ_A the characteristic function of the set A . Hence, using assumptions (3.2), it follows that

$$\begin{aligned} \int_{\{u_1 > u_2\}} \left([|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2] (t \nabla u_1 e^{tu_1} - t \nabla u_2 e^{tu_2}) - \mu [|\nabla u_1|^p - |\nabla u_2|^p] (e^{tu_1} - e^{tu_2}) \right) dx \\ \leq \int_{\{u_1 > u_2\}} (f(x, u_1) - f(x, u_2)) (e^{tu_1} - e^{tu_2}) dx \leq 0. \end{aligned}$$

Observe that

$$\begin{aligned} (3.3) \quad & \int_{\{u_1 > u_2\}} \left[|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2 \right] (t \nabla u_1 e^{tu_1} - t \nabla u_2 e^{tu_2}) dx \\ & - \mu \int_{\{u_1 > u_2\}} \left[|\nabla u_1|^p - |\nabla u_2|^p \right] (e^{tu_1} - e^{tu_2}) dx \\ & = \int_{\{u_1 > u_2\}} e^{tu_1} \left[|\nabla u_1|^p (t - \mu) + \mu |\nabla u_2|^p - t |\nabla u_2|^{p-2} \nabla u_2 \nabla u_1 \right] dx \\ & \quad + \int_{\{u_1 > u_2\}} e^{tu_2} \left[|\nabla u_2|^p (t - \mu) + \mu |\nabla u_1|^p - t |\nabla u_1|^{p-2} \nabla u_1 \nabla u_2 \right] dx. \end{aligned}$$

Next, as $\nabla e^{tu_i} = t \nabla u_i e^{tu_i}$, $i = 1, 2$, we have

$$|\nabla u_i|^p = \frac{|\nabla e^{tu_i}|^p}{t^p e^{tpu_i}} \quad i = 1, 2.$$

Hence, using the above identities, and as $\frac{\mu}{t-\mu} = p-1$ and $\frac{t}{t-\mu} = p$, it follows that,

$$\begin{aligned} e^{tu_1} \left[|\nabla u_1|^p (t - \mu) + \mu |\nabla u_2|^p - t |\nabla u_2|^{p-2} \nabla u_2 \nabla u_1 \right] \\ = \frac{t - \mu}{t^p e^{t(p-1)u_1}} \left[|\nabla e^{tu_1}|^p + (p-1) \left(\frac{e^{tu_1}}{e^{tu_2}} \right)^p |\nabla e^{tu_2}|^p - p \left(\frac{e^{tu_1}}{e^{tu_2}} \right)^{p-1} |\nabla e^{tu_2}|^{p-2} \nabla e^{tu_2} \nabla e^{tu_1} \right], \\ e^{tu_2} \left[|\nabla u_2|^p (t - \mu) + \mu |\nabla u_1|^p - t |\nabla u_1|^{p-2} \nabla u_1 \nabla u_2 \right] \\ = \frac{t - \mu}{t^p e^{t(p-1)u_2}} \left[|\nabla e^{tu_2}|^p + (p-1) \left(\frac{e^{tu_2}}{e^{tu_1}} \right)^p |\nabla e^{tu_1}|^p - p \left(\frac{e^{tu_2}}{e^{tu_1}} \right)^{p-1} |\nabla e^{tu_1}|^{p-2} \nabla e^{tu_1} \nabla e^{tu_2} \right]. \end{aligned}$$

Then, by (3.3), we have

$$\begin{aligned} (3.4) \quad & \int_{\{u_1 > u_2\}} \frac{t - \mu}{t^p e^{t(p-1)u_1}} \left[|\nabla e^{tu_1}|^p + (p-1) \left(\frac{e^{tu_1}}{e^{tu_2}} \right)^p |\nabla e^{tu_2}|^p - p \left(\frac{e^{tu_1}}{e^{tu_2}} \right)^{p-1} |\nabla e^{tu_2}|^{p-2} \nabla e^{tu_2} \nabla e^{tu_1} \right] dx \\ & + \int_{\{u_1 > u_2\}} \frac{t - \mu}{t^p e^{t(p-1)u_2}} \left[|\nabla e^{tu_2}|^p + (p-1) \left(\frac{e^{tu_2}}{e^{tu_1}} \right)^p |\nabla e^{tu_1}|^p - p \left(\frac{e^{tu_2}}{e^{tu_1}} \right)^{p-1} |\nabla e^{tu_1}|^{p-2} \nabla e^{tu_1} \nabla e^{tu_2} \right] dx \leq 0. \end{aligned}$$

By Picone's inequality (Proposition 2.4), we know that both brackets in (3.4) are positive and are equal to zero if and only if $e^{tu_1} = k e^{tu_2}$ for some $k \in \mathbb{R}$. As $t - \mu > 0$, thanks to (3.4), we deduce the existence of $k \in \mathbb{R}$ such that

$$(3.5) \quad e^{tu_1} = k e^{tu_2} \quad \text{in } \{u_1 > u_2\}.$$

Since u_1 and u_2 are continuous on $\overline{\Omega}$ and satisfy $u_1 - u_2 \leq 0$ on $\partial\Omega$, we deduce that $u_1 = u_2$ on $\partial\{u_1 > u_2\}$. Hence, (3.5) applied to $x \in \partial\{u_1 > u_2\}$, implies $k = 1$. This implies that $u_1 = u_2$ in $\{u_1 > u_2\}$, which proves $u_1 \leq u_2$, as desired. \square

Corollary 3.2. *Assume that (A_1) holds and suppose $\lambda \leq 0$. If $u_1, u_2 \in W^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$ are respectively a lower and an upper solution of (P_λ) , then $u_1 \leq u_2$.*

Proof. Define the function $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ given by

$$f(x, s) = \lambda c(x) |s|^{p-2} s + h(x).$$

Since (A_1) holds and $\lambda \leq 0$, f is a L^1 -Carathéodory function which satisfies (3.2). Consequently, the proposition follows from Theorem 3.1 \square

The following result guarantees the regularity that we need to apply the previous comparison principle.

Lemma 3.3. *Assume that (A_1) holds and suppose $\lambda \leq 0$. Then, any solution of (P_λ) belongs to $C^{0,\tau}(\overline{\Omega})$.*

Proof. This follows directly from [32, Theorem IX-2.2]. \square

Proof of Theorem 1.2. The proof is just the combination of Corollary 3.2 and Lemma 3.3. \square

Remark 3.2. It is important to note that this comparison and uniqueness results do not hold in general for solution belonging only to $W_0^{1,p}(\Omega)$. See [38, Example 1.1]

Remark 3.3. The following counter-example (see [39, p.7]) shows that there is no hope to obtain, when $p \neq 2$, a comparison principle like [7, Corollary 3.1]. For $N = 2$ and $R > 0$, consider the following problem on the ball

$$\begin{cases} -\Delta_4 u = |\nabla u|^2 & \text{in } B(0, R), \\ u = 0 & \text{on } \partial B(0, R). \end{cases}$$

We easily see that $u_1 = 0$ and $u_2 = \frac{1}{8}(R^2 - |x|^2)$ are both solutions of the above problem belonging to $W_0^{1,4}(B(0, R)) \cap L^\infty(B(0, R))$.

4. A PRIORI LOWER BOUND AND EXISTENCE OF A LOWER SOLUTION

As explained in the introduction, the aim of this section is to find a lower solution below every upper solution of problem (P_λ) . First of all, we show that under a rather mild assumption (in particular no sign on c is required) the solutions to (P_λ) admit a lower bound. Precisely we consider problem (P_λ) assuming now

$$(4.1) \quad \begin{cases} \Omega \subset \mathbb{R}^N, N \geq 2, \text{ is a bounded domain with } \partial\Omega \text{ of class } C^{0,1}. \\ c \text{ and } h \text{ belong to } L^q(\Omega) \text{ for some } q > \max\{N/p, 1\}, \\ \mu \in L^\infty(\Omega) \text{ satisfies } 0 < \mu_1 \leq \mu(x) \leq \mu_2. \end{cases}$$

Adapting the proof of [17, Lemma 3.1], based in turn on ideas of [4], we obtain

Lemma 4.1. *Under the assumptions (4.1), for any $\lambda \geq 0$, there exists a constant $M_\lambda > 0$ with $M_\lambda := M(N, p, q, |\Omega|, \lambda, \mu_1, \|c^+\|_q, \|h^-\|_q) > 0$ such that, every $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ upper solution of (P_λ) satisfies*

$$\min_{\Omega} u > -M_\lambda.$$

Proof. Let us split the proof in two steps.

Step 1: *There exists a positive constant $M_1 = M_1(p, q, N, |\Omega|, \lambda, \mu_1, \|c^+\|_q, \|h^-\|_q) > 0$ such that $\|u^-\| \leq M_1$.*

First of all, observe that for every function $u \in W^{1,p}(\Omega)$, it follows that

$$(4.2) \quad \nabla((u^-)^{\frac{p+1}{p}}) = \frac{p+1}{p}(u^-)^{1/p} \nabla u^-, \quad \text{and so,} \quad |\nabla u^-|^p u^- = \left(\frac{p}{p+1}\right)^p |\nabla(u^-)^{\frac{p+1}{p}}|^p.$$

Suppose that $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is an upper solution of (P_λ) and let us consider $\varphi = u^-$ as a test function. Under the assumptions (4.1), it follows that

$$(4.3) \quad \begin{aligned} - \int_{\Omega} |\nabla u^-|^p dx &\geq -\lambda \int_{\Omega} c(x) |u^-|^p dx + \int_{\Omega} \mu(x) |\nabla u^-|^p u^- dx + \int_{\Omega} h(x) u^- dx \\ &\geq -\lambda \int_{\Omega} c^+(x) |u^-|^p dx + \mu_1 \int_{\Omega} |\nabla u^-|^p u^- dx - \int_{\Omega} h^-(x) u^- dx. \end{aligned}$$

By (4.2) and (4.3), we have that

$$(4.4) \quad \mu_1 \left(\frac{p}{p+1}\right)^p \int_{\Omega} |\nabla(u^-)^{\frac{p+1}{p}}|^p dx + \int_{\Omega} |\nabla u^-|^p dx \leq \lambda \int_{\Omega} c^+(x) |u^-|^p dx + \int_{\Omega} h^-(x) u^- dx.$$

Firstly, we apply Young's inequality and, for every $\varepsilon > 0$, it follows that

$$\begin{aligned} \int_{\Omega} c^+(x)|u^-|^p dx &= \int_{\Omega} (c^+(x))^{1/p} |u^-|^{1/p} (c^+(x))^{\frac{p-1}{p}} |u^-|^{\frac{(p+1)(p-1)}{p}} dx \\ &\leq C(\varepsilon) \int_{\Omega} c^+(x) u^- dx + \varepsilon \int_{\Omega} c^+(x) ((u^-)^{\frac{p+1}{p}})^p dx \end{aligned}$$

Moreover, applying Hölder and Sobolev inequalities, observe that

$$\int_{\Omega} c^+(x) ((u^-)^{\frac{p+1}{p}})^p dx \leq \|c^+\|_q \| (u^-)^{\frac{p+1}{p}} \|_{\frac{qp}{q-1}}^p \leq S \|c^+\|_q \|\nabla(u^-)^{\frac{p+1}{p}}\|_p^p$$

with S the constant from the embedding from $W_0^{1,p}(\Omega)$ into $L^{\frac{qp}{q-1}}(\Omega)$. Hence, choosing ε small enough to ensure that $\varepsilon S \lambda \|c^+\|_q \leq \frac{\mu_1}{2} \left(\frac{p}{p+1}\right)^p$ and substituting in (4.4), we apply again Hölder and Sobolev inequalities and we find a constant $C = C(\mu_1, \lambda, \|c^+\|_q, p, q, |\Omega|, N)$ such that

$$\frac{\mu_1}{2} \left(\frac{p}{p+1}\right)^p \|\nabla(u^-)^{\frac{p+1}{p}}\|_p^p + \|\nabla u^-\|_p^p \leq (\|h^-\|_q + C(\varepsilon)\|c^+\|_q) \|u^-\|_{\frac{q}{q-1}} \leq C(\|h^-\|_q + \|c^+\|_q) \|\nabla u^-\|_p.$$

This allows to conclude that

$$\|u^-\| \leq \left(C(\|h^-\|_q + \|c^+\|_q) \right)^{\frac{1}{p-1}} =: M_1.$$

Step 2: Conclusion.

Since (4.1) holds, every $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ upper solution of (P_λ) satisfies

$$(4.5) \quad -\Delta_p u \geq \lambda c(x) |u|^{p-2} u - h^-(x), \quad \text{in } \Omega.$$

Moreover, observe that 0 is also an upper solution of (4.5). Hence, since the minimum of two upper solution is an upper solution (see [14, Corollary 3.3]), it follows that $\min(u, 0)$ is an upper solution of (4.5). Furthermore, observe that $\min(u, 0)$ is an upper solution of

$$-\Delta_p u \geq \lambda c^+(x) |u|^{p-2} u - h^-(x), \quad \text{in } \Omega.$$

Hence, applying [39, Theorem 6.1.2], we have the existence of $M_2 = M_2(N, p, \lambda, |\Omega|, \|c^+\|_q) > 0$ and $M_3 = M_3(N, p, \lambda, |\Omega|, \|c^+\|_q) > 0$ such that

$$\sup_{\Omega} u^- \leq M_2 [\|u^-\|_p + \|h^-\|_q] \leq M_3 [\|u^-\| + \|h^-\|_q].$$

Finally, the result follows by Step 1. \square

Remark 4.1.

- Observe that the lower bound does not depend on h^+ and c^- . In particular, we have the same lower bound for all $h \geq 0$ and all $c \leq 0$.
- Since c does not have a sign, there is no loss of generality in assuming $\lambda \geq 0$. If we consider $\lambda \leq 0$, we recover the same result with M_λ depending on $\|c^-\|_q$ instead of $\|c^+\|_q$.

