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DECAY RATE OF THE TIMOSHENKO SYSTEM WITH ONE BOUNDARY DAMPING

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Abstract. In this paper, we study the indirect boundary stabilization of the Timoshenko system with only one dissipation law. This system, which models the dynamics of a beam, is a hyperbolic system with two wave speeds. Assuming that the wave speeds are equal, we prove exponential stability. Otherwise, we show that the decay rate is of exponential or polynomial type. Note that the results hold without the technical assumptions on the coefficients coming from the multiplier method: a sharp analysis of the behaviour of the resolvent operator along the imaginary axis is performed to avoid those artificial restrictions.

1. **Introduction.** In this paper we study the boundary stabilization of the following Timoshenko system :

$$\begin{aligned} u_{tt} - (u_x + y)_x &= 0 && \text{in } (0,1)\times(0,\infty), && (1)\\ y_{tt} - a y_{xx} + b(u_x + y) &= 0 && \text{in } (0,1)\times(0,\infty), && (2)\\ u_x(0,t) = y(0,t) = u(1,t) &= 0 && \text{in } (0,\infty) && (3) \end{aligned}$$

with the following initial conditions

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ y(x,0) = y_0(x),\ y_t(x,0) = y_1(x) \quad \text{in } (0,1) \quad (4)$$

and the boundary dissipation law

$$b^{-1} a y_x(1,t) = -\beta y_t(1,t) \qquad \text{in } (0,\infty), \quad (5)$$

where a, b and β are strictly positive constants.

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In this problem the functions u and y denote, respectively, the transverse displacement of the beam and the rotation angle of the filament.

Denote by ρ, I_ρ, EI, κ, $\omega(x,t)$ and $\varphi(x,t)$, the mass density, the moment of mass inertia, the rigidity coefficient, the shear modulus of the elastic beam, the lateral displacement at location x and time t and the bending angle at location x and time t respectively. Then (1)-(5) coincides with the systems in [11], [12], [15], [33]... with $u(x,t) = \omega\left(x, \sqrt{\frac{\kappa}{\rho}}t\right)$, $y(x,t) = -\varphi\left(x, \sqrt{\frac{\kappa}{\rho}}t\right)$, $a = \frac{(EI)\rho}{\kappa I_\rho}$, $b = \frac{\rho}{I_\rho}$. The same effort to explain the dimensionless expression of the problem is done in [35] with slightly different notation.

Let (u, y) be a regular solution of system (1)-(5). Its associated total energy is defined by

$$E(t) = \frac{1}{2}\int_0^1 (|u_t|^2 + b^{-1}|y_t|^2 + b^{-1}a|y_x|^2 + |u_x + y|^2)dx. \tag{6}$$

Then a classical computation using parts integration gives:

$$\frac{d}{dt}E(t) = -\beta|y_t(1,t)|^2 \leq 0. \tag{7}$$

Hence system (1)-(5) is dissipative in the sense that its associated energy is non increasing with respect to time.

Let us now mention some known results related to the boundary stabilization of the Timoshenko beam. Kim and Renardy proved the exponential stability of the system under two boundary controls in [17]. In [5], Ammar-Khodja and his coauthors studied the decay rate of the energy of the nonuniform Timoshenko beam with two boundary controls acting in the rotation-angle equation. Under the equal speed wave propagation condition (that is to say, if $a = 1$, in our modelization), they established exponential decay results up to an unknown finite dimensional space of initial data. In addition, they showed that the equal speed wave propagation condition is necessary for the exponential stability.
In [8], the system (1)-(5) was studied but with the boundary condition $u(0,t) = 0$ instead of $u_x(0,t) = 0$. Under the equal speed condition ($a = 1$) and if b is outside a discrete set of exceptional values, a polynomial energy decay rate is obtained using a spectral analysis. On the other hand, if $\sqrt{a}$ is a rational number and if b is outside another discrete set of exceptional values, a polynomial type decay rate is proved to hold using a frequency domain approach.
The main goal of this work is to obtain the energy decay rate if $\sqrt{a}$ is a rational number but without assumptions on b (except (C_1) and/or (C_2), introduced in Theorem 2.2) and with the boundary condition $u_x(0,t) = 0$ instead of $u(0,t) = 0$. The result is proved by means of suitable estimates of the resolvent operator norm along the imaginary axis. The technique we choose involves the Laplace transform which is an innovative technical choice to our knowledge.

If $\sqrt{a}$ is not a rational number, the obtention of the energy decay rate is reduced to a non trivial number theory problem and we conjecture that the decay rate of the energy in that case is very small. The problem is still open.

2. **Well-posedness and strong stability.** In this section we study the existence, uniqueness and strong stability of the solution of system (1)-(5). Since the studied problem is similar to that seen in [8], for shortness we only give the results. Let us set

$\Omega = (0,1),\ H_R^1(\Omega) = \{u \in H^1(\Omega) : u(1) = 0\}$ and $H_L^1(\Omega) = \{y \in H^1(\Omega) : y(0) = 0\}$.

Define the energy space $\mathcal{H}$ as follows

$$\mathcal{H} = H_R^1(\Omega) \times L^2(\Omega) \times H_L^1(\Omega) \times L^2(\Omega), \tag{8}$$

with the inner product defined by

$$(U, U_1)_{\mathcal{H}} = \int_0^1 (v\overline{v_1} + b^{-1} z\overline{z_1} + ab^{-1} y_x \overline{y_{1x}} + (u_x + y)(\overline{u_{1x} + y_1}))dx, \tag{9}$$

for all $U = (u, v, y, z)$, $U_1 = (u_1, v_1, y_1, z_1) \in \mathcal{H}$.

Here again a and b are strictly positive constants (as in the introduction).

Remark 1. The norm $(U,U)_{\mathcal{H}}^{\frac{1}{2}}$ induced by (9) is equivalent to the usual norm of $\mathcal{H}$.

For shortness we denote by $\|\,.\,\|$ the $L^2(\Omega)$-norm.
Now, we define a linear unbounded operator $\mathcal{A} : D(\mathcal{A}) \to \mathcal{H}$ by:

$$D(\mathcal{A}) = \{U \in \mathcal{H} : u, y \in H^2(\Omega), v \in H_R^1(\Omega), z \in H_L^1(\Omega),\ (b^{-1}a)y_x(1) = \beta z(1),\ u_x(0)\}, \tag{10}$$

$$\mathcal{A}(u, v, y, z) = \big(v, (u_x + y)_x, z, ay_{xx} - b(u_x + y)\big), \quad \forall\, U = (u, v, y, z) \in D(\mathcal{A}). \tag{11}$$

Then we rewrite formally System (1)-(5) into the evolution equation

$$\begin{cases} U_t = \mathcal{A}U, \\ U(0) = U_0, \quad U_0 \in \mathcal{H}, \end{cases} \tag{12}$$

with $U = (u, u_t, y, y_t)$.

Proposition 1. *The operator $\mathcal{A}$ is m-dissipative in the energy space $\mathcal{H}$.*

Remark 2. Note that the dissipativeness holds since we can check using integrations by parts:

$$\Re(\mathcal{A}U, U)_{\mathcal{H}} = -\beta|z(1)|^2 \le 0, \forall\, U = (u, v, y, z) \in \mathcal{D}(\mathcal{A}). \tag{13}$$

Using Lumer-Phillips Theorem (see [26], Theorem 1.4.3), the operator $\mathcal{A}$ generates a C_0-semigroup of contractions $e^{t\mathcal{A}}$ on $\mathcal{H}$. Then, we have the following results.

Theorem 2.1. *(Existence and uniqueness)*
(1) If $U_0 \in \mathcal{D}(\mathcal{A})$, then system (12) has a unique strong solution

$$U \in C^0(\mathbb{R}_+, \mathcal{D}(\mathcal{A}) \cap C^1(\mathbb{R}_+, \mathcal{H}).$$

(2) If $U_0 \in \mathcal{H}$, then system (12) has a unique weak solution

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

Now, we have the following general strong stability result.

Theorem 2.2. *(Strong stability)*
Let (C_1) and (C_2) be the following conditions:

$$\begin{aligned} &b \neq \frac{(ak_1^2 - k_2^2)(k_1^2 - ak_2^2)}{(a+1)(k_1^2 + k_2^2)}\pi^2, \forall k_1, k_2 \in \frac{1}{2} + \mathbb{N}, k_2 < k_1, && (C_1) \\ &b \neq k^2\pi^2, \forall k \in \mathbb{N}^*. && (C_2) \end{aligned}$$

If $a \neq 1$, System (1)-(5) is strongly stable if and only if the coefficient b satisfies (C_1).

If $a = 1$, System (1)-(5) is strongly stable if and only if the coefficient b satisfies (C_1) and (C_2).

Proof. This result is analogous to Theorem 2.4 of [8]. The proof has to be adapted but since there is no new difficulty in the calculations, we do not give the details here. □

Remark 3. If the coefficient b does not satisfy the required condition(s), the operator $\mathcal{A}$ has a finite number of purely imaginary eigenvalues with explicit eigenvectors. In that case we can show the strong and polynomial or exponential stability in the space orthogonal to these eigenvectors that is invariant under the action of $\mathcal{A}$.

3. **Explicit expression for the resolvent.** In this section we give an explicit expression of the resolvent $(\lambda I - \mathcal{A})^{-1}$ and prove some useful estimates. In fact such estimates are useful since later on, we will use a result of [10] (Theorem 2.4) which involves the norm operator of $(\lambda I - \mathcal{A})^{-1}$ with $\lambda = i\mu,\ \mu \in \mathbb{R}$.
Let $U_1 = (u_1, v_1, y_1, z_1) \in \mathcal{H}$, we look for a solution $U \in \mathcal{D}(\mathcal{A})$ of

$$(i\mu - \mathcal{A})U = U_1,\ \mu \in \mathbb{R}. \tag{14}$$

Note that if the conditions (C_1) and/or (C_2) of Theorem 2.2 hold, Problem (14) admits an unique solution for all $\mu \in \mathbb{R}$. If it is not the case, the existence and uniqueness remain true for large enough values of $|\mu|$ (see Remark 3).

The explicit expression for the resolvent we give in next Proposition 2 involves the restriction on $[0;1]$ of the classical convolution product of two functions on $\mathbb{R}$. Let us recall the definition and establish useful properties.

Lemma 3.1. *(A technical lemma)*
Let $\psi \in C^\infty([0,\infty[)$ and f in $L^2(0;1)$ be two functions and define their convolution product $\psi \star f$ on $[0,1]$ by :

$$(\psi \star f)(x) = \int_0^x \psi(x-s)f(s)ds, \forall x \in [0;1]. \tag{15}$$

Then the following two properties hold:

1. *$(\psi \star f) \in H^1(0;1)$ and its derivative is:*
$$(\psi \star f)'(x) = \int_0^x \psi'(x-s)f(s)ds + \psi(0)f(x), \forall x \in [0;1]. \tag{16}$$
2. *If $\psi(0) = 0$ is also assumed, then $(\psi \star f) \in H^2(0;1)$ and its second derivative is:*

$$(\psi \star f)''(x) = \int_0^x \psi''(x-s)f(s)ds + \psi'(0)f(x), \forall x \in [0;1]. \tag{17}$$

Proof. 1. The functions ψ extended by 0 on $(-\infty, 0)$ and f extended by 0 on $\mathbb{R}$ outside $[0,1]$ are still called ψ and f respectively. Then the convolution product defined by (15) is extended by the classical convolution product on $\mathbb{R}$ i.e by

$$(\psi \star f)(x) = \int_R \psi(x-s) f(s) ds, \forall x \in \mathbb{R}. \tag{18}$$

It is well known that $(\psi \star f)' = (\psi')_{\text{dist}} \star f$ where $(\psi')_{\text{dist}}$ is the derivative of ψ in the distributional sense. Due to the property of ψ and its extension on $\mathbb{R}$ we have

$$(\psi')_{\text{dist}} = \psi' + \psi(0)\delta_0,$$

where δ_0 is the Dirac distribution at $x = 0$. The property (16) follows from this remark.