Proposition 4.2. *Under the assumptions (A_1) , for any $\lambda \in \mathbb{R}$, there exists $\underline{u}_\lambda \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ lower solution of (P_λ) such that, for every β upper solution of (P_λ) , we have $\underline{u}_\lambda \leq \min\{0, \beta\}$.*

Proof. We need to distinguish in our proof the cases $\lambda \leq 0$ and $\lambda \geq 0$. First we assume that $\lambda \leq 0$. By Lemma 4.1, we have a constant $M > 0$ such that every upper solution β of (P_λ) satisfies $\beta \geq -M$. Let α be the solution of

$$-\Delta_p u = -h^-(x), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

It is then easy to prove that $\underline{u} = \alpha - M \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a lower solution of (P_λ) with $\underline{u} \leq -M$. By the choice of M , this implies that $\underline{u} \leq \bar{u}$ for every upper solution \bar{u} of (P_λ) .

Now, when $\lambda \geq 0$ we first introduce the auxiliary problem

$$(4.6) \quad \begin{cases} -\Delta_p u = \lambda c(x)|u|^{p-2}u + \mu|\nabla u|^p - h^-(x) - 1, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

Thanks to the previous lemma, there exists $M_\lambda > 0$ such that, for every $\beta_1 \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ upper solution of (4.6), we have $\beta_1 \geq -M_\lambda$. Now, for $k > M_\lambda$, we introduce the problem

$$(4.7) \quad \begin{cases} -\Delta_p u = -\lambda c(x)k^{p-1} - h^-(x) - 1, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and denote by α_λ its solution. Since $-\lambda c(x)k^{p-1} - h^-(x) - 1 < 0$, the comparison principle (see for instance [37, Lemma A.0.7]) implies that $\alpha_\lambda \leq 0$. Observe that, for every β_1 upper solution of (P_λ) , we have that

$$-\Delta_p \beta_1 \geq \lambda c(x)|\beta_1|^{p-2}\beta_1 + \mu|\nabla \beta_1|^p - h^-(x) - 1 \geq -\lambda c(x)k^{p-1} - h^-(x) - 1 = -\Delta_p \alpha_\lambda.$$

Consequently, it follows that

$$\begin{cases} -\Delta_p \beta_1 \geq -\Delta_p \alpha_\lambda, & \text{in } \Omega, \\ \beta_1 \geq \alpha_\lambda = 0, & \text{on } \partial\Omega, \end{cases}$$

and, applying again the comparison principle, that $\beta_1 \geq \alpha_\lambda$.

Now, we introduce the problem

$$(4.8) \quad \begin{cases} -\Delta_p u = \lambda c(x)|\tilde{T}_k(u)|^{p-2}\tilde{T}_k(u) + \mu|\nabla u|^p - h^-(x) - 1, & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where

$$\tilde{T}_k(s) = \begin{cases} -k, & \text{if } s \leq -k, \\ s, & \text{if } s > -k. \end{cases}$$

Observe that β_1 and 0 are upper solutions of (4.8). Recalling that the minimum of two upper solution is an upper solution (see [14, Corollary 3.3]), it follows that $\bar{\beta} = \min\{0, \beta_1\}$ is an upper solution of (4.8). As α_λ is a lower solution of (4.8) with $\alpha_\lambda \leq \bar{\beta}$, applying Theorem 2.1, we conclude the existence of \underline{u}_λ minimum solution of (4.8) with $\alpha_\lambda \leq \underline{u}_\lambda \leq \bar{\beta} = \min\{0, \beta_1\}$.

As, for every upper solution β of (P_λ) , β is an upper solution of (4.8), we have $\alpha_\lambda \leq \beta$. Recalling that \underline{u}_λ is the minimum solution of (4.8) with $\alpha_\lambda \leq \underline{u}_\lambda \leq 0$, we deduce that $\underline{u}_\lambda \leq \beta$.

It remains to prove that \underline{u}_λ is a lower solution of (P_λ) . First, observe that \underline{u}_λ is an upper solution of (4.6). By construction, this implies that $\underline{u}_\lambda \geq -M_\lambda > -k$. Consequently, \underline{u}_λ is a solution of (4.6) and so, a lower solution of (P_λ) . \square

5. THE FUNCTIONAL SETTING

Let us introduce some auxiliary functions which are going to play an important role in the rest of the work. Define

$$(5.1) \quad g(s) = \begin{cases} \left| \frac{p-1}{\mu} \left(1 + \frac{\mu}{p-1}s\right) \ln \left(1 + \frac{\mu}{p-1}s\right) \right|^{p-2} \frac{p-1}{\mu} \left(1 + \frac{\mu}{p-1}s\right) \ln \left(1 + \frac{\mu}{p-1}s\right), & s > -\frac{p-1}{\mu}, \\ 0, & s \leq -\frac{p-1}{\mu}, \end{cases}$$

$$G(s) = \int_0^s g(t) dt \quad \text{and} \quad H(s) = \frac{1}{p}g(s)s - G(s).$$

In the following lemma we prove some properties of these functions.

Lemma 5.1.

- i) The function g is continuous on \mathbb{R} , satisfies $g > 0$ on \mathbb{R}^+ and there exists $D > 0$ with $-D \leq g \leq 0$ on \mathbb{R}^- . Moreover, $G \geq 0$ on \mathbb{R} .
- ii) For any $\delta > 0$, there exists $\bar{c} = \bar{c}(\delta, \mu, p) > 0$ such that, for any $s > \frac{p-1}{\mu}$, $g(s) \leq \bar{c}s^{p-1+\delta}$.
- iii) $\lim_{s \rightarrow +\infty} g(s)/s^{p-1} = +\infty$ and $\lim_{s \rightarrow +\infty} G(s)/s^p = +\infty$.
- iv) There exists $R > 0$ such that the function H satisfies $H(s) \leq \left(\frac{s}{t}\right)^{p-1}H(t)$, for $R \leq s \leq t$.

v) The function H is bounded on \mathbb{R}^- .

Proof. i) By definition, it is obvious that g is continuous, $g > 0$ on \mathbb{R}^+ and g is bounded and $g \leq 0$ on \mathbb{R}^- . This implies also that $G \geq 0$ by integration.

ii) First of all, recall that for any $\varepsilon > 0$ there exists $c = c(\varepsilon) > 0$ such that $\ln(s) \leq c(\varepsilon)s^\varepsilon$ for all $s \in (1, \infty)$. This implies that, for any $\delta > 0$,

$$\lim_{s \rightarrow +\infty} \frac{g(s)}{s^{p-1+\delta}} = \lim_{s \rightarrow +\infty} \left(\frac{(p-1)(1 + \frac{\mu}{p-1}s)}{\mu s} \right)^{p-1} \frac{(\ln(1 + \frac{\mu}{p-1}s))^{p-1}}{s^\delta} = 0.$$

Hence, there exists $R > \frac{p-1}{\mu}$ such that, for all $s > R$,

$$\frac{g(s)}{s^{p-1+\delta}} \leq 1.$$

As the function $\frac{g(s)}{s^{p-1+\delta}}$ is continuous on the compact set $[\frac{p-1}{\mu}, R]$, we have a constant $C > 0$ with

$$\frac{g(s)}{s^{p-1+\delta}} \leq C \quad \text{on } [\frac{p-1}{\mu}, R].$$

The result follows for $\bar{C} = \max(C, 1)$.

iii) As

$$\lim_{s \rightarrow +\infty} \frac{\frac{p-1}{\mu} (1 + \frac{\mu}{p-1}s) \ln(1 + \frac{\mu}{p-1}s)}{s} = +\infty,$$

and $p > 1$, we easily deduce that

$$\lim_{s \rightarrow +\infty} \frac{g(s)}{s^{p-1}} = +\infty$$

and, by L'Hospital's rule

$$\lim_{s \rightarrow +\infty} \frac{G(s)}{s^p} = +\infty.$$

iv) First of all, integrating by parts, we observe that, for any $s \geq 0$,

$$G(s) = \left(\frac{p-1}{\mu} \right)^p \left[\frac{1}{p} \left(1 + \frac{\mu}{p-1}s \right)^p \left(\ln \left(1 + \frac{\mu}{p-1}s \right) \right)^{p-1} - \frac{\mu}{p} \int_0^s \left(1 + \frac{\mu}{p-1}t \right)^{p-1} \left(\ln \left(1 + \frac{\mu}{p-1}t \right) \right)^{p-2} dt \right],$$

and so, for any $s \geq 0$, it follows that

$$H(s) = \frac{1}{p} \left(\frac{p-1}{\mu} \right)^p \left[\mu \int_0^s \left(1 + \frac{\mu}{p-1}t \right)^{p-1} \left(\ln \left(1 + \frac{\mu}{p-1}t \right) \right)^{p-2} dt - \left(1 + \frac{\mu}{p-1}s \right)^{p-1} \left(\ln \left(1 + \frac{\mu}{p-1}s \right) \right)^{p-1} \right].$$

To prove iv), we show that the function $\varphi(s) := \frac{H(s)}{s^{p-1}}$ is non-decreasing on $[R, +\infty)$ for some $R > 0$. Observe that

$$\varphi'(s) = \frac{1}{s^p} [H'(s)s - (p-1)H(s)].$$

Hence, we just need to prove that $H'(s)s - (p-1)H(s) \geq 0$ for $s \geq R$. After some simple computations, we see that it is enough to prove the existence of $R > 0$ such that, for all $s \geq R$, $\kappa(s) \geq 0$ where

$$\begin{aligned} \kappa(s) = & \left(1 + \frac{\mu}{p-1}s \right)^{p-2} \left(\ln \left(1 + \frac{\mu}{p-1}s \right) \right)^{p-2} \left(\left(\frac{\mu s}{p-1} \right)^2 + \ln \left(1 + \frac{\mu}{p-1}s \right) \right) \\ & - \mu \int_0^s \left(1 + \frac{\mu}{p-1}t \right)^{p-1} \left(\ln \left(1 + \frac{\mu}{p-1}t \right) \right)^{p-2} dt. \end{aligned}$$

Observe that

$$\begin{aligned} \kappa'(s) = & \frac{\mu}{p-1} \left(1 + \frac{\mu}{p-1}s \right)^{p-3} \left(\ln \left(1 + \frac{\mu}{p-1}s \right) \right)^{p-3} \\ & \left[(p-2) \left(\frac{\mu s}{p-1} - \ln \left(1 + \frac{\mu}{p-1}s \right) \right)^2 + \left(\frac{\mu s}{p-1} \right)^2 \ln \left(1 + \frac{\mu}{p-1}s \right) \right]. \end{aligned}$$

Hence, we distinguish two cases:

- i) In case $p \geq 2$, it is obvious that $\kappa'(s) > 0$, for any $s > 0$. This implies that κ is increasing and so, that $\kappa(s) > 0$ for $s > 0$, since $\kappa(0) = 0$.
- ii) If $1 < p < 2$, as $\lim_{s \rightarrow \infty} \kappa'(s) = +\infty$, there exists $R_1 > 0$ such that, for any $s \geq R_1$, we have $\kappa'(s) > 1$ and hence, there exists $R_2 \geq R_1$ such that $\kappa(s) > 0$, for any $s \geq R_2$.

In any case, we can conclude the existence of $R \geq 0$ such that $\kappa(s) > 0$ for any $s \geq R$. Consequently, there exists $R > 0$ such that $\varphi'(s) > 0$, for $s \geq R$, which means that φ is non-decreasing for $s \geq R$ and hence H satisfies $H(s) \leq \left(\frac{s}{t}\right)^{p-1} H(t)$, for $R \leq s \leq t$.

v) This follows directly from the definition of the functions g and G . □

Next, we define the function

$$(5.2) \quad \alpha_\lambda = \frac{p-1}{\mu} \left(e^{\frac{\mu}{p-1} \underline{u}_\lambda} - 1 \right) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

where $\underline{u}_\lambda \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is the lower solution of (P_λ) obtained in Proposition 4.2. Before going further, since $\underline{u}_\lambda \leq 0$, observe that $0 \geq \alpha_\lambda \geq -\frac{p-1}{\mu} + \varepsilon$ for some $\varepsilon > 0$.

Now, for any $\lambda \in \mathbb{R}$, let us consider the auxiliary problem

$$(Q_\lambda) \quad -\Delta_p v = f_\lambda(x, v), \quad v \in W_0^{1,p}(\Omega),$$

where

$$(5.3) \quad f_\lambda(x, s) = \begin{cases} \lambda c(x)g(s) + \left(1 + \frac{\mu}{p-1}s\right)^{p-1} h(x), & \text{if } s \geq \alpha_\lambda(x), \\ \lambda c(x)g(\alpha_\lambda(x)) + \left(1 + \frac{\mu}{p-1}\alpha_\lambda(x)\right)^{p-1} h(x), & \text{if } s \leq \alpha_\lambda(x), \end{cases}$$

where g is defined by (5.1). In the following lemma, we prove some properties of the solutions of (Q_λ) .

Lemma 5.2. *Assume that (A_1) holds. Then, it follows that:*

- i) *Every solution of (Q_λ) belongs to $L^\infty(\Omega)$.*
- ii) *Every solution v of (Q_λ) satisfies $v \geq \alpha_\lambda$.*
- iii) *A function $v \in W_0^{1,p}(\Omega)$ is a solution of (Q_λ) if, and only if, the function*

$$u = \frac{p-1}{\mu} \ln \left(1 + \frac{\mu}{p-1} v \right) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$$

is a solution of (P_λ) .