2. (17) is a consequence of (16). □

Note that weaker assumptions could be made on ψ for this lemma ($\psi \in C^2([0,1])$ is sufficient) but it will be applied to the functions $w_{i,j}$ in the next Proposition and they belong to $C^2(\mathbb{R}) \cap C^\infty([0,\infty[)$.

Proposition 2. *(Explicit expression for the resolvent of the operator)*
Let a, b and β be strictly positive real numbers and μ a real number.
Let the spaces $\mathcal{H}$ and $D(\mathcal{A})$ be defined by (8) and (10).
Denote by $\pm i q_j, j = 1,2$ the four distinct roots of the polynomial Δ defined by

$$\Delta(p) := a p^4 + (a+1)\mu^2 p^2 + \mu^4 - b\mu^2 \tag{19}$$

(q_1 and q_2 are positive real numbers depending on a, b and μ and they are supposed to satisfy $q_2(\mu) < q_1(\mu)$, for $\mu > 0$).
Define the functions ϕ, w_{11}, w_{12}, w_{21}, w_{22} and Φ on $\mathbb{R}$, by:

$$\begin{cases} \phi(x) &= -\dfrac{\sin(q_1 x)}{q_1} + \dfrac{\sin(q_2 x)}{q_2}, x \geq 0 \\ \phi(x) &= 0 \qquad\qquad\qquad\qquad , x < 0 \end{cases} \tag{20}$$

$$\begin{cases} w_{11}(x) &= (\mu^2 - b)\phi(x) + a\phi''(x), \\ w_{12}(x) &= -\phi'(x), \\ w_{21}(x) &= b\,\phi'(x), \\ w_{22}(x) &= \mu^2\phi(x) + \phi''(x), \end{cases} \tag{21}$$

$$\begin{aligned} \Phi(\mu) =& (w_{1,1})_x(1)(b^{-1}a(w_{2,2})_x(1) - \beta i\mu w_{2,2}(1)) \\ & - w_{1,2}(1)(b^{-1}a(w_{2,1})_{xx}(1) - \beta i\mu (w_{2,1})_x(1)). \end{aligned} \tag{22}$$

Then ϕ belongs to $C^2(\mathbb{R}) \cap C^\infty([0;+\infty))$ and

$$\phi(0) = \phi'(0) = \phi''(0) = 0, \phi^{(3)}(0^+) = q_1^2 - q_2^2.$$

Assume that conditions (C_1) and/or (C_2) of Theorem 2.2 hold and that $|\mu|^2 > b$, and let $U_1 = (u_1, v_1, y_1, z_1)$ in $\mathcal{H}$, then the solution $U = (u, v, y, z) \in \mathcal{D}(\mathcal{A})$ of $(i\mu - \mathcal{A})U = U_1$ is given by:

$$u = u_p + u_r;\ v = i\mu u - u_1;\ y = y_p + y_r;\ z = i\mu y - y_1 \tag{23}$$

where

$$(f_1, g_1, h_1) := (-v_1 - i\mu u_1, -z_1 - i\mu y_1, -y_1(1)) \in L^2(\Omega) \times L^2(\Omega) \times \mathbb{C}, \tag{24}$$

$$\left\{ \begin{array}{lcl} u_p(x) & = & \dfrac{1}{a(q_1^2 - q_2^2)} \left[w_{1,1} \star f_1 + w_{1,2} \star g_1 \right] \\ y_p(x) & = & \dfrac{1}{a(q_1^2 - q_2^2)} \left[w_{2,1} \star f_1 + w_{2,2} \star g_1 \right] \end{array} \right. \tag{25}$$

$$\left\{ \begin{array}{lcl} u_r(x) & = & \alpha_1 w_{11}'(x) + \alpha_2 w_{12}(x) \\ y_r(x) & = & \alpha_1 w_{21}'(x) + \alpha_2 w_{22}(x). \end{array} \right. \tag{26}$$

with

$$\gamma_1 = -u_p(1), \quad \gamma_2 = h_1 - \left(b^{-1} a (y_p)_x(1) - \beta i \mu y_p(1)\right), \tag{27}$$

$$\begin{aligned} \alpha_1 =& \gamma_1 \frac{b^{-1} a (w_{22})_x(1) - \beta i \mu w_{22}(1)}{\Phi(\mu)} - \gamma_2 \frac{w_{12}(1)}{\Phi(\mu)}, \\ \alpha_2 =& -\gamma_1 \frac{b^{-1} a (w_{21})_{xx}(1) - \beta i \mu (w_{21})_x(1)}{\Phi(\mu)} + \gamma_2 \frac{w_{11}(1)}{\Phi(\mu)}. \end{aligned} \tag{28}$$

The following lemma will be useful to prove the latter proposition.

Lemma 3.2. *(A property of ϕ)*
The function ϕ defined by (20) satisfies:

$$a\phi^{(4)} + (a+1)\mu^2 \phi'' + \mu^2(\mu^2 - b)\phi = 0. \tag{29}$$

Proof. The successive derivatives of ϕ are, for any $x \geq 0$:

$$\left\{ \begin{array}{l} \phi'(x) = -\cos(q_1 x) + \cos(q_2 x) \\ \phi''(x) = q_1 \sin(q_1 x) - q_2 \sin(q_2 x) \\ \phi^{(3)}(x) = q_1^2 \cos(q_1 x) - q_2^2 \cos(q_2 x) \\ \phi^{(4)}(x) = -q_1^3 \sin(q_1 x) + q_2^3 \sin(q_2 x). \end{array} \right. \tag{30}$$

Then

$$\begin{aligned} a\phi^{(4)} + (a+1)\mu^2 \phi'' + \mu^2(\mu^2 - b)\phi = & -\frac{a q_1^4 - (a+1)\mu^2 q_1^2 + \mu^2(\mu^2 - b)}{q_1} \sin(q_1 x) \\ & + \frac{a q_2^4 - (a+1)\mu^2 q_2^2 + \mu^2(\mu^2 - b)}{q_2} \sin(q_2 x). \end{aligned}$$

Now, by definition, (iq_1) and (iq_2) are both roots of the polynomial Δ given by (19).
Thus $a q_1^4 - (a+1)\mu^2 q_1^2 + \mu^2(\mu^2 - b) = a q_2^4 - (a+1)\mu^2 q_2^2 + \mu^2(\mu^2 - b) = 0.$ □

Proof of Proposition 2: Problem (14) is equivalent to find $U \in \mathcal{D}(\mathcal{A})$ such that

$$\left\{ \begin{array}{l} i\mu u - v = u_1, \\ i\mu v - u_{xx} - y_x = v_1, \\ i\mu y - z = y_1, \\ i\mu z - a y_{xx} + b y + b u_x = z_1, \end{array} \right. \tag{31}$$

Eliminating v and z we have to find $(u, y) \in H^2(\Omega)^2$ satisfying

$$\left\{\begin{array}{ll} (i) & \mu^2 u + u_{xx} + y_x = f_1, \\ (ii) & \mu^2 y + a y_{xx} - b y - b u_x = g_1, \\ (iii) & y(0) = 0, \\ (iv) & u_x(0) = u(1) = 0, \\ (v) & b^{-1} a y_x(1) - \beta i \mu y(1) = h_1, \end{array}\right. \tag{32}$$

where $(f_1, g_1, h_1) = (-v_1 - i\mu u_1, -z_1 - i\mu y_1, -y_1(1)) \in L^2(\Omega) \times L^2(\Omega) \times \mathbb{C}$.

Step 1: First we check that (u_p, y_p) given by (25) satisfies the first two equations of (32).

Using Lemma 3.1 as well as $w_{1,1}(0) = 0$, $w'_{1,1}(0) = a\phi^{(3)}(0^+) = a(q_1^2 - q_2^2)$, $w_{1,2}(0) = 0$ and $w'_{1,2}(0) = 0$, it holds:

$$u_{p,xx}(x) = \frac{1}{a(q_1^2 - q_2^2)} \left[w''_{1,1} \star f_1 + w''_{1,2} \star g_1 \right] + a(q_1^2 - q_2^2) f_1. \tag{33}$$

Likewise, using $w_{2,1}(0) = w_{2,2}(0) = 0$, it holds:

$$y_{p,x}(x) = \frac{1}{a(q_1^2 - q_2^2)} \left[w'_{2,1} \star f_1 + w'_{2,2} \star g_1 \right]. \tag{34}$$

Thus

$$\begin{aligned} \mu^2 u_p + u_{p,xx} + y_{p,x} = & \frac{1}{a(q_1^2 - q_2^2)} \Big[\left(\mu^2 w_{1,1} + w''_{1,1} + w'_{2,1} \right) \star f_1 \\ & + a(q_1^2 - q_2^2) f_1 + \left(\mu^2 w_{1,2} + w''_{1,2} + w'_{2,2} \right) \star g_1 \Big]. \end{aligned} \tag{35}$$

Now, ϕ has been constructed such that

$$a\phi^{(4)} + (a+1)\mu^2 \phi'' + \mu^2(\mu^2 - b)\phi = 0 \quad \text{which implies} \quad \mu^2 w_{1,1} + w''_{1,1} + w'_{2,1} = 0$$

(see Lemma 3.2). That is why $\mu^2 u_p + u_{p,xx} + y_{p,x} = \dfrac{1}{a(q_1^2 - q_2^2)} a(q_1^2 - q_2^2) f_1 = f_1$, since $\mu^2 w_{1,2} + w''_{1,2} + w'_{2,2} = 0$ (simple calculation).

For the same reasons, after some calculations, the following equation is proved to hold: $\mu^2 y_p + a y_{p,xx} - b y_p - b u_{p,x} = g_1$.

Step 2: We check that (u_r, y_r) given by (26) satisfies the following two equations:

$$\left\{\begin{array}{ll} (i) & \mu^2 u + u_{xx} + y_x = 0, \\ (ii) & \mu^2 y + a y_{xx} - b y - b u_x = 0. \end{array}\right. \tag{36}$$

$$\begin{aligned} \mu^2 u_r + u_{r,xx} + y_{r,x} &= \mu^2(\alpha_1 w'_{11}(x) + \alpha_2 w_{12}(x)) + \alpha_1 w_{11}^{(3)}(x) + \alpha_2 w'_{12}(x) \\ &\quad + \alpha_1 w''_{21}(x) + \alpha_2 w'_{22}(x) \\ &= \alpha_1(\mu^2 w'_{11}(x) + w_{11}^{(3)}(x) + w''_{21}(x)) + \alpha_2(\mu^2 w'_{12}(x) \\ &\quad + w_{12}^{(3)}(x) + w''_{22}(x)) \end{aligned} \tag{37}$$

Then $\mu^2 u_r + u_{r,xx} + y_{r,x} = 0$ for the reasons invoked in Step 1 ((3) and $\mu^2 w_{1,2} + w''_{1,2} + w'_{2,2} = 0$). Likewise (u_r, y_r) satisfies the second equation of (36).

From these first two steps, we deduce that (u, y) given by (23) satisfies the first two equations of (32).