Proof. i) This follows directly from [32, Theorem IV-7.1].

ii) First of all, observe that α_λ is a lower solution of (Q_λ) . For a solution $v \in W_0^{1,p}(\Omega)$ of (Q_λ) , we have $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ by the previous step and, for all $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\varphi \geq 0$,

$$\int_{\Omega} [|\nabla v|^{p-2} \nabla v - |\nabla \alpha_\lambda|^{p-2} \nabla \alpha_\lambda] \nabla \varphi \, dx \geq \int_{\Omega} [f_\lambda(x, v) - f_\lambda(x, \alpha_\lambda)] \varphi \, dx.$$

Now, since there exist constants $d_1, d_2 > 0$ such that for all $\xi, \eta \in \mathbb{R}^N$,

$$(5.4) \quad \langle |\xi|^{p-2} \xi - |\eta|^{p-2} \eta, \xi - \eta \rangle \geq \begin{cases} d_1 (|\xi| + |\eta|)^{p-2} |\xi - \eta|^2, & \text{if } 1 < p < 2, \\ d_2 |\xi - \eta|^p, & \text{if } p \geq 2, \end{cases}$$

(see for instance [37, Lemma A.0.5]), we choose $\varphi = (\alpha_\lambda - v)^+$ and obtain that

$$0 \geq \int_{\{\alpha_\lambda \geq v\}} [|\nabla v|^{p-2} \nabla v - |\nabla \alpha_\lambda|^{p-2} \nabla \alpha_\lambda] \nabla (\alpha_\lambda - v) \, dx \geq \int_{\{\alpha_\lambda \geq v\}} [f_\lambda(x, v) - f_\lambda(x, \alpha_\lambda)] (\alpha_\lambda - v) \, dx = 0.$$

Consequently, using again (5.4), we deduce that $\alpha_\lambda = v$ in $\{\alpha_\lambda \geq v\}$ and so, that $v \geq \alpha_\lambda$.

iii) Suppose that $v \in W_0^{1,p}(\Omega)$ is a solution of (Q_λ) . The first parts, *i*), *ii*) imply that $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is such that $v \geq \alpha_\lambda \geq -\frac{p-1}{\mu} + \varepsilon$ with $\varepsilon > 0$ and hence $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Let us prove that u is a (weak) solution of (P_λ) . Let ϕ be an arbitrary function belonging to $C_0^\infty(\Omega)$ and define $\varphi = \phi / (1 + \frac{\mu}{p-1}v)^{p-1}$. It follows that $\varphi \in W_0^{1,p}(\Omega)$. As $e^{\frac{\mu u}{p-1}} = 1 + \frac{\mu}{p-1}v$, we have the following identity

$$\begin{aligned} \int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla \varphi \, dx &= \int_{\Omega} e^{\mu u} |\nabla u|^{p-2} \nabla u \left(\frac{\nabla \phi}{\left(1 + \frac{\mu}{p-1}v\right)^{p-1}} - \frac{\mu \phi \nabla v}{\left(1 + \frac{\mu}{p-1}v\right)^p} \right) dx \\ &= \int_{\Omega} \frac{e^{\mu u}}{\left(1 + \frac{\mu}{p-1}v\right)^{p-1}} |\nabla u|^{p-2} \nabla u \left(\nabla \phi - \frac{\mu \phi \nabla \left(\frac{p-1}{\mu} (e^{\frac{\mu}{p-1}u} - 1) \right)}{1 + \frac{\mu}{p-1}v} \right) dx \\ &= \int_{\Omega} |\nabla u|^{p-2} \nabla u (\nabla \phi - \mu \phi \nabla u) \, dx = \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \phi \, dx - \mu \int_{\Omega} |\nabla u|^p \phi \, dx. \end{aligned}$$

On the other hand, by definition of g , observe that

$$\begin{aligned} \int_{\Omega} \left[\lambda c(x) g(v) + \left(1 + \frac{\mu}{p-1}v\right)^{p-1} h(x) \right] \varphi \, dx \\ &= \int_{\Omega} \left[\lambda c(x) \left| \frac{p-1}{\mu} \ln \left(1 + \frac{\mu}{p-1}v\right) \right|^{p-2} \left(\frac{p-1}{\mu} \ln \left(1 + \frac{\mu}{p-1}v\right) \right) + h(x) \right] \phi \, dx \\ &= \int_{\Omega} [\lambda c(x) |u|^{p-2} u + h(x)] \phi \, dx. \end{aligned}$$

As v is a solution of (Q_λ) we deduce from these two identities that

$$\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \phi \, dx = \int_{\Omega} [\lambda c(x) |u|^{p-2} u + \mu |\nabla u|^p + h(x)] \phi \, dx,$$

and so, u is a solution of (P_λ) , as desired.

In the same way, assume that $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ is a solution of (P_λ) . By Proposition 4.2 we know that $u \geq \underline{u}_\lambda$. Hence, it follows that $v = \frac{p-1}{\mu} (e^{\frac{\mu u}{p-1}} - 1) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and satisfies $v \geq \alpha_\lambda \geq -\frac{p-1}{\mu} + \varepsilon$ for some $\varepsilon > 0$. Arguing exactly as before, we deduce that v is a solution of (Q_λ) . \square

Remark 5.1. Arguing exactly as in the proof of Lemma 5.2, *iii*), we can show that $v_1 \in H^1(\Omega) \cap L^\infty(\Omega)$ (respectively $v_2 \in H^1(\Omega) \cap L^\infty(\Omega)$) is a lower solution (respectively an upper solution) of (Q_λ) if, and only if, the function

$$u_1 = \frac{p-1}{\mu} \ln \left(1 + \frac{\mu}{p-1} v_1\right) \quad \left(\text{respectively } u_2 = \frac{p-1}{\mu} \ln \left(1 + \frac{\mu}{p-1} v_2\right) \right)$$

is a lower solution (respectively an upper solution) of (P_λ) .

The interest of problem (Q_λ) comes from the fact that it has a variational formulation. We can obtain the solutions of (Q_λ) as critical points of the functional $I_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined as

$$(5.5) \quad I_\lambda(v) = \frac{1}{p} \int_{\Omega} |\nabla v|^p \, dx - \int_{\Omega} F_\lambda(x, v) \, dx,$$

where we define $F_\lambda(x, s) = \int_0^s f_\lambda(x, t) \, dt$ i.e.

$$(5.6) \quad F_\lambda(x, s) = \lambda c(x) G(s) + \frac{p-1}{\mu p} \left(1 + \frac{\mu}{p-1} s\right)^p h(x), \quad \text{if } s \geq \alpha_\lambda(x),$$

and

$$(5.7) \quad \begin{aligned} F_\lambda(x, s) &= \left[\lambda c(x) g(\alpha_\lambda(x)) + \left(1 + \frac{\mu}{p-1} \alpha_\lambda(x)\right)^{p-1} h(x) \right] (s - \alpha_\lambda) \\ &\quad + \lambda c(x) G(\alpha_\lambda(x)) + \frac{p-1}{\mu p} \left(1 + \frac{\mu}{p-1} \alpha_\lambda(x)\right)^p h(x), \quad \text{if } s \leq \alpha_\lambda(x). \end{aligned}$$

Observe that under the assumptions (A_1) , since g has subcritical growth (see Lemma 5.1), $I \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ (see for example [22] page 356).

Lemma 5.3. *Assume that (A_1) holds and let $\lambda \in \mathbb{R}$ be arbitrary, Then, any bounded Cerami sequence for I_λ admits a convergent subsequence.*

Proof. Let $\{v_n\} \subset W_0^{1,p}(\Omega)$ be a bounded Cerami sequence for I_λ at level $d \in \mathbb{R}$. We are going to show that, up to a subsequence, $v_n \rightarrow v \in W_0^{1,p}(\Omega)$ for a $v \in W_0^{1,p}(\Omega)$.

Since $\{v_n\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$, up to a subsequence, we can assume that $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$, $v_n \rightarrow v$ in $L^r(\Omega)$, for $1 \leq r < p^*$, and $v_n \rightarrow v$ a.e. in Ω . First of all, recall that $\langle I'_\lambda(v_n), v_n - v \rangle \rightarrow 0$ with

$$\begin{aligned} \langle I'_\lambda(v_n), v_n - v \rangle &= \int_\Omega |\nabla v_n|^{p-2} \nabla v_n \nabla (v_n - v) \, dx - \int_{\{v_n \geq \alpha_\lambda\}} \lambda c(x) g(v_n) (v_n - v) \, dx \\ &\quad - \int_{\{v_n \geq \alpha_\lambda\}} \left(1 + \frac{\mu}{p-1} v_n\right)^{p-1} (v_n - v) h(x) \, dx - \int_{\{v_n \leq \alpha_\lambda\}} f_\lambda(x, \alpha_\lambda(x)) (v_n - v) \, dx. \end{aligned}$$

Let $0 < \delta < (\frac{p}{N} - \frac{1}{q})p^*$, $r < p^*$ and $s < \frac{p^*}{p-1+\delta}$ such that $\frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 1$. Using Lemma 5.1 ii), and the Sobolev embedding as well as Hölder inequality, we have that

$$\begin{aligned} \left| \lambda \int_{\{v_n \geq \alpha_\lambda\}} c(x) g(v_n) (v_n - v) \, dx \right| &\leq |\lambda| \int_\Omega |c(x)| |g(v_n)| |v_n - v| \, dx \leq |\lambda| \|c\|_q \|g(v_n)\|_s \|v_n - v\|_r \\ &\leq D |\lambda| \|c\|_q (1 + \|v_n\|_{(\frac{p-1+\delta}{p-1+\delta})s}^{p-1+\delta}) \|v_n - v\|_r \\ &\leq DS |\lambda| \|c\|_q (1 + \|v_n\|^{p-1+\delta}) \|v_n - v\|_r. \end{aligned}$$

Since $\|v_n\|$ is bounded and $v_n \rightarrow v$ in $L^r(\Omega)$, for $1 \leq r < p^*$, we obtain

$$\lambda \int_{\{v_n \geq \alpha_\lambda\}} c(x) g(v_n) (v_n - v) \, dx \rightarrow 0.$$

Arguing in the same way, we have

$$\int_{\{v_n \geq \alpha_\lambda\}} \left(1 + \frac{\mu}{p-1} v_n\right)^{p-1} (v_n - v) h(x) \, dx + \int_{\{v_n \leq \alpha_\lambda\}} f_\lambda(x, \alpha_\lambda(x)) (v_n - v) \, dx \rightarrow 0.$$

So, we deduce that

$$(5.8) \quad \int_\Omega |\nabla v_n|^{p-2} \nabla v_n \nabla (v_n - v) \, dx \rightarrow 0.$$

Hence, applying [22, Theorem 10], we conclude that $v_n \rightarrow v$ in $W_0^{1,p}(\Omega)$, as desired. \square

6. SHARP EXISTENCE RESULTS ON THE LIMIT COERCIVE CASE

In this section, following ideas from [8, Section 3], we prove Theorem 1.1. As a preliminary step, considering $\mu > 0$ constant, we introduce

$$m_{p,\lambda} := \begin{cases} \inf_{u \in W_\lambda} \int_\Omega \left(|\nabla u|^p - \left(\frac{\mu}{p-1}\right)^{p-1} h(x) |u|^p \right) dx, & \text{if } W_\lambda \neq \emptyset, \\ +\infty, & \text{if } W_\lambda = \emptyset. \end{cases}$$

where

$$W_\lambda := \{w \in W_0^{1,p}(\Omega) : \lambda c(x) w(x) = 0 \text{ a.e. } x \in \Omega, \|w\| = 1\}$$

and we define

$$m := \inf_{u \in W_0^{1,p}(\Omega)} I_\lambda(u) \in \mathbb{R} \cup \{-\infty\}.$$

Proposition 6.1. *Assume that (A_1) holds, $\lambda \leq 0$ and that $m_{p,\lambda} > 0$. Then m is finite and it is reached by a function $v \in W_0^{1,p}(\Omega)$. Consequently the problem (P_λ) has a solution.*

Proof. To prove that I_λ has a global minimum since, by Lemma 5.3, any bounded Cerami sequence has a convergent subsequence it suffices to show that I_λ is coercive. Having found a global minimum $v \in W_0^{1,p}(\Omega)$ we deduce, by Lemma 5.2, that $u = \frac{p-1}{\mu} \ln(1 + \frac{\mu}{p-1}v)$ is a solution of (P_λ) . To show that I_λ is coercive we consider an arbitrary sequence $\{v_n\} \subset W_0^{1,p}(\Omega)$ such that $\|v_n\| \rightarrow \infty$ and we prove that

$$\lim_{n \rightarrow \infty} I_\lambda(v_n) = +\infty.$$

Assume by contradiction that, along a subsequence, $I_\lambda(v_n)$ is bounded from above and hence

$$(6.1) \quad \limsup_{n \rightarrow \infty} \frac{I_\lambda(v_n)}{\|v_n\|^p} \leq 0.$$

We introduce the sequence $w_n = \frac{v_n}{\|v_n\|}$, for all $n \in \mathbb{N}$ and observe that, up to a subsequence $w_n \rightharpoonup w$ weakly in $W_0^{1,p}(\Omega)$, $w_n \rightarrow w$ in $L^r(\Omega)$, for $1 \leq r < p^*$, and $w_n \rightarrow w$ a.e. in Ω . We consider two cases:

Case 1): $w^+ \notin W_\lambda$. In that case, the set $\Omega_0 = \{x \in \Omega : \lambda c(x)w^+(x) \neq 0\} \subset \Omega$ has non-zero measure and so, it follows that $v_n(x) = w_n(x)\|v_n\| \rightarrow \infty$ a.e. in Ω_0 . Hence, taking into account that $G \geq 0$ and $\lim_{s \rightarrow +\infty} G(s)/s^p = +\infty$ (see Lemma 5.1) and using Fatou's Lemma, we have

$$(6.2) \quad \begin{aligned} \limsup_{n \rightarrow \infty} \int_{\Omega} \frac{\lambda c(x)G(v_n)}{|v_n|^p} |w_n|^p dx &\leq \limsup_{n \rightarrow \infty} \int_{\Omega_0} \frac{\lambda c(x)G(v_n)}{|v_n|^p} |w_n|^p dx \\ &\leq \int_{\Omega_0} \limsup_{n \rightarrow \infty} \frac{\lambda c(x)G(v_n)}{|v_n|^p} |w_n|^p dx = -\infty. \end{aligned}$$

On the other hand, observe that for any $v \in W_0^{1,p}(\Omega)$, we can rewrite

$$\begin{aligned} I_\lambda(v) &= \frac{1}{p} \int_{\Omega} |\nabla v|^p dx - \int_{\Omega} \lambda c(x)G(v) dx + \int_{\{v \leq \alpha_\lambda\}} \lambda c(x)G(v) dx \\ &\quad - \frac{p-1}{p\mu} \int_{\{v \geq \alpha_\lambda\}} \left(1 + \frac{\mu}{p-1}v\right)^p h(x) dx - \int_{\{v \leq \alpha_\lambda\}} F_\lambda(x, v) dx. \end{aligned}$$

Hence, considering together (6.1) and (6.2), we obtain

$$0 \geq \limsup_{n \rightarrow \infty} \frac{I_\lambda(v_n)}{\|v_n\|^p} \geq \liminf_{n \rightarrow \infty} \frac{I_\lambda(v_n)}{\|v_n\|^p} \geq -C - \limsup_{n \rightarrow \infty} \int_{\Omega} \frac{\lambda c(x)G(v_n)}{\|v_n\|^p} dx = +\infty,$$

and so, *Case 1)* cannot occur.