Step 3: Let us now check the boundary conditions of (32).

1. At $x = 0$: $y_p(0) = \dfrac{1}{a(q_1^2 - q_2^2)} \left[w_{2,1} \star f_1 + w_{2,2} \star g_1\right](0) = 0$ by definition of the convolution product.
   And $y_r(0) = \alpha_1 w'_{21}(0) + \alpha_2 w_{22}(0) = 0$, since $\phi(0) = \phi''(0) = 0$. Then $y(0) = y_p(0) + y_r(0) = 0$. Now

   $u_{p,x}(0) = \dfrac{1}{a(q_1^2 - q_2^2)} \left[w'_{1,1} \star f_1 + w'_{1,2} \star g_1\right](0) + w_{1,1}(0) f_1(0) + w_{1,2}(0) g_1(0) = 0$ since $w_{1,1}(0) = 0$ and $w_{1,2}(0) = 0$.
   And $u_{r,x}(0) = \alpha_1 w''_{11}(0) + \alpha_2 w'_{12}(0) = 0$ (cf. the proof of Lemma 3.2). Then $u_x(0) = u_{p,x}(0) + u_{r,x}(0) = 0$.
2. At $x = 1$: since γ_1 and γ_2 are defined by (27), the conditions for u and y at $x = 1$ are satisfied if, and only if $(\alpha_1, \alpha_2) \in \mathbb{C}^2$ is solution of

$$\begin{pmatrix} (w_{1,1})_x(1) & w_{1,2}(1) \\ b^{-1}a(w_{2,1})_{xx}(1) - \beta i\mu (w_{2,1})_x(1) & b^{-1}a(w_{2,2})_x(1) - \beta i\mu w_{2,2}(1) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}. \tag{38}$$

Let $M(\mu)$ be the matrix of the previous system and Φ defined

$$\Phi(\mu) = \det(M(\mu)) \tag{39}$$

then for all $\mu \in \mathbb{R}$, $\Phi(\mu) \neq 0$, otherwise $\lambda = i\mu$ would not be in the resolvent of $\mathcal{A}$. By definition, the expression of $\Phi(\mu)$ is given by (22).
Consequently the solution of (38) is:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \frac{1}{\Phi(\mu)} \begin{pmatrix} b^{-1}a(w_{2,2})_x(1) - \beta i\mu w_{2,2}(1) & -w_{1,2}(1) \\ -b^{-1}a(w_{2,1})_{xx}(1) + \beta i\mu (w_{2,1})_x(1) & w_{1,1}(1) \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix}. \tag{40}$$

that is to say α_1 and α_2 are given by (28).

Note that, the existence of γ_1 and γ_2 requires some regularity of u_p and y_p at $x = 1$. In fact, due to Lemma 3.1, u_p and y_p belong to $H^2(0;1)$ since $w_{11}(0) = w_{12}(0) = w_{21}(0) = w_{22}(0) = 0$. □

4. **First estimates for the resolvent operator norm.** This section is devoted to the study of the behaviour of $\Phi(\mu)$ as $\mu \to +\infty$ (see (22) for the definition of $\Phi(\mu)$). More precisely we try to find the best real number l such that for μ large enough

$$\mu^l |\Phi(\mu)| \geq c > 0,$$

where c is a constant independent of μ. Note that $\Phi(\mu)$ is the determinant of the matrix $M(\mu)$ which involves $\phi^{(j)}(1), j = 0, ..., 3$, where ϕ is given by (20) and consequently $\Phi(\mu)$ is oscillating. Hence the basic idea is to isolate the oscillating terms and to get the asymptotic behaviour of their coefficients. By definition, the expression of $\Phi(\mu)$ is:

$$\begin{aligned}\Phi(\mu) =&(w_{1,1})_x(1)(b^{-1}a(w_{2,2})_x(1)-\beta i\mu w_{2,2}(1))\\ &-w_{1,2}(1)(b^{-1}a(w_{2,1})_{xx}(1)+\beta i\mu(w_{2,1})_x(1)).\end{aligned} \tag{41}$$

Using (21) leads to an expression involving the function ϕ introduced in (20):

$$\Re\Phi(\mu) = b^{-1}a\left\{[(\mu^2-b)\phi'(1)+a\phi^{(3)}(1)][\mu^2\phi'(1)+\phi^{(3)}(1)]+b\phi'(1)\phi^{(3)}(1)\right\}, \tag{42}$$

$$\Im\Phi(\mu) = -\beta\mu\left\{[(\mu^2-b)\phi'(1)+a\phi^{(3)}(1)][\mu^2\phi(1)+\phi^{(2)}(1)]+b\phi'(1)\phi^{(2)}(1)\right\}. \tag{43}$$

Our idea is to use the Laplace transform. Indeed a straightforward computation gives:

$$\frac{1}{a(q_1^2-q_2^2)}\mathcal{L}(\phi)(p) = \frac{1}{\Delta(p)} \tag{44}$$

where q_1 and q_2 are defined by (19) in Proposition 2 and $\mathcal{L}(\phi)(p) = \int_0^{+\infty} e^{-px}\,\phi\,(x)\,dx$, for p such that $\Re(p)>0$.