Case 2): $w^+ \in W_\lambda$. First of all, since $\lambda c \leq 0$ and $G \geq 0$ (see Lemma 5.1), observe that for any $v \in W_0^{1,p}(\Omega)$,

$$\begin{aligned} I_\lambda(v) &\geq \frac{1}{p} \int_{\Omega} \left(|\nabla v|^p - \left(\frac{\mu}{p-1}\right)^{p-1} h(x)(v^+)^p\right) dx - \frac{1}{p} \left(\frac{\mu}{p-1}\right)^{p-1} \int_{\{v \geq \alpha_\lambda\}} h(x)(v^-)^p dx \\ &\quad - \frac{p-1}{\mu p} \int_{\{v \geq \alpha_\lambda\}} \left[\left(1 + \frac{\mu}{p-1}v\right)^p - \left(\frac{\mu}{p-1}\right)^p |v|^p\right] h(x) dx - \int_{\{v \leq \alpha_\lambda\}} F_\lambda(x, v) dx. \end{aligned}$$

Moreover, observe that

$$(6.3) \quad \begin{aligned} &\frac{1}{p} \left| \int_{\{v \geq \alpha_\lambda\}} \left[\left(1 + \frac{\mu}{p-1}v\right)^p - \left(\frac{\mu}{p-1}\right)^p |v|^p\right] h(x) dx \right| \\ &= \left| \int_{\{v \geq \alpha_\lambda\}} \left(\int_0^1 \left|s + \frac{\mu}{p-1}v\right|^{p-2} \left(s + \frac{\mu}{p-1}v\right) ds\right) h(x) dx \right| \\ &\leq \int_{\Omega} \left(1 + \frac{\mu}{p-1}|v|\right)^{p-1} |h(x)| dx \leq D \|h\|_q (1 + \|v\|^{p-1}), \end{aligned}$$

for some constant $D > 0$. Thus, for any $v \in W_0^{1,p}(\Omega)$, it follows that

$$(6.4) \quad \begin{aligned} I_\lambda(v) &\geq \frac{1}{p} \int_{\Omega} \left(|\nabla v|^p - \left(\frac{\mu}{p-1}\right)^{p-1} h(x)(v^+)^p\right) dx - \frac{1}{p} \left(\frac{\mu}{p-1}\right)^{p-1} \int_{\{v \geq \alpha_\lambda\}} h(x)(v^-)^p dx \\ &\quad - D \|h\|_q (1 + \|v\|^{p-1}) - \int_{\{v \leq \alpha_\lambda\}} F_\lambda(x, v) dx. \end{aligned}$$

Hence, using that by the definition of F_λ (see (5.6) and (5.7)) there exists $m \in L^q(\Omega)$, $q > \max\{N/p, 1\}$, such that, for a.e. $x \in \Omega$ and all $s \leq 0$,

$$(6.5) \quad |F_\lambda(x, s)| \leq m(x)(1 + |s|),$$

and applying (6.1) and (6.4), we deduce, as $w^+ \in W_\lambda$, that

$$0 \geq \limsup_{n \rightarrow \infty} \frac{I_\lambda(v_n)}{\|v_n\|^p} \geq \liminf_{n \rightarrow \infty} \frac{I_\lambda(v_n)}{\|v_n\|^p} \geq \frac{1}{p} \int_\Omega \left(|\nabla w|^p - \left(\frac{\mu}{p-1}\right)^{p-1} h(x)(w^+)^p \right) dx \geq \frac{1}{p} \min\{1, m_{p,\lambda}\} \|w\|^p \geq 0,$$

and so, that

$$\lim_{n \rightarrow \infty} \frac{I_\lambda(v_n)}{\|v_n\|^p} = 0 \quad \text{and} \quad w \equiv 0.$$

Finally, taking into account that $w_n \rightarrow 0$ in $L^r(\Omega)$, for $1 \leq r < p^*$, we obtain the contradiction

$$0 = \lim_{n \rightarrow \infty} \frac{I_\lambda(v_n)}{\|v_n\|^p} \geq \frac{1}{p}.$$

Hence, *Case 2)* cannot occur. \square

Proof of Theorem 1.1. To prove this result, we look for a couple of lower and upper solutions (α, β) of (P_λ) with $\alpha \leq \beta$ and then we apply Theorem 2.1. First, assume that both $\|\mu^+\|_\infty > 0$ and $\|\mu^-\|_\infty > 0$. Observe that any solution of

$$(6.6) \quad -\Delta_p u = \lambda c(x)|u|^{p-2}u + \|\mu^+\|_\infty |\nabla u|^p + h(x), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

is an upper solution of (P_λ) and, any solution of

$$(6.7) \quad -\Delta_p u = \lambda c(x)|u|^{p-2}u - \|\mu^-\|_\infty |\nabla u|^p + h(x), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

is a lower solution of (P_λ) . Now, since $m_{p,\lambda}^+ > 0$, Proposition 6.1 ensures the existence of $\beta \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ solution of (6.6). In the same way, $m_{p,\lambda}^- > 0$ implies the existence of $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ solution of

$$-\Delta_p v = \lambda c(x)|v|^{p-2}v + \|\mu^-\|_\infty |\nabla v|^p - h(x), \quad v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

and hence $\alpha = -v$ is a solution of (6.7). Moreover, Lemma 3.3 implies $\alpha, \beta \in W_0^{1,p}(\Omega) \cap \mathcal{C}(\overline{\Omega})$. Hence, since α is a lower solution of (6.6), it follows that $\alpha \leq \beta$, thanks to Theorem 3.1. Thus, we can apply Theorem 2.1 to conclude the proof. Now note that if $\|\mu^+\|_\infty = 0$, (6.6) reduces to

$$(6.8) \quad -\Delta_p u = \lambda c(x)|u|^{p-2}u + h(x), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

which has a solution by [22, Theorem 13]. This solution corresponds again to an upper solution to (P_λ) . Similarly, we can justify the existence of the lower solution when $\|\mu^-\|_\infty = 0$. \square

7. A NECESSARY AND SUFFICIENT CONDITION FOR THE EXISTENCE OF A SOLUTION TO (P_0)

In this section we prove Theorem 1.3. First of all, following the ideas of [8], inspired in turn in ideas of [2], we find a necessary condition for the existence of a solution of (P_0) . Recall that the problem (P_0) is given by

$$(P_0) \quad -\Delta_p u = \mu |\nabla u|^p + h(x), \quad u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

Proposition 7.1. *Assume that (A_1) holds and suppose that (P_0) has a solution. Then m_p defined by (1.1) satisfies $m_p > 0$.*

Proof. Assume that (P_0) has a solution $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Then, for any $\phi \in \mathcal{C}_0^\infty(\Omega)$, it follows that

$$(7.1) \quad \int_\Omega |\nabla u|^{p-2} \nabla u \nabla (|\phi|^p) dx - \mu \int_\Omega |\nabla u|^p |\phi|^p dx - \int_\Omega h(x) |\phi|^p dx = 0.$$

Now, applying Young's inequality, observe that

$$\begin{aligned} \int_\Omega |\nabla u|^{p-2} \nabla u \nabla (|\phi|^p) dx &= p \int_\Omega |\phi|^{p-2} \phi |\nabla u|^{p-2} \nabla u \nabla \phi dx \leq p \int_\Omega |\phi|^{p-1} |\nabla u|^{p-1} |\nabla \phi| dx \\ &\leq \mu \int_\Omega |\phi|^p |\nabla u|^p dx + \left(\frac{p-1}{\mu}\right)^{p-1} \int_\Omega |\nabla \phi|^p dx. \end{aligned}$$

Hence, substituting in (7.1), multiplying by $\left(\frac{\mu}{p-1}\right)^{p-1}$ and using the density of $C_0^\infty(\Omega)$ in $W_0^{1,p}(\Omega)$, we obtain

$$(7.2) \quad \int_{\Omega} \left(|\nabla \phi|^p - \left(\frac{\mu}{p-1}\right)^{p-1} h(x) |\phi|^p \right) dx \geq 0, \quad \forall \phi \in W_0^{1,p}(\Omega).$$

Arguing by contradiction, assume that

$$\inf \left\{ \int_{\Omega} \left(|\nabla \phi|^p - \left(\frac{\mu}{p-1}\right)^{p-1} h(x) |\phi|^p \right) dx : \phi \in W_0^{1,p}(\Omega), \|\phi\| = 1 \right\} = 0.$$

By standard arguments there exists $\phi_0 \in C^{0,\tau}(\bar{\Omega})$ for some $\tau \in (0, 1)$, with $\phi_0 > 0$ in Ω such that

$$(7.3) \quad \int_{\Omega} |\nabla \phi_0|^p dx = \left(\frac{\mu}{p-1}\right)^{p-1} \int_{\Omega} h(x) |\phi_0|^p dx.$$

Now, substituting the above identity in (7.1) with $\phi = \phi_0$, we have that

$$(7.4) \quad \int_{\Omega} \left(|\nabla \phi_0|^p + (p-1) \left(\frac{\mu}{p-1}\right)^p \phi_0^p |\nabla u|^p - p \left(\frac{\mu}{p-1}\right)^{p-1} \phi_0^{p-1} |\nabla u|^{p-2} \nabla u \nabla \phi_0 \right) dx = 0.$$

Finally, observe that

$$\frac{\mu}{p-1} \nabla u = \frac{1}{e^{\frac{\mu}{p-1}u}} \nabla e^{\frac{\mu}{p-1}u}.$$

Hence, by substituting in (7.4), we deduce that

$$(7.5) \quad \int_{\Omega} \left(|\nabla \phi_0|^p + (p-1) \left(\frac{\phi_0}{e^{\frac{\mu}{p-1}u}}\right)^p |\nabla e^{\frac{\mu}{p-1}u}|^p - p \left(\frac{\phi_0}{e^{\frac{\mu}{p-1}u}}\right)^{p-1} |\nabla e^{\frac{\mu}{p-1}u}|^{p-2} \nabla e^{\frac{\mu}{p-1}u} \nabla \phi_0 \right) dx = 0.$$

Applying Proposition 2.4, this proves the existence of $k \in \mathbb{R}$ such that

$$\phi_0 = k e^{\frac{\mu}{p-1}u}.$$

As $\phi_0 = 0$ and $e^{\frac{\mu}{p-1}u} = 1$ on $\partial\Omega$, this implies that $k = 0$ which contradicts the fact that $\phi_0 > 0$ in Ω . \square

Proof of Theorem 1.3. The proof is just the combination of Proposition 7.1 and of Remark 1.1. \square

8. ON THE CERAMI CONDITON AND THE MOUNTAIN-PASS GEOMETRY

We are going to show that, for any $\lambda > 0$, the Cerami sequences for I_λ at any level are bounded. The proof is inspired by [31], see also [29]. Nevertheless it requires to develop some new ideas. In view of Lemma 5.3, this will imply that I_λ satisfies the Cerami condition at any level $d \in \mathbb{R}$.

Lemma 8.1. *Fixed $\lambda > 0$ arbitrary, assume that (A_1) holds and suppose that $m_p > 0$ with m_p defined by (1.1). Then, the Cerami sequences for I_λ at any level $d \in \mathbb{R}$ are bounded.*

Proof. Let $\{v_n\} \subset W_0^{1,p}(\Omega)$ be a Cerami sequence for I_λ at level $d \in \mathbb{R}$. First we claim that $\{v_n^-\}$ is bounded. Indeed since $\{v_n\}$ is a Cerami sequence, we have that

$$(8.1) \quad \langle I'_\lambda(v_n), v_n^- \rangle = - \int_{\Omega} |\nabla v_n^-|^p dx - \int_{\Omega} f_\lambda(x, v_n) v_n^- dx \rightarrow 0$$

from which, since $f_\lambda(x, s)$ is bounded on $\Omega \times \mathbb{R}^-$, the claim follows. To prove that $\{v_n^+\}$ is also bounded we assume by contradiction that $\|v_n\| \rightarrow \infty$. We define

$$\Omega_n^+ = \{x \in \Omega : v_n(x) \geq 0\} \quad \text{and} \quad \Omega_n^- = \Omega \setminus \Omega_n^+,$$

and introduce the sequence $\{w_n\} \subset W_0^{1,p}(\Omega)$ given by $w_n = v_n / \|v_n\|$. Observe that $\{w_n\} \subset W_0^{1,p}(\Omega)$ is bounded in $W_0^{1,p}(\Omega)$. Hence, up to a subsequence, it follows that $w_n \rightharpoonup w$ in $W_0^{1,p}(\Omega)$, $w_n \rightarrow w$ strongly in $L^r(\Omega)$ for $1 \leq r < p^*$, and $w_n \rightarrow w$ a.e. in Ω . We split the proof in several steps.

Step 1: $cw \equiv 0$.