Lemma 4.1. *(Another expression for Φ)*
Recall that q_1 and q_2 are defined by (19) and ϕ by (20). Let R and $\delta(p,\mu)$ be defined by

$$R(\mu) := a(r_1^2-r_2^2),\ \textit{with}\ r_j^2 = \frac{q_j^2}{\mu^2} \tag{45}$$

$$\delta(p,\mu) := (ap^2+1)(p^2+1)-b/\mu^2 = a(p^2+r_1^2)(p^2+r_2^2). \tag{46}$$

Also define

$$\begin{aligned}&J_0(\mu) := \frac{\mu}{R(\mu)}\phi(1);\ J_1(\mu) := \frac{1}{R(\mu)}\phi'(1);\ J_2(\mu) = \frac{1}{\mu R(\mu)}\phi''(1)\\ &\textit{and}\ J_3(\mu) := \frac{1}{\mu^2 R(\mu)}\phi^{(3)}(1).\end{aligned} \tag{47}$$

Then another expression for the function Φ introduced in Proposition 2 (see (22)) is:

$$\Phi(\mu) = \mu^2R^2(\mu)\left[\mu^2(J_1+aJ_3)(b^{-1}a(J_1+J_3)-i\beta(J_0+J_2))-bJ_1(b^{-1}aJ_1-i\beta J_0)\right]. \tag{48}$$

Proof. Calculations. □

Lemma 4.2. *(Decomposition of a rational fraction) For any $n\in\mathbb{N}$, the decomposition of the following rational fraction in $\mathbb{C}$ is, for any $p\in\mathbb{C}-\{\pm i\}$:*

$$\frac{1}{(p^2+1)^{2n+2}} = \frac{1}{(p-i)^{2n+2}(p+i)^{2n+2}} = \sum_{m=1}^{2n+2}\left(\frac{A_{m,n}}{(p-i)^m}+\frac{\overline{A_{m,n}}}{(p+i)^m}\right) \tag{49}$$

where $A_{2n+2,n} := \dfrac{(-1)^{n+1}}{2^{2n+2}}$, $A_{2n+1,n} := \dfrac{(-1)^{n+1}(n+1)i}{2^{2n+2}}$ *and, for any integer* k *in* $[2, 2n+1]$*:*

$$A_{2n+2-k,n} := \frac{(-1)^{n+1+k}\begin{pmatrix} 2n+1+k \\ k \end{pmatrix}}{(i^k)2^{2n+2+k}}. \tag{50}$$

Moreover, for any $n \in \mathbb{N}$ *and any integer* m*, such that* $1 \le m \le n$*:* $|A_{m,n}| \le 1$.

Proof. These results are obtained by classical methods for the decomposition of rational fractions. The expression for $A_{2n+2-k,n}$ is calculated by evaluating at $p = i$:

$$\frac{1}{k!}\partial_p^k\left[(p-i)^{2n+2}\frac{1}{(p^2+1)^{2n+2}}\right]. \quad \square \tag{51}$$

Lemma 4.3. *(Asymptotic behaviour of the functions* $J_k, k = 0;1;2;3$ *for large values of* μ*, if* $a = 1$*)*
Assume that $a = 1$*. Let the functions* $J_k, k = 0;1;2;3$ *be defined by* (47) *with* $\delta(p,\mu) := (p^2+1)^2 - b/\mu^2$*. The two expressions* s_1 *and* s_2 *are defined as follows:*

$$s_1 := -\frac{1}{\sqrt{b}}\sin\left(\frac{\sqrt{b}}{2}\right) \text{ and } s_2 := -\frac{1}{2}\cos\left(\frac{\sqrt{b}}{2}\right) + \frac{1}{8}\sqrt{b}\sin\left(\frac{\sqrt{b}}{2}\right). \tag{52}$$

It holds:

$$J_0(\mu) := s_1\mu\cos(\mu) + s_2\sin(\mu) + O\left(\frac{1}{\mu}\right) \tag{53}$$

$$J_1(\mu) := -s_1\mu\sin(\mu) + (s_1+s_2)\cos(\mu) + O\left(\frac{1}{\mu}\right) \tag{54}$$

$$J_2(\mu) := -s_1\mu\cos(\mu) - (2s_1+s_2)\sin(\mu) + O\left(\frac{1}{\mu}\right) \tag{55}$$

$$J_3(\mu) := s_1\mu\sin(\mu) - (3s_1+s_2)\cos(\mu) + O\left(\frac{1}{\mu}\right) \tag{56}$$

where $O(x)$ *is bounded with* x *as* x *tends to* 0.

Proof. The Laplace transforms of ϕ and of its successive derivatives (see (30)) are well-known:

$$\begin{cases} \mathcal{L}(\phi)(p) = -\dfrac{1}{p^2+q_1^2} + \dfrac{1}{p^2+q_2^2} = \dfrac{q_1^2-q_2^2}{(p^2+q_1^2)(p^2+q_2^2)} = \dfrac{R(\mu)}{\delta(p/\mu,\mu)} \\ \mathcal{L}(\phi')(p) = -\dfrac{p}{p^2+q_1^2} + \dfrac{p}{p^2+q_2^2} = p\dfrac{q_1^2-q_2^2}{(p^2+q_1^2)(p^2+q_2^2)} = p\dfrac{R(\mu)}{\delta(p/\mu,\mu)} \\ \mathcal{L}(\phi'')(p) = p^2\dfrac{R(\mu)}{\delta(p/\mu,\mu)} \\ \mathcal{L}(\phi^{(3)})(p) = p^3\dfrac{R(\mu)}{\delta(p/\mu,\mu)} \end{cases}$$

Recall that $R(\mu)$ and $\delta(p,\mu)$ are defined by (45) and (46).
Since $\delta(p,\mu) := (p^2+1)^2 - b/\mu^2$ $(a = 1)$, it holds, for μ such that $\mu^2 > b$ and p, such that $\Re(p) \ge 2$:

$$\frac{1}{\delta(p,\mu)} = \sum_{n\geq 0} \frac{b^n}{\mu^{2n}} \frac{1}{(p^2+1)^{2n+2}}. \tag{57}$$

Recall that we are looking for the asymptotic behaviour of the functions $J_k, k = 0; 1; 2; 3$ for large values of μ. Thus the assumption $\mu^2 > b$ is not restrictive.