As $\|v_n^-\|$ is bounded and by assumption $\|v_n\| \rightarrow \infty$, clearly $w^- \equiv 0$. It remains to show that $cw^+ \equiv 0$. Assume by contradiction that $cw^+ \not\equiv 0$ i.e., defining $\Omega^+ := \{x \in \Omega : c(x)w(x) > 0\}$, we assume $|\Omega^+| > 0$. Since $\|v_n\| \rightarrow \infty$ and $\langle I'_\lambda(v_n), v_n \rangle \rightarrow 0$, it follows that

$$(8.2) \quad \frac{\langle I'_\lambda(v_n), v_n \rangle}{\|v_n\|^p} \rightarrow 0.$$

First of all, observe that

$$(8.3) \quad \begin{aligned} \langle I'_\lambda(v_n), v_n \rangle &= \|v_n\|^p - \left(\frac{\mu}{p-1}\right)^{p-1} \int_{\Omega_n^+} h(x)|v_n|^p dx - \int_{\Omega_n^+} \lambda c(x)g(v_n)v_n dx - \int_{\Omega_n^-} f_\lambda(x, v_n)v_n dx \\ &\quad - \int_{\Omega_n^+} \left[\left(1 + \frac{\mu}{p-1}v_n\right)^{p-1} - \left(\frac{\mu}{p-1}\right)^{p-1}|v_n|^{p-2}v_n \right] v_n h(x) dx. \end{aligned}$$

Now, since $f_\lambda(x, s)$ is bounded on $\Omega \times \mathbb{R}^-$, we deduce that

$$(8.4) \quad \frac{1}{\|v_n\|^p} \int_{\Omega_n^-} f_\lambda(x, v_n)v_n dx \rightarrow 0.$$

Moreover, using that $w_n \rightarrow w$ in $L^r(\Omega)$, $1 \leq r < p^*$, with $w^- \equiv 0$, we have

$$(8.5) \quad \frac{1}{\|v_n\|^p} \int_{\Omega_n^+} |v_n|^p h(x) dx = \int_{\Omega_n^+} |w_n|^p h(x) dx \rightarrow \int_{\Omega} w^p h(x) dx,$$

Next, we are going to show that

$$(8.6) \quad \frac{1}{\|v_n\|^p} \int_{\Omega_n^+} \left[\left(1 + \frac{\mu}{p-1}v_n\right)^{p-1} - \left(\frac{\mu}{p-1}\right)^{p-1}|v_n|^{p-2}v_n \right] v_n h(x) dx \rightarrow 0.$$

Observe that

$$\int_{\Omega_n^+} \left[\left(1 + \frac{\mu}{p-1}v_n\right)^{p-1} - \left(\frac{\mu}{p-1}\right)^{p-1}v_n^{p-1} \right] v_n h(x) dx = (p-1) \int_{\Omega_n^+} \left[\int_0^1 \left(s + \frac{\mu}{p-1}v_n\right)^{p-2} ds \right] v_n h(x) dx.$$

We consider separately the case $p \geq 2$ and the case $1 < p < 2$. In case $p \geq 2$, there exists $D > 0$ such that

$$\left| \int_{\Omega_n^+} \left[\int_0^1 \left(s + \frac{\mu}{p-1}v_n\right)^{p-2} ds \right] v_n h(x) dx \right| \leq \frac{p-1}{\mu} \int_{\Omega_n^+} \left(1 + \frac{\mu}{p-1}v_n\right)^{p-1} |h(x)| dx \leq D \|h\|_q (1 + \|v_n\|^{p-1}).$$

On the other hand, in case $1 < p < 2$, we have a constant $D > 0$ with

$$\left| \int_{\Omega_n^+} \left[\int_0^1 \left(s + \frac{\mu}{p-1}v_n\right)^{p-2} ds \right] v_n h(x) dx \right| \leq \frac{p-1}{\mu} \int_{\Omega_n^+} \left(\frac{\mu}{p-1}v_n\right)^{p-1} |h(x)| dx \leq D \|h\|_q \|v_n\|^{p-1}.$$

The claim (8.6) follows then directly from the above inequalities. So, substituting (8.3), (8.4), (8.5) and (8.6) in (8.2) and using that g is bounded on \mathbb{R}^- , we deduce that

$$(8.7) \quad \lambda \int_{\Omega} c(x) \frac{g(v_n)}{v_n^{p-1}} w_n^p dx \rightarrow 1 - \left(\frac{\mu}{p-1}\right)^{p-1} \int_{\Omega} w^p h(x) dx.$$

Let us prove that this is a contradiction. By Lemma 5.1, we know that $\lim_{s \rightarrow +\infty} g(s)/s^{p-1} = +\infty$ and as $w_n \rightarrow w > 0$ a.e. in Ω^+ , it follows that

$$c(x) \frac{g(v_n)}{v_n^{p-1}} w_n^p \rightarrow +\infty \text{ a.e. in } \Omega^+.$$

Since $|\Omega^+| > 0$, we have

$$(8.8) \quad \int_{\Omega^+} c(x) \frac{g(v_n)}{v_n^{p-1}} w_n^p dx \rightarrow +\infty.$$

On the other hand, as $g \geq 0$ on \mathbb{R}^+ , $\frac{g(s)}{s^{p-1}}$ is bounded on \mathbb{R}^- and $\|w_n\|_{pq'}$ is bounded, we have

$$(8.9) \quad \int_{\Omega \setminus \Omega^+} c(x) \frac{g(v_n)}{v_n^{p-1}} w_n^p dx \geq -D.$$

So (8.8) and (8.9) together give a contradiction with (8.7). Consequently, we conclude that $cw \equiv 0$.

Step 2: Let us introduce a new functional $J_\lambda : W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined as

$$J_\lambda(v) = I_\lambda(v) - \frac{p-1}{\mu p} \int_{\{v \geq \alpha_\lambda\}} \left[\left(1 + \frac{\mu}{p-1}v\right)^p - \left(\frac{\mu}{p-1}\right)^p |v|^p \right] h^-(x) dx$$

and let us introduce the sequence $\{z_n\} \subset W_0^{1,p}(\Omega)$ defined by $z_n = t_n v_n$, where $t_n \in [0, 1]$ satisfies

$$J_\lambda(z_n) = \max_{t \in [0,1]} J_\lambda(t v_n),$$

(if t_n is not unique we choose its smallest possible value). We claim that

$$\lim_{n \rightarrow \infty} J_\lambda(z_n) = +\infty.$$

We argue again by contradiction. Suppose the existence of $M < +\infty$ such that

$$(8.10) \quad \liminf_{n \rightarrow \infty} J_\lambda(z_n) \leq M,$$

and introduce a sequence $\{k_n\} \subset W_0^{1,p}(\Omega)$, defined as

$$k_n = \left(\frac{2pM}{m_p}\right)^{\frac{1}{p}} w_n = \left(\frac{2pM}{m_p}\right)^{\frac{1}{p}} \frac{v_n}{\|v_n\|}.$$

Let us prove, taking M bigger if necessary, that for n large enough we have

$$(8.11) \quad J_\lambda(k_n) > \frac{3}{2}M.$$

As $\left(\frac{2pM}{m_p}\right)^{\frac{1}{p}} \frac{1}{\|v_n\|} \in [0, 1]$ for n large enough, this will give the contradiction

$$\frac{3}{2}M \leq \liminf_{n \rightarrow \infty} J_\lambda(k_n) \leq \liminf_{n \rightarrow \infty} J_\lambda(z_n) \leq M.$$

First of all, observe that $k_n \rightarrow k := \left(\frac{2pM}{m_p}\right)^{\frac{1}{p}} w$ in $W_0^{1,p}(\Omega)$, $k_n \rightarrow k$ in $L^r(\Omega)$, for $1 \leq r < p^*$, and $k_n \rightarrow k$ a.e. in Ω . By the properties of G (see Lemma 5.1) together with $k \geq 0$ and $c k \equiv 0$, it is easy to prove that

$$(8.12) \quad \int_{\{k_n \geq \alpha_\lambda\}} c(x)G(k_n) dx \rightarrow \int_{\Omega} c(x)G(k) dx = 0.$$

As $w^- \equiv 0$, we have $\chi_n^- \rightarrow 0$ a.e. in Ω where χ_n^- is the characteristic function of Ω_n^- . Recall (see (6.5)) that we have $m \in L^q(\Omega)$, $q > \max\{N/p, 1\}$, such that, for a.e. $x \in \Omega$ and all $s \leq 0$,

$$|F_\lambda(x, s)| \leq m(x)(1 + |s|).$$

This implies that

$$(8.13) \quad \int_{\{k_n \leq \alpha_\lambda\}} F_\lambda(x, k_n) dx \rightarrow 0, \quad \text{as well as} \quad \int_{\{k_n \leq \alpha_\lambda\}} |k_n|^p h(x) dx \rightarrow 0.$$

Taking into account (8.12) and (8.13) we obtain that

$$(8.14) \quad \begin{aligned} J_\lambda(k_n) &= \frac{1}{p} \int_{\Omega} \left(|\nabla k_n|^p - \left(\frac{\mu}{p-1}\right)^{p-1} h(x) |k_n|^p \right) dx \\ &\quad - \frac{p-1}{p\mu} \int_{\{k_n \geq \alpha_\lambda\}} \left[\left(1 + \frac{\mu}{p-1}k_n\right)^p - \left(\frac{\mu}{p-1}\right)^p |k_n|^p \right] h^+(x) dx + o(1). \end{aligned}$$

Now, observe that, by definition of m_p ,

$$(8.15) \quad \frac{1}{p} \int_{\Omega} \left(|\nabla k_n|^p - \left(\frac{\mu}{p-1}\right)^{p-1} h(x) |k_n|^p \right) dx \geq \frac{1}{p} m_p \|k_n\|^p = 2M.$$

Furthermore, arguing as in (6.3), observe that

$$\frac{1}{p} \left| \int_{\{k_n \geq \alpha_\lambda\}} \left[\left(1 + \frac{\mu}{p-1}k_n\right)^p - \left(\frac{\mu}{p-1}\right)^p |k_n|^p \right] h^+(x) dx \right| \leq C \|h^+\|_q \left(1 + \left(\frac{2pM}{m_p}\right)^{\frac{p-1}{p}}\right),$$

where C is independent of M . This implies that

$$J_\lambda(k_n) \geq 2M - C\|h^+\|_q \left(1 + \left(\frac{2pM}{m_p}\right)^{\frac{p-1}{p}}\right) + o(1),$$

and, taking M bigger if necessary, for any $n \in \mathbb{N}$ large enough, (8.11) follows.

Step 3: For $n \in \mathbb{N}$ large enough, $t_n \in (0, 1)$.

By the definition of J_λ and using that

$$\left(1 + \frac{\mu}{p-1}s\right)^p - \left(\frac{\mu}{p-1}\right)^p s^p \geq 0, \quad \forall s \geq 0,$$

observe that

$$\begin{aligned} J_\lambda(v_n) &\leq I_\lambda(v_n) - \frac{p-1}{p\mu} \int_{\{\alpha_\lambda \leq v_n \leq 0\}} \left[\left(1 + \frac{\mu}{p-1}v_n\right)^p - \left(\frac{\mu}{p-1}\right)^p |v_n|^p \right] h^-(x) dx \\ &\leq I_\lambda(v_n) + \frac{1}{p} \left(\frac{\mu}{p-1}\right)^{p-1} \int_{\{\alpha_\lambda \leq v_n \leq 0\}} |v_n|^p h^-(x) dx \\ &\leq I_\lambda(v_n) + \frac{1}{p} \left(\frac{\mu}{p-1}\right)^{p-1} \|\alpha_\lambda\|_\infty^p \|h^-\|_1. \end{aligned}$$

Consequently, since $I_\lambda(v_n) \rightarrow d$, there exists $D > 0$ such that, for all $n \in \mathbb{N}$, $J_\lambda(v_n) \leq D$. Thus, taking into account that $J_\lambda(0) = -\frac{p-1}{p\mu}\|h^+\|_1$ and $J_\lambda(t_n v_n) \rightarrow +\infty$, we conclude that $t_n \in (0, 1)$ for n large enough.

Step 4: *Conclusion.*

First of all, as $t_n \in (0, 1)$ for n large enough, by the definition of z_n , observe that $\langle J'_\lambda(z_n), z_n \rangle = 0$, for those n . Thus, it follows that

$$\begin{aligned} J_\lambda(z_n) &= J_\lambda(z_n) - \frac{1}{p} \langle J'_\lambda(z_n), z_n \rangle \\ &= \lambda \int_{\{z_n \geq \alpha_\lambda\}} c(x)H(z_n) dx - \frac{p-1}{p\mu} \int_{\{z_n \geq \alpha_\lambda\}} \left(1 + \frac{\mu}{p-1}z_n\right)^{p-1} h^+(x) dx \\ &\quad - \int_{\{z_n \leq \alpha_\lambda\}} \left[F_\lambda(x, z_n) - \frac{1}{p} f_\lambda(x, z_n) z_n \right] dx. \end{aligned}$$

Using the definition of $f_\lambda(x, s)$ for $s \leq \alpha_\lambda(x)$ and the fact that $\|z_n^-\|$ is bounded, we easily deduce the existence of $D_1 > 0$ such that, for all n large enough,

$$(8.16) \quad \frac{p-1}{p\mu} \int_\Omega \left|1 + \frac{\mu}{p-1}z_n\right|^{p-2} \left(1 + \frac{\mu}{p-1}z_n\right) h^+(x) dx \leq -J_\lambda(z_n) + \lambda \int_\Omega c(x)H(z_n) dx + D_1.$$

Now, since $\{v_n\}$ is a Cerami sequence, observe that (again for n large enough)

$$\begin{aligned} d+1 &\geq I_\lambda(v_n) - \frac{1}{p} \langle I'_\lambda(v_n), v_n \rangle \\ &= \lambda \int_{\{v_n \geq \alpha_\lambda\}} c(x)H(v_n) dx - \frac{p-1}{p\mu} \int_{\{v_n \geq \alpha_\lambda\}} \left(1 + \frac{\mu}{p-1}v_n\right)^{p-1} h(x) dx \\ &\quad - \int_{\{v_n \leq \alpha_\lambda\}} \left[F_\lambda(x, v_n) - \frac{1}{p} f_\lambda(x, v_n) v_n \right] dx \end{aligned}$$

and, as above, there exists a constant $D_2 > 0$ such that

$$(8.17) \quad \lambda \int_\Omega c(x)H(v_n) dx \leq \frac{p-1}{p\mu} \int_\Omega \left|1 + \frac{\mu}{p-1}v_n\right|^{p-2} \left(1 + \frac{\mu}{p-1}v_n\right) h(x) dx + D_2.$$

Moreover, observe that

$$\begin{aligned}
& \int_{\Omega} \left| 1 + \frac{\mu}{p-1} v_n \right|^{p-2} \left(1 + \frac{\mu}{p-1} v_n \right) h(x) dx \\
&= \int_{\Omega} \left| 1 + \frac{\mu}{p-1} v_n \right|^{p-2} \left(1 + \frac{\mu}{p-1} v_n \right) h^+(x) dx - \int_{\Omega} \left| 1 + \frac{\mu}{p-1} v_n \right|^{p-2} \left(1 + \frac{\mu}{p-1} v_n \right) h^-(x) dx \\
&= \frac{1}{t_n^{p-1}} \int_{\Omega} \left| t_n + \frac{\mu}{p-1} z_n \right|^{p-2} \left(t_n + \frac{\mu}{p-1} z_n \right) h^+(x) dx - \int_{\Omega} \left| 1 + \frac{\mu}{p-1} v_n \right|^{p-2} \left(1 + \frac{\mu}{p-1} v_n \right) h^-(x) dx \\
&\leq \frac{1}{t_n^{p-1}} \int_{\Omega} \left| 1 + \frac{\mu}{p-1} z_n \right|^{p-2} \left(1 + \frac{\mu}{p-1} z_n \right) h^+(x) dx - \int_{\Omega} \left| 1 + \frac{\mu}{p-1} v_n \right|^{p-2} \left(1 + \frac{\mu}{p-1} v_n \right) h^-(x) dx.
\end{aligned}$$

Considering together this inequality with (8.16) and (8.17), we obtain that

$$\begin{aligned}
(8.18) \quad \lambda \int_{\Omega} c(x) H(v_n) dx &\leq D_2 - \frac{J_{\lambda}(z_n)}{t_n^{p-1}} + \frac{\lambda}{t_n^{p-1}} \int_{\Omega} c(x) H(z_n) dx + \frac{D_1}{t_n^{p-1}} \\
&\quad - \frac{p-1}{p\mu} \int_{\Omega} \left| 1 + \frac{\mu}{p-1} v_n \right|^{p-2} \left(1 + \frac{\mu}{p-1} v_n \right) h^-(x) dx.
\end{aligned}$$

Now, since H is bounded on \mathbb{R}^- , there exists $D_3 > 0$ such that, for all $n \in \mathbb{N}$,

$$(8.19) \quad \int_{\Omega_n^-} c(x) H(z_n) dx \leq D_3.$$

On the other hand, using *iv*) of Lemma 5.1, it follows that

$$(8.20) \quad \int_{\Omega_n^+} c(x) H(z_n) dx \leq t_n^{p-1} \int_{\Omega_n^+} c(x) H(v_n) dx + D_4,$$

for some positive constant D_4 . Hence, substituting (8.19) and (8.20) in (8.18), it follows that

$$\lambda \int_{\Omega_n^-} c(x) H(v_n) dx \leq D_5 - \frac{J_{\lambda}(z_n) - D_6}{t_n^{p-1}} - \frac{p-1}{p\mu} \int_{\Omega} \left| 1 + \frac{\mu}{p-1} v_n \right|^{p-2} \left(1 + \frac{\mu}{p-1} v_n \right) h^-(x) dx.$$

Arguing as in the previous steps, observe that

$$\int_{\Omega} \left| 1 + \frac{\mu}{p-1} v_n \right|^{p-2} \left(1 + \frac{\mu}{p-1} v_n \right) h^-(x) dx \geq - \int_{\Omega_n^-} \left| 1 + \frac{\mu}{p-1} v_n \right|^{p-1} h^-(x) dx \geq -D_7 \|h^-\|_q (1 + \|v_n^-\|^{p-1}),$$

and so, we have that

$$\lambda \int_{\Omega_n^-} c(x) H(v_n) dx \leq D_5 - \frac{J_{\lambda}(z_n) - D_6}{t_n^{p-1}} + D_7 \|h^-\|_q (1 + \|v_n^-\|^{p-1}).$$

By Step 1, we know that $\|v_n^-\|$ is bounded and Step 4 shows that $J_{\lambda}(z_n) \rightarrow \infty$. Recall also that, by Step 4, $t_n \in (0, 1)$. This implies that

$$(8.21) \quad \lambda \int_{\Omega_n^-} c(x) H(v_n) dx \rightarrow -\infty$$

which contradicts the fact that H is bounded on \mathbb{R}^- . This allows to conclude that the Cerami sequences for I_{λ} at level $d \in \mathbb{R}$ are bounded. \square

Now, we turn to the verification of the mountain pass geometry when $\lambda \geq 0$ is small.

Lemma 8.2. *Assume that (A_1) holds and suppose that $m_p > 0$. For $\lambda \geq 0$ small enough, there exists $r > 0$ such that $I_{\lambda}(v) > I_{\lambda}(0)$ for $\|v\| = r$.*

Proof. For an arbitrary fixed $r > 0$, let $v \in W_0^{1,p}(\Omega)$ be such that $\|v\| = r$. We can write

$$\begin{aligned} I_\lambda(v) &= \frac{1}{p} \int_\Omega \left(|\nabla v|^p - \left(\frac{\mu}{p-1}\right)^{p-1} (v^+)^p h(x) \right) dx - \frac{1}{p} \left(\frac{\mu}{p-1}\right)^{p-1} \int_{\{\alpha_\lambda \leq v \leq 0\}} |v|^p h(x) dx \\ &\quad - \frac{p-1}{p\mu} \int_{\{v \geq \alpha_\lambda\}} \left[\left(1 + \frac{\mu}{p-1} v\right)^p - \left(\frac{\mu}{p-1}\right)^p |v|^p \right] h(x) dx \\ &\quad - \int_{\{v \leq \alpha_\lambda\}} h(x) \left[\left(1 + \frac{\mu}{p-1} \alpha_\lambda\right)^{p-1} (v - \alpha_\lambda) + \frac{p-1}{p\mu} \left(1 + \frac{\mu}{p-1} \alpha_\lambda\right)^p \right] dx \\ &\quad - \lambda \left(\int_{\{v \geq \alpha_\lambda\}} c(x) G(v) dx + \int_{\{v \leq \alpha_\lambda\}} c(x) [g(\alpha_\lambda)(v - \alpha_\lambda) + G(\alpha_\lambda)] dx \right). \end{aligned}$$

Now, observe that, as above,

$$\left| \int_{\{v \geq \alpha_\lambda\}} \left[\left(1 + \frac{\mu}{p-1} v\right)^p - \left(\frac{\mu}{p-1}\right)^p |v|^p \right] h(x) dx \right| \leq p \int_\Omega \left(1 + \frac{\mu}{p-1} |v|\right)^{p-1} |h(x)| dx \leq D_1(1 + r^{p-1}),$$

with D_1 independent of λ . In the same way, using the fact that $\alpha_\lambda \in [-\frac{p-1}{\mu}, 0]$, we deduce that

$$\begin{aligned} &\left| -\frac{1}{p} \left(\frac{\mu}{p-1}\right)^{p-1} \int_{\{\alpha_\lambda \leq v \leq 0\}} |v|^p h(x) dx \right| \leq D_2, \\ &\left| - \int_{\{v \leq \alpha_\lambda\}} h(x) \left[\left(1 + \frac{\mu}{p-1} \alpha_\lambda\right)^{p-1} (v - \alpha_\lambda) + \frac{p-1}{p\mu} \left(1 + \frac{\mu}{p-1} \alpha_\lambda\right)^p \right] dx \right| \leq D_3 + D_4 r, \end{aligned}$$

with D_2, D_3 and D_4 independent of λ . Finally, observe that

$$\begin{aligned} \int_\Omega \left(|\nabla v|^p - \left(\frac{\mu}{p-1}\right)^{p-1} (v^+)^p h(x) \right) dx &= \int_\Omega \left(|\nabla v^+|^p - \left(\frac{\mu}{p-1}\right)^{p-1} (v^+)^p h(x) \right) dx + \int_\Omega |\nabla v^-|^p dx \\ &\geq m_p \|v^+\|^p + \|v^-\|^p \geq \min\{1, m_p\} \|v\|^p = \min\{1, m_p\} r^p. \end{aligned}$$

So, we obtain that

$$(8.22) \quad \begin{aligned} I_\lambda(v) &\geq \frac{1}{p} \min\{1, m_p\} r^p - D_1 r^{p-1} - D_4 r - D_5, \\ &\quad - \lambda \left(\int_{\{v \geq \alpha_\lambda\}} c(x) G(v) dx + \int_{\{v \leq \alpha_\lambda\}} c(x) [g(\alpha_\lambda)(v - \alpha_\lambda) + G(\alpha_\lambda)] dx \right), \end{aligned}$$

where the constants D_i are independent of λ . Moreover, observe that for r large enough,

$$(8.23) \quad \frac{1}{p} \min\{1, m_p\} r^p - D_1 r^{p-1} - D_4 r - D_5 \geq \frac{1}{2p} \min\{1, m_p\} r^p + I_\lambda(0).$$

On the other hand, by Lemma 5.1, for every $\delta > 0$,

$$(8.24) \quad \left| \left(\int_{\{v \geq \alpha_\lambda\}} c(x) G(v) dx + \int_{\{v \leq \alpha_\lambda\}} c(x) [g(\alpha_\lambda)(v - \alpha_\lambda) + G(\alpha_\lambda)] dx \right) \right| \leq D_6 r^{p+\delta} + D_7 r + D_8,$$

for some constant D_6, D_7, D_8 independent of λ . Hence, for λ small enough, we have

$$(8.25) \quad \lambda \left(\int_{\{v \geq \alpha_\lambda\}} c(x) G(v) dx + \int_{\{v \leq \alpha_\lambda\}} c(x) [g(\alpha_\lambda)(v - \alpha_\lambda) + G(\alpha_\lambda)] dx \right) \leq \frac{1}{4p} \min\{1, m_p\} r^p,$$

and so, gathering (8.22), (8.23) and (8.25), we conclude that

$$I_\lambda(v) \geq \frac{1}{4p} \min\{1, m_p\} r^p + I_\lambda(0) > I_\lambda(0).$$

□

Lemma 8.3. *Assume that (A_1) holds and that $m_p > 0$. For any $\lambda > 0$, $M > 0$, and $r > 0$, there exists $w \in W_0^{1,p}(\Omega)$ such that $\|w\| > r$ and $I_\lambda(w) \leq -M$.*

Proof. Consider $v \in C_0^\infty(\Omega)$ such that $v \geq 0$ and $cv \not\equiv 0$ and let us take $t \in \mathbb{R}^+$, $t \geq 1$. First of all, as $\alpha_\lambda \leq 0$, observe that

$$\begin{aligned} I_\lambda(tv) &\leq \frac{1}{p} t^p \int_\Omega \left(|\nabla v|^p - \left(\frac{\mu}{p-1}\right)^{p-1} |v|^p h(x) \right) dx - \lambda t^p \int_\Omega c(x) v^p \frac{G(tv)}{t^p v^p} dx \\ &\quad + \frac{p-1}{p\mu} \int_\Omega \left[\left(1 + \frac{\mu}{p-1} tv\right)^p - \left(\frac{\mu}{p-1}\right)^p (tv)^p \right] h^-(x) dx. \end{aligned}$$

As above, we have

$$\frac{1}{p} \int_\Omega \left[\left(1 + \frac{\mu}{p-1} tv\right)^p - \left(\frac{\mu}{p-1}\right)^p (tv)^p \right] h^-(x) dx \leq t^{p-1} \int_\Omega \left(1 + \frac{\mu}{p-1} v\right)^{p-1} h^-(x) dx.$$

Hence we obtain

$$I_\lambda(tv) \leq t^p \left[\frac{1}{p} \int_\Omega \left(|\nabla v|^p - \left(\frac{\mu}{p-1}\right)^{p-1} |v|^p h(x) \right) dx - \lambda \int_\Omega c(x) v^p \frac{G(tv)}{t^p v^p} dx + \frac{1}{t} \frac{p-1}{\mu} \left\| 1 + \frac{\mu}{p-1} v \right\|_\infty^{p-1} \|h^-\|_1 \right].$$

Now, since by Lemma 5.1, we have

$$\lim_{t \rightarrow \infty} \lambda \int_\Omega c(x) v^p \frac{G(tv)}{(tv)^p} dx = \infty,$$

we deduce that $\lim_{t \rightarrow \infty} I_\lambda(tv) = -\infty$ from which the lemma follows. \square

Proposition 8.4. *Assume that (A_1) holds and suppose that $m_p > 0$. Moreover, suppose that $\lambda \geq 0$ is small enough in order to ensure that the conclusion of Lemma 8.2 holds. Then, I_λ possesses a critical point $v \in B(0, r)$ with $I_\lambda(v) \leq I_\lambda(0)$, which is a local minimum of I_λ .*

Proof. From Lemma 8.2, we see that there exists $r > 0$ such that

$$m := \inf_{v \in B(0, r)} I_\lambda(v) \leq I_\lambda(0) \quad \text{and} \quad I_\lambda(v) > I_\lambda(0) \quad \text{if} \quad \|v\| = r.$$

Let $\{v_n\} \subset B(0, r)$ be such that $I_\lambda(v_n) \rightarrow m$. Since $\{v_n\}$ is bounded, up to a subsequence, it follows that $v_n \rightharpoonup v \in W_0^{1,p}(\Omega)$. By the weak lower semicontinuity of the norm and of the functional I_λ , we have

$$\|v\| \leq \liminf_{n \rightarrow \infty} \|v_n\| \leq r \quad \text{and} \quad I_\lambda(v) \leq \liminf_{n \rightarrow \infty} I_\lambda(v_n) = m \leq I_\lambda(0).$$

Finally, as $I_\lambda(v) > I_\lambda(0)$ if $\|v\| = r$, we deduce that $v \in B(0, r)$ is a local minimum of I_λ . \square

Proof of Theorem 1.4. Assume that $\lambda > 0$ is small enough in order to ensure that the conclusion of Lemma 8.2 holds. By Proposition 8.4 we have a first critical point, which is a local minimum of I_λ . On the other hand, since the Cerami condition holds, in view of Lemmata 8.2. and 8.3, we can apply Theorem 2.8 and obtain a second critical point of I_λ at the mountain-pass level. This gives two different solutions of (Q_λ) . Finally, by Lemma 5.2, we obtain two solutions of (P_λ) . \square

9. PROOF OF THEOREMS 1.5 AND 1.6

In this section, we assume the stronger assumption (A_2) . In that case, we are able to improve our results on the non-coercive case.

Proposition 9.1. *Assume that (A_2) holds with $h \not\geq 0$. Then, for every $\lambda > 0$, there exists $v \in C_0^{1,\tau}(\overline{\Omega})$, for some $0 < \tau < 1$, with $v \ll 0$, which is a local minimum of I_λ in the $W_0^{1,p}$ -topology and a solution of (Q_λ) with $v \geq \alpha_\lambda$ (with α_λ defined by (5.2)).*

Proof. First of all, observe that, as $h \not\geq 0$, we have $m_p > 0$ and hence, by Theorem 1.3, (P_0) has a solution $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. By Lemma 5.2,

$$v_0 = \frac{p-1}{\mu} \left(e^{\frac{\mu}{p-1} u_0} - 1 \right) \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

is then a weak solution of

$$\begin{cases} -\Delta_p v_0 = \left(1 + \frac{\mu}{p-1} v_0\right)^{p-1} h(x) \leq 0, & \text{in } \Omega, \\ v_0 = 0, & \text{on } \partial\Omega. \end{cases}$$

As moreover, $\left(1 + \frac{\mu}{p-1} v_0\right)^{p-1} h(x) \in L^\infty(\Omega)$, it follows from [21, 35] that $v_0 \in C_0^{1,\tau}(\overline{\Omega})$, for some $\tau \in (0, 1)$ and, by the strong maximum principle (see [40]), that $v_0 \ll 0$. Now, we split the rest of the proof in three steps.