For any $n \in \mathbb{N}$, the decomposition is, for any $p \in \mathbb{C} - \{\pm i\}$:

$$\frac{1}{(p^2+1)^{2n+2}} = \frac{1}{(p-i)^{2n+2}(p+i)^{2n+2}} = \sum_{m=1}^{2n+2} \left(\frac{A_{m,n}}{(p-i)^m} + \frac{\overline{A_{m,n}}}{(p+i)^m} \right) \tag{58}$$

$$\Rightarrow \frac{1}{(p^2+1)^{2n+2}} = \sum_{m=1}^{2n+2} \left(A_{m,n} \mathcal{L}\left[\frac{x^{m-1}}{(m-1)!} e^{ix} \right](p) + \overline{A_{m,n}} \mathcal{L}\left[\frac{x^{m-1}}{(m-1)!} e^{-ix} \right](p) \right) \tag{59}$$

where $A_{m,n}$ is given in Lemma 4.2 for any integer $n \geq 0$ and any integer m such that $1 \leq m \leq 2n+2$.
Now, μ is fixed such that $\mu^2 > b$. Then, for any p such that $\Re(p) \geq 2$:

$$\begin{aligned}
\mathcal{L}(\phi)(p) &= \frac{R(\mu)}{\delta(p/\mu,\mu)} \\
&= \frac{R(\mu)}{\mu} \sum_{n\geq 0} \mathcal{L}\left[\frac{b^n}{\mu^{2n}} \sum_{m=1}^{2n+2} \frac{(\mu x)^{m-1}}{(m-1)!} \left(A_{m,n} e^{i\mu x} + \overline{A_{m,n}} e^{-i\mu x} \right) \right](p) \\
&= \frac{R(\mu)}{\mu} \sum_{n\geq 0} \frac{b^n}{\mu^{2n}} \sum_{m=1}^{2n+2} \int_{x=0}^{\infty} e^{-px} \left(A_{m,n} e^{i\mu x} + \overline{A_{m,n}} e^{-i\mu x} \right) \frac{(\mu x)^{m-1}}{(m-1)!} dx.
\end{aligned}$$

Moreover, for a fixed μ such that $\mu^2 > b$ and p such that $\Re(p) \geq \max\{2; |\mu|+1\}$:

$$\begin{aligned}
&\sum_{n\geq 0} \frac{b^n}{\mu^{2n}} \sum_{m=1}^{2n+2} \int_{x=0}^{\infty} \left| e^{-px} \left(A_{m,n} e^{i\mu x} + \overline{A_{m,n}} e^{-i\mu x} \right) \frac{(\mu x)^{m-1}}{(m-1)!} \right| dx \\
\leq\ & \sum_{n\geq 0} \frac{b^n}{\mu^{2n}} \sum_{m=1}^{2n+2} \int_{x=0}^{\infty} 2e^{-\Re(p)x} \frac{(|\mu| x)^{m-1}}{(m-1)!} dx \\
\leq\ & \sum_{n\geq 0} \frac{b^n}{\mu^{2n}} \int_{x=0}^{\infty} 2e^{-\Re(p)x} \sum_{m=1}^{\infty} \frac{(|\mu| x)^{m-1}}{(m-1)!} dx \\
=\ & \sum_{n\geq 0} \frac{b^n}{\mu^{2n}} \int_{x=0}^{\infty} 2e^{(|\mu|-\Re(p))x} dx \\
\leq\ & 2\sum_{n\geq 0} \frac{b^n}{\mu^{2n}} \\
<\ & +\infty.
\end{aligned} \tag{60}$$

Thus, by Fubini/Tonelli Theorems, the integral and the summation can be interchanged. That is to say, for a fixed μ such that $\mu^2 > b$ and p such that $\Re(p) \geq \max\{2; |\mu|+1\}$:

$$\begin{aligned}\mathcal{L}(\phi)(p) &:= \int_{x=0}^{+\infty} e^{-px}\,\phi\,(x)\,dx \\ &= \int_{x=0}^{\infty} e^{-px}\frac{R(\mu)}{\mu}\sum_{n\geq 0}\frac{b^n}{\mu^{2n}}\sum_{m=1}^{2n+2}\left(A_{m,n}e^{i\mu x}+\overline{A_{m,n}}e^{-i\mu x}\right)\frac{(\mu x)^{m-1}}{(m-1)!}dx.\end{aligned} \tag{61}$$

For any fixed μ such that $\mu^2 > b$, let us define h on $\mathbb{R}^+$ by:

$$\begin{aligned}h(x) :=& \frac{R(\mu)}{\mu}\sum_{n\geq 0}\frac{b^n}{\mu^{2n}}\sum_{m=1}^{2n+2}\left(A_{m,n}e^{i\mu x}+\overline{A_{m,n}}e^{-i\mu x}\right)\frac{(\mu x)^{m-1}}{(m-1)!} \\ =& 2\frac{R(\mu)}{\mu}\sum_{n\geq 0}\left[\sum_{m=1}^{2n+2}\frac{b^n}{\mu^{2n-m+1}}\frac{x^{m-1}}{(m-1)!}\Re\left(A_{m,n}e^{i\mu x}\right)\right].\end{aligned} \tag{62}$$

Lemma 4.2 implies:

$$\left|\sum_{m=1}^{2n+2}\frac{b^n}{\mu^{2n-m+1}}\frac{x^{m-1}}{(m-1)!}\Re\left(A_{m,n}e^{i\mu x}\right)\right| \leq (2n+2)\left(\frac{b}{\mu^2}\right)^n e^x. \tag{63}$$

For any fixed μ such that $\mu^2 > b$, the series $(2n+2)\left(\frac{b}{\mu^2}\right)^n$ converges. Hence the continuity of h on $\mathbb{R}^+$ and in particular at $x = 1$.
Thus, (61) means that ϕ and h are continuous functions on $\mathbb{R}^+$ and that their Laplace transforms coincide for p, such that $\Re(p) \geq \max\{2; |\mu| + 1\}$.
Thus $\phi(x) = h(x)$, for any $x > 0$ and, in particular at $x = 1$.
That is to say, for any μ such that $\mu^2 > b$:

$$J_0(\mu) = 2\sum_{n\geq 0}\left[\sum_{m=1}^{2n+2}\frac{b^n}{\mu^{2n-m+1}}\frac{1}{(m-1)!}\Re\left(A_{m,n}e^{i\mu}\right)\right].$$