Step 1: 0 is a strict upper solution of (Q_λ) .

Observe that 0 is an upper solution of (Q_λ) . In order to prove that 0 is strict, let $v \leq 0$ be a solution of (Q_λ) . As $g \leq 0$ on \mathbb{R}^- (see Lemma 5.1), it follows that v is a lower solution of (Q_0) and so, thanks to the comparison principle, see Corollary 3.2, $v \leq v_0 \ll 0$. Hence, 0 is a strict upper solution of (Q_λ) .

Step 2: (Q_λ) has a strict lower solution $\underline{\alpha} \ll 0$.

By construction $\underline{\alpha} = \alpha_\lambda - 1$ is a lower solution of (Q_λ) . Moreover, as every solution v of (Q_λ) satisfies $v \geq \alpha_\lambda \gg \underline{\alpha}$, we conclude that $\underline{\alpha}$ is a strict lower solution of (Q_λ) .

Step 3: *Conclusion.*

By Corollary 2.6, Proposition 2.7, and Lemma 5.2, we have the existence of $v \in W_0^{1,p}(\Omega) \cap C_0^{1,\tau}(\overline{\Omega})$, local minimum of I_λ and solution of (Q_λ) such that $\alpha_\lambda \leq v \ll 0$ as desired. \square

Proof of the first part of Theorem 1.5. By Proposition 9.1, there exists a first critical point, which is a local minimum of I_λ . By Theorem 2.9 and since the Cerami condition holds, we have two options. If we are in the first case, then together with Lemma 8.3, we see that I_λ has the mountain-pass geometry and by Theorem 2.8, we have the existence of a second solution. In the second case, we have directly the existence of a second solution of (Q_λ) . Then by Lemma 5.2 we conclude to the existence of two solutions to (P_λ) . \square

Now, we consider the case $h \geq 0$.

Lemma 9.2. *Assume that (A_2) holds and suppose that $h \geq 0$. Recall that γ_1 denotes the first eigenvalue of (1.2). It follows that:*

- i) For any $0 \leq \lambda < \gamma_1$, any solution u of the problem (P_λ) satisfies $u \gg 0$.*
- ii) For $\lambda = \gamma_1$, the problem (P_λ) has no solution.*
- iii) For $\lambda > \gamma_1$, the problem (P_λ) has no non-negative solution.*

Proof. Observe first that, taking u^- as test function in (P_λ) , we obtain

$$(9.1) \quad - \int_{\Omega} (|\nabla u^-|^p - \lambda c(x) |u^-|^p) dx = \int_{\Omega} (\mu |\nabla u|^p u^- + h(x) u^-) dx.$$

i) For $\lambda < \gamma_1$, there exists $\varepsilon > 0$ such that, for every $u \in W_0^{1,p}(\Omega)$,

$$\int_{\Omega} (|\nabla u|^p - \lambda c(x) |u|^p) dx \geq \varepsilon \|u\|^p.$$

Consequently, as $h \geq 0$ and $\mu > 0$, we have that

$$0 \geq -\varepsilon \|u^-\|^p \geq - \int_{\Omega} (|\nabla u^-|^p - \lambda c(x) |u^-|^p) dx = \int_{\Omega} (\mu |\nabla u|^p u^- + h(x) u^-) dx \geq 0,$$

which implies that $u^- = 0$ and so that $u \geq 0$. Hence $-\Delta_p u \geq 0$ and by the strong maximum principle (see [40]), we have $u \gg 0$.

ii) In case $\lambda = \gamma_1$ we have, for every $u \in W_0^{1,p}(\Omega)$,

$$(9.2) \quad \int_{\Omega} (|\nabla u|^p - \gamma_1 c(x) |u|^p) dx \geq 0.$$

Assume by contradiction that (P_λ) has a solution u . By (9.1) and (9.2), and using that $h \geq 0$ and $\mu > 0$, we have in particular

$$\int_{\Omega} (|\nabla u^-|^p - \gamma_1 c(x) |u^-|^p) dx = 0.$$

This implies that $u^- = k\varphi_1$ for some $k \in \mathbb{R}$ and φ_1 the first eigenfunction of (1.2) and hence, either $u \equiv 0$ or $u \ll 0$. As $h \not\equiv 0$, the first case cannot occur as 0 is not a solution of (P_λ) . In the second case, as $h \not\leq 0$, we have

$$\int_{\Omega} h(x)u^- dx > 0$$

which contradicts (9.1), (9.2) and $\mu > 0$.

iii) Suppose by contradiction that u is a non-negative solution of (P_λ) . As in the proof of i), we prove $u \gg 0$ and hence, there exists $D_1 > 0$ such that $u \geq D_1 d$ with $d(x) = \text{dist}(x, \partial\Omega)$. Let $\varphi_1 > 0$ be the first eigenfunction of (1.2). As $\varphi_1 \in C^1(\overline{\Omega})$, we have $D_2 > 0$ such that $\varphi_1 \leq D_2 d$. This implies that $\frac{\varphi_1}{u} \in L^\infty(\Omega)$ and $\frac{\varphi_1^p}{u^{p-1}} \in W_0^{1,p}(\Omega)$ with

$$\nabla\left(\frac{\varphi_1^p}{u^{p-1}}\right) = p\left(\frac{\varphi_1}{u}\right)^{p-1}\nabla\varphi_1 - (p-1)\left(\frac{\varphi_1}{u}\right)^p\nabla u.$$

Hence we can take $\frac{\varphi_1^p}{u^{p-1}}$ as test function in (P_λ) and we have that

$$\lambda \int_{\Omega} c(x)\varphi_1^p dx + \int_{\Omega} [\mu|\nabla u|^p + h(x)]\frac{\varphi_1^p}{u^{p-1}} dx = \int_{\Omega} \nabla\left(\frac{\varphi_1^p}{u^{p-1}}\right)|\nabla u|^{p-2}\nabla u dx.$$

On the other hand, applying Proposition 2.4, we obtain

$$\gamma_1 \int_{\Omega} c(x)\varphi_1^p dx = \int_{\Omega} |\nabla\varphi_1|^p dx \geq \int_{\Omega} \nabla\left(\frac{\varphi_1^p}{u^{p-1}}\right)|\nabla u|^{p-2}\nabla u dx.$$

Consequently, gathering together both inequalities, we have the contradiction

$$(9.3) \quad 0 \geq (\gamma_1 - \lambda) \int_{\Omega} c(x)\varphi_1^p dx \geq \int_{\Omega} [\mu|\nabla u|^p + h(x)]\frac{\varphi_1^p}{u^{p-1}} dx > 0.$$

□

Corollary 9.3. *Assume that (A_2) holds. If, for some $\lambda > 0$, (P_λ) has a solution $u_\lambda \geq 0$ then (P_0) has a solution.*

Proof. Observe that u_λ is an upper solution of (P_0) . By Proposition 4.2, we know that (P_0) has a lower solution α with $\alpha \leq u_\lambda$. The conclusion follows from Theorem 2.1. □

Corollary 9.4. *Assume that (A_2) holds with $h \not\leq 0$. If (P_λ) has a solution for some $\lambda \in (0, \gamma_1)$, then (P_0) has a solution.*

Proof. If (P_λ) has a solution u , by Lemma 9.2, we have $u \gg 0$. The result follows from Corollary 9.3. □

Proposition 9.5. *Assume that (P_0) has a solution $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and suppose that (A_2) holds with $h \not\leq 0$. Then there exists $\bar{\lambda} < \gamma_1$ such that:*

- i) *For every $0 < \lambda < \bar{\lambda}$, there exists $v \in C_0^{1,\tau}(\overline{\Omega})$, for some $0 < \tau < 1$, with $v \gg 0$, which is a local minimum of I_λ in the $W_0^{1,p}$ -topology and a solution of (Q_λ) .*
- ii) *For $\lambda = \bar{\lambda}$, there exists $u \in C_0^{1,\tau}(\overline{\Omega})$, for some $0 < \tau < 1$, with $u \geq u_0$, which is a solution of (P_λ) .*
- iii) *For $\lambda > \bar{\lambda}$, the problem (P_λ) has no non-negative solution.*

Proof. Defining

$$\bar{\lambda} = \sup\{\lambda : (P_\lambda) \text{ has a non-negative solution } u_\lambda\},$$

we directly obtain that, for $\lambda > \bar{\lambda}$, the problem (P_λ) has no non-negative solution and, by Lemma 9.2 ii), we see that $\bar{\lambda} \leq \gamma_1$. Moreover, arguing exactly as in the first part of Proposition 9.1, we deduce that

$$v_0 = \frac{p-1}{\mu} (e^{\frac{\mu}{p-1}u_0} - 1) \in W_0^{1,p}(\Omega) \cap C_0^1(\overline{\Omega})$$

satisfies $v_0 \gg 0$. Now, fix $\lambda \in (0, \bar{\lambda})$.

Step 1: *0 is a strict lower solution of (Q_λ) .*

The proof of this step follows the corresponding one of Proposition 9.1.

Step 2: (Q_λ) has a strict upper solution.

By the definition of $\bar{\lambda}$ we can find $\delta \in (\lambda, \bar{\lambda})$ and a non-negative solution u_δ of (P_δ) . As above, we easily see that

$$v_\delta = \frac{p-1}{\mu} (e^{\frac{\mu}{p-1} u_\delta} - 1) \in W_0^{1,p}(\Omega) \cap C_0^1(\bar{\Omega})$$

is a non-negative upper solution of (Q_λ) and $v_\delta \gg 0$. Moreover, if v is a solution of (Q_λ) with $v \leq v_\delta$, Theorem 2.2 implies that $v \ll v_\delta$. Hence, v_δ is a strict upper solution of (P_λ) .

Step 3: *Proof of i).*

The conclusion follows as in Proposition 9.1.

Step 4: *Existence of a solution for $\lambda = \bar{\lambda}$.*

Let $\{\lambda_n\}$ be a sequence with $\lambda_n < \bar{\lambda}$ and $\lambda_n \rightarrow \bar{\lambda}$ and $\{v_n\}$ be the corresponding sequence of minimum of I_{λ_n} obtained in i). This implies that $\langle I'_{\lambda_n}(v_n), \varphi \rangle = 0$ for all $\varphi \in W_0^{1,p}(\Omega)$. By the above construction, we also have

$$I_{\lambda_n}(v_n) \leq I_{\lambda_n}(0) = -\frac{p-1}{p\mu} \int_{\Omega} h(x) dx.$$

Arguing exactly as in Lemmata 8.1 and 5.3, we prove easily the existence of $v \in W_0^{1,p}(\Omega)$ such that $v_n \rightarrow v$ in $W_0^{1,p}(\Omega)$ with v a solution of (Q_λ) for $\lambda = \bar{\lambda}$. As $v_n \geq 0$ we obtain also $v \geq 0$, and, by Lemma 5.2, we have the existence of a solution u of (P_λ) with $u \geq 0$. As u is then an upper solution of (P_0) , we conclude that $u \geq u_0$.

Step 5: $\bar{\lambda} < \gamma_1$.

As by Lemma 9.2, the problem (P_λ) has no solution for $\lambda = \gamma_1$, this follows from Step 4. \square

Proof of the second part of Theorem 1.5. By Lemma 9.2, we have $u_0 \gg 0$. Let us consider $\bar{\lambda} \in (0, \gamma_1)$ given by Proposition 9.5. Hence, for $\lambda < \bar{\lambda}$, there exists a first critical point u_1 , which is a local minimum of I_λ . We then argue as in the proof of the first part to obtain the second solution u_2 of (P_λ) . By Lemma 9.2, these two solutions satisfy $u_i \gg 0$ and, by Theorem 3.1, we conclude that $u_i \geq u_0$. Now, for $\lambda = \bar{\lambda}$, respectively $\lambda > \bar{\lambda}$, the result follows respectively from Proposition 9.5 ii) and iii). \square

Proof of Theorem 1.6. Part 1: Case $\lambda \in (0, \gamma_1)$.

Step 1: *There exists $k > 0$ such that $(P_{\lambda,k})$ has at least one solution.*

Let $\lambda_0 \in (\lambda, \gamma_1)$ and δ small enough such that

$$\lambda_0 s^{p-1} \geq \lambda \left(\frac{p-1}{\mu} \left(1 + \frac{\mu}{p-1} s \right) \ln \left(1 + \frac{\mu}{p-1} s \right) \right)^{p-1}, \quad \forall s \in [0, \delta].$$

Define w as a solution of

$$(9.4) \quad -\Delta_p w = \lambda_0 c(x) |w|^{p-2} w + h(x), \quad w \in W_0^{1,p}(\Omega).$$

As $\lambda_0 < \gamma_1$, we have $w \gg 0$.

For l small enough, $\tilde{\beta} = lw$ satisfies $0 \leq \tilde{\beta} \leq \delta$ and, for k such that $l^{p-1} \geq \left(1 + \frac{\mu}{p-1} \delta \right)^{p-1} k$, it is easy to prove that $\beta = \frac{p-1}{\mu} \ln \left(1 + \frac{\mu}{p-1} \tilde{\beta} \right)$ is an upper solution of $(P_{\lambda,k})$ with $\beta \geq 0$. As 0 is a lower solution of $(P_{\lambda,k})$, the claim follows from Theorem 2.1.

Step 2: *For $k \geq k_0$, the problem $(P_{\lambda,k})$ has no solution.*

Let u be a solution of $(P_{\lambda,k})$. By Lemma 9.2, we have $u \gg 0$. This implies that u is an upper solution of $(P_{0,k})$. As 0 is a lower solution of $(P_{0,k})$, by Theorem 2.1, the problem $(P_{0,k})$ has a solution and hence, by Proposition 7.1, $m_p > 0$ which means that $k < k_0$. This implies that, for $k \geq k_0$, the problem $(P_{\lambda,k})$ has no solution.

Step 3: $\bar{k} = \sup\{k \in (0, k_0) : (P_{\lambda, k}) \text{ has at least one solution}\} < k_0$.

Assume by contradiction that $\bar{k} = k_0$. Let $\{k_n\}$ be an increasing sequence such that $k_n \rightarrow \bar{k}$, $k_n \geq \frac{1}{2}\bar{k}$ and there exists $\{u_n\}$ a sequence of solutions of (P_{λ, k_n}) . As in the previous step we have that u_n is an upper solution of $(P_{0, \frac{1}{2}\bar{k}})$. By Theorem 3.1, we know that $u_n \geq u_0$ with $u_0 \gg 0$ the solution of $(P_{0, \frac{1}{2}\bar{k}})$. Now, let $\phi \in W_0^{1,p}(\Omega) \cap C_0^1(\bar{\Omega})$ with $\phi \gg 0$ and

$$\left(\frac{p-1}{\mu}\right)^{p-1} \int_{\Omega} |\nabla \phi|^p dx = k_0 \int_{\Omega} h(x) \phi^p dx.$$

Using ϕ^p as test function and applying Young inequality as in the proof of Proposition 7.1, it follows that

$$\begin{aligned} \left(\frac{p-1}{\mu}\right)^{p-1} \int_{\Omega} |\nabla \phi|^p dx &\geq \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla (|\phi|^p) dx - \mu \int_{\Omega} |\phi|^p |\nabla u_n|^p dx \\ &= \lambda \int_{\Omega} c(x) |u_n|^{p-2} u_n \phi^p dx + k_n \int_{\Omega} h(x) \phi^p dx \\ &\geq \lambda \int_{\Omega} c(x) |u_0|^{p-2} u_0 \phi^p dx + k_n \int_{\Omega} h(x) \phi^p dx. \end{aligned}$$

Passing to the limit, we have the contradiction

$$\left(\frac{p-1}{\mu}\right)^{p-1} \int_{\Omega} |\nabla \phi|^p dx \geq \lambda \int_{\Omega} c(x) |u_0|^{p-2} u_0 \phi^p dx + \left(\frac{p-1}{\mu}\right)^{p-1} \int_{\Omega} |\nabla \phi|^p dx.$$

Step 4: For $k > \bar{k}$, the problem $(P_{\lambda, k})$ has no solution and for $k < \bar{k}$, the problem $(P_{\lambda, k})$ has at least two solutions u_1, u_2 with $u_i \gg 0$.

The first statement is obvious by definition of \bar{k} . Now, for $k < \bar{k}$, let $\tilde{k} \in (k, \bar{k})$ such that $(P_{\lambda, \tilde{k}})$ has a solution \tilde{u} . By Lemma 9.2, we have $\tilde{u} \gg 0$. Then, it is easy to observe that $\beta_1 = \left(\frac{k}{\tilde{k}}\right)^{\frac{1}{p-1}} \tilde{u}$ and $\beta_2 = \tilde{u}$ are both upper solutions of $(P_{\lambda, k})$ with $0 \ll \beta_1 \ll \beta_2$.

Observe that 0 is a strict lower solution of $(P_{\lambda, k})$. As $\beta_1 \gg 0$ is an upper solution of $(P_{\lambda, k})$, by Theorem 2.1, the problem $(P_{\lambda, k})$ has a minimum solution u_1 with $0 \ll u_1 \leq \beta_1$.

In order to prove the existence of the second solution, observe that if β_2 is not strict, it means that $(P_{\lambda, k})$ has a solution u_2 with $u_2 \leq \beta_2$ but $u_2 \not\ll \beta_2$. Then $u_2 \neq u_1$ and we have our two solutions. If β_2 is strict, we argue as in the proof of Theorem 1.5.

Step 5: The function $\bar{k}(\lambda)$ is non-increasing.

Let us consider $\lambda_1 < \lambda_2$, $\tilde{k} < \bar{k}(\lambda_2)$ and $\tilde{u} \gg 0$ a solution of $(P_{\lambda_2, \tilde{k}})$. It is easy to prove that \tilde{u} is an upper solution of $(P_{\lambda_1, \tilde{k}})$. As 0 is a lower solution of $(P_{\lambda_1, \tilde{k}})$ with $0 \leq \tilde{u}$, by Theorem 2.1, the problem $(P_{\lambda_1, \tilde{k}})$ has a solution. This implies that $\bar{k}(\lambda_1) \geq \bar{k}(\lambda_2)$.

Part 2: Case $\lambda = \gamma_1$.

By Lemma 9.2, we know that the problem (P_{γ_1}) has no solution for $k > 0$. Moreover, by (9.1), we see that if (P_{γ_1}) with $h \equiv 0$ has a non-trivial solution, then $u \geq 0$ and hence, by the strong maximum principle $u \gg 0$. Arguing as in the proof of iii) of Lemma 9.2, we obtain the same contradiction (9.3).

Part 3: Case $\lambda > \gamma_1$.

Step 1: There exists $k > 0$ such that $(P_{\lambda, k})$ has at least one solution $u \ll 0$.

By Proposition 2.3 with $\bar{h} = h$, there exists $\delta_0 > 0$ such that, for $\lambda \in (\gamma_1, \gamma_1 + \delta_0)$, the solution of

$$(9.5) \quad -\Delta_p w = \lambda c(x) |w|^{p-2} w + h(x), \quad w \in W_0^{1,p}(\Omega),$$

satisfies $w \ll 0$. Let us fix $\lambda_0 \in (\gamma_1, \min(\gamma_1 + \delta_0, \lambda))$ and δ small enough such that

$$\lambda_0 |s|^{p-2} s \geq \lambda \left| \frac{p-1}{\mu} \left(1 + \frac{\mu}{p-1} s\right) \ln \left(1 + \frac{\mu}{p-1} s\right) \right|^{p-2} \frac{p-1}{\mu} \left(1 + \frac{\mu}{p-1} s\right) \ln \left(1 + \frac{\mu}{p-1} s\right), \quad \forall s \in [-\delta, 0].$$

Define w as a solution of

$$(9.6) \quad -\Delta_p w = \lambda_0 c(x) |w|^{p-2} w + h(x), \quad u \in W_0^{1,p}(\Omega).$$

As $\gamma_1 < \lambda_0 < \gamma_1 + \delta_0$, we have $w \ll 0$.

For l small enough, $\tilde{\beta} = lw$ satisfies $-\min(\delta, \frac{p-1}{\mu}) < \tilde{\beta} \leq 0$ and, for $k \leq l^{p-1}$, it is easy to prove that $\beta = \frac{p-1}{\mu} \ln(1 + \frac{\mu}{p-1} \tilde{\beta})$ is an upper solution of $(P_{\lambda,k})$ with $\beta \ll 0$. By Proposition 4.2, $(P_{\lambda,k})$ has a lower solution α with $\alpha \leq \beta$ and the claim follows from Theorem 2.1.

Step 2: For k large enough, the problem $(P_{\lambda,k})$ has no solution.

Otherwise, let u be a solution of $(P_{\lambda,k})$. By Lemma 4.1 and Remark 4.1, we have $M_\lambda > 0$ such that, for all $k > 0$, the corresponding solution u satisfies $u \geq -M_\lambda$. Let $\phi \in C_0^1(\bar{\Omega})$ with $\phi \gg 0$. Using ϕ^p as test function, by Young inequality as in the proof of Proposition 7.1, it follows that

$$\begin{aligned} \left(\frac{p-1}{\mu}\right)^{p-1} \int_{\Omega} |\nabla \phi|^p dx &\geq \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla(\phi^p) dx - \mu \int_{\Omega} \phi^p |\nabla u|^p dx \\ &= \lambda \int_{\Omega} c(x) |u|^{p-2} u \phi^p dx + k \int_{\Omega} h(x) \phi^p dx \\ &\geq -\lambda M^{p-1} \int_{\Omega} c(x) \phi^p dx + k \int_{\Omega} h(x) \phi^p dx. \end{aligned}$$

which is a contradiction for k large enough.

Step 3: Define $\tilde{k}_1 = \sup\{k > 0 : (P_{\lambda,k}) \text{ has at least one solution } u \ll 0\}$. For $k < \tilde{k}_1$, the problem $(P_{\lambda,k})$ has at least two solutions with $u_1 \ll 0$ and $\min u_2 < 0$.

For $k < \tilde{k}_1$, let $\tilde{k} \in (k, \tilde{k}_1)$ such that $(P_{\lambda,\tilde{k}})$ has a solution $\tilde{u} \ll 0$. It is then easy to observe that $\beta_1 = \tilde{u}$ and $\beta_2 = \left(\frac{k}{\tilde{k}}\right)^{\frac{1}{p-1}} \tilde{u}$ are both upper solutions of $(P_{\lambda,k})$ with $\beta_1 \ll \beta_2 \ll 0$.

By Proposition 4.2, $(P_{\lambda,k})$ has a lower solution α with $\alpha \leq \beta_1$ and hence, by Theorem 2.1, the problem $(P_{\lambda,k})$ has a minimum solution u_1 with $\alpha \leq u_1 \leq \beta_1$.

In order to prove the existence of the second solution, observe that if β_2 is not strict, it means that $(P_{\lambda,k})$ has a solution u_2 with $u_2 \leq \beta_2$ but $u_2 \not\ll \beta_2$. Then $u_2 \neq u_1$ and we have our two solutions. If β_2 is strict, we argue as in the proof of Theorem 1.5.

Step 4: Define $\tilde{k}_2 = \sup\{k > 0 : (P_{\lambda,k}) \text{ has at least one solution}\}$. For $k > \tilde{k}_2$, the problem $(P_{\lambda,k})$ has no solution and, in case $\tilde{k}_1 < \tilde{k}_2$, for all $k \in (\tilde{k}_1, \tilde{k}_2)$, the problem $(P_{\lambda,k})$ has at least one solution u with $u \not\ll 0$ and $\min u < 0$.

The first statement follows directly from the definition of \tilde{k}_2 . In case $\tilde{k}_1 < \tilde{k}_2$, for $k \in (\tilde{k}_1, \tilde{k}_2)$, let $\tilde{k} \in (k, \tilde{k}_2)$ such that $(P_{\lambda,\tilde{k}})$ has a solution \tilde{u} . Observe that \tilde{u} is an upper solution of $(P_{\lambda,k})$. Again, Proposition 4.2 gives us a lower solution α of $(P_{\lambda,k})$ with $\alpha \leq \tilde{u}$ and hence, by Theorem 2.1, the problem $(P_{\lambda,k})$ has a solution u . By definition of \tilde{k}_1 , we have that $u \not\ll 0$ and by Lemma 9.2, we know that $\min u < 0$.

Step 5: The function $\tilde{k}_1(\lambda)$ is non-decreasing.

Let us consider $\lambda_1 < \lambda_2$, $k < \tilde{k}_1(\lambda_1)$ and $u \ll 0$ a solution of $(P_{\lambda_1,k})$. It is easy to prove that u is an upper solution of $(P_{\lambda_2,k})$. Again, applying Proposition 4.2 and Theorem 2.1, we prove that the problem $(P_{\lambda_2,k})$ has a solution $u \ll 0$. This implies that $\tilde{k}_1(\lambda_1) \leq \tilde{k}_1(\lambda_2)$. \square

APPENDIX A. SUFFICIENT CONDITION

Lemma A.1. Given $f \in L^r(\Omega)$, $r > \max\{N/p, 1\}$ if $p \neq N$ and $1 < r < \infty$ if $p = N$, let us consider

$$E_f(u) = \left(\int_{\Omega} (|\nabla u|^p - f(x)|u|^p) dx \right)^{\frac{1}{p}}$$

for an arbitrary $u \in W_0^{1,p}(\Omega)$. It follows that:

- i) If $1 < p < N$ and $\|f^+\|_{N/p} < S_N$, $E_f(u)$ is an equivalent norm in $W_0^{1,p}(\Omega)$.
- ii) If $p = N$ and $\|f^+\|_r < S_{N,r}$, $E_f(u)$ is an equivalent norm in $W_0^{1,p}(\Omega)$.
- iii) If $p > N$ and $\|f^+\|_1 < S_N$, $E_f(u)$ is an equivalent norm in $W_0^{1,p}(\Omega)$.

where, for $p \neq N$, S_N denotes the optimal constant in the Sobolev inequality, i.e.

$$S_N = \inf \{ \|\nabla u\|_p^p : u \in W_0^{1,p}(\Omega), \|u\|_{p^*} = 1 \},$$

and, for $p = N$,

$$S_{N,r} = \inf \left\{ \|\nabla u\|_p^p : u \in W_0^{1,p}(\Omega), \|u\|_{\frac{Nr}{r-1}} = 1 \right\}.$$

Proof. We give the proof for $1 < p < N$. The other cases can be done in the same way. First of all, by applying Hölder and Sobolev's inequalities, observe that, for any $h \in L^{\frac{N}{p}}(\Omega)$, it follows that

$$\int_{\Omega} h(x)|u|^p dx \leq \|h\|_{\frac{N}{p}} \|u\|_{p^*}^p \leq \frac{1}{S_N} \|h\|_{\frac{N}{p}} \|\nabla u\|_p^p.$$

On the one hand, by using this inequality, observe that

$$\int_{\Omega} \left(|\nabla u|^p - f(x)|u|^p \right) dx \leq \|u\|^p \left(1 + \frac{\|f\|_{\frac{N}{p}}}{S_N} \right).$$

On the other hand, following the same argument, we obtain that

$$\int_{\Omega} \left(|\nabla u|^p - f(x)|u|^p \right) dx \geq \int_{\Omega} \left(|\nabla u|^p - f^+(x)|u|^p \right) dx \geq \|u\|^p \left(1 - \frac{\|f^+\|_{\frac{N}{p}}}{S_N} \right) = A \|u\|^p$$

with $A > 0$ since $\|f^+\|_{\frac{N}{p}} < S_N$. The result follows. \square

As an immediate Corollary, we have a sufficient condition to ensure that $m_p > 0$.

Corollary A.2. *Recall that m_p is defined by (1.1). Under the assumptions (A_1) , it follows that:*

- i) *If $1 < p < N$, then $\|h^+\|_{N/p} < \left(\frac{p-1}{\mu}\right)^{p-1} S_N$ implies $m_p > 0$.*
- ii) *If $p = N$, then $\|h^+\|_q < \left(\frac{p-1}{\mu}\right)^{p-1} S_{N,q}$ implies $m_p > 0$.*
- iii) *If $p > N$, then $\|h^+\|_1 < \left(\frac{p-1}{\mu}\right)^{p-1} S_N$ implies $m_p > 0$.*

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