Only the last two terms of the sum are interesting for the asymptotic behaviour of J_0 i.e. $m = 2n + 2$ and $m = 2n + 1$:

$$\begin{aligned}J_0(\mu) =& 2\sum_{n\geq 0}\left[\mu\frac{b^n}{(2n+1)!}\Re\left(A_{2n+2,n}e^{i\mu}\right)+\frac{b^n}{(2n)!}\Re\left(A_{2n+1,n}e^{i\mu}\right)\right] \\ &+2\sum_{n\geq 1}\sum_{m=1}^{2n}\left(\frac{b}{\mu^2}\right)^n\frac{\mu^{m-1}}{(m-1)!}\Re\left(A_{m,n}e^{i\mu}\right).\end{aligned} \tag{64}$$

Now the first two terms are calculated (the computations for the sums are left to the reader), leading to:

$$\begin{aligned}&2\sum_{n\geq 0}\left[\mu\frac{b^n}{(2n+1)!}\Re\left(A_{2n+2,n}e^{i\mu}\right)+\frac{b^n}{(2n)!}\Re\left(A_{2n+1,n}e^{i\mu}\right)\right] \\ &= \sum_{n\geq 0}\left(\frac{b^n}{(2n+1)!}\frac{(-1)^{n+1}}{2^{2n+1}}\right)\mu\cos(\mu)+\sum_{n\geq 0}\left(\frac{b^n}{(2n)!}\frac{(-1)^{n+1}}{2^{2n+1}}(n+1)\right)\sin(\mu) \\ &= s_1\mu\cos(\mu)+s_2\sin(\mu).\end{aligned} \tag{65}$$

The rest is: $R(b,\mu) := 2\sum_{n\geq 1}\sum_{m=1}^{2n}\left(\frac{b}{\mu^2}\right)^n \frac{\mu^{m-1}}{(m-1)!}\Re\left(A_{m,n}e^{i\mu}\right)$. Let us prove that $R(b,\mu) = O\left(\frac{1}{\mu}\right)$.
Lemma 4.2 implies:

$$\begin{aligned}
|R(b,\mu)| &\leq 2\sum_{n\geq 1}\sum_{m=1}^{2n}\left(\frac{b}{\mu^2}\right)^n \frac{\mu^{m-1}}{(m-1)!} = \sum_{k\geq 0}\left(\frac{\mu^{2k}}{(2k)!}+\frac{\mu^{2k+1}}{(2k+1)!}\right)\sum_{n\geq k+1}\left(\frac{b}{\mu^2}\right)^n \\
&= \sum_{k\geq 0}\left(\frac{\mu^{2k}}{(2k)!}+\frac{\mu^{2k+1}}{(2k+1)!}\right)\frac{b^{k+1}}{\mu^{2k+2}-b\mu^{2k}} = \frac{1}{\mu^2-b}\sum_{k\geq 0}\frac{b^{k+1}}{(2k)!} \\
&+\frac{\mu}{\mu^2-b}\sum_{k\geq 0}\frac{b^{k+1}}{(2k+1)!}.
\end{aligned} \tag{66}$$

Hence the result for $R(b,\mu)$ and J_0.
Similar computations lead to the asymptotic behaviours (54), (55) and (56). □

Lemma 4.4. *(Asymptotic behaviour of the functions $J_k, k = 0;1;2;3$ for large values of μ, if $a \neq 1$)*
Let the functions $J_k, k = 0;1;2;3$ be defined by (47) with
$\delta(p,\mu) := (ap^2+1)(p^2+1) - b/\mu^2$. Assume that $a \neq 1$. It holds:

$$\begin{aligned}
J_0(\mu) :=& \frac{1}{a-1}\left(\sqrt{a}\sin\left(\frac{\mu}{\sqrt{a}}\right)-\sin(\mu)\right) \\
&-\frac{1}{\mu}\left(\frac{b}{2(a-1)^2}\right)\left(a\cos\left(\frac{\mu}{\sqrt{a}}\right)+\cos(\mu)\right)+\frac{1}{\mu}\epsilon\left(\frac{1}{\mu}\right)
\end{aligned} \tag{67}$$

$$\begin{aligned}
J_1(\mu) :=& \frac{1}{a-1}\left(\cos\left(\frac{\mu}{\sqrt{a}}\right)-\cos(\mu)\right) \\
&+\frac{1}{\mu}\left(\frac{b}{2(a-1)^2}\right)\left(\sqrt{a}\sin\left(\frac{\mu}{\sqrt{a}}\right)+\sin(\mu)\right)+\frac{1}{\mu}\epsilon\left(\frac{1}{\mu}\right)
\end{aligned} \tag{68}$$

$$\begin{aligned}
J_2(\mu) :=& \frac{1}{a-1}\left(-\frac{1}{\sqrt{a}}\sin\left(\frac{\mu}{\sqrt{a}}\right)+\sin(\mu)\right) \\
&+\frac{1}{\mu}\left(\frac{b}{2(a-1)^2}\right)\left(\cos\left(\frac{\mu}{\sqrt{a}}\right)+\cos(\mu)\right)+\frac{1}{\mu}\epsilon\left(\frac{1}{\mu}\right)
\end{aligned} \tag{69}$$

$$\begin{aligned}
J_3(\mu) :=& \frac{1}{a-1}\left(-\frac{1}{a}\cos\left(\frac{\mu}{\sqrt{a}}\right)+\cos(\mu)\right) \\
&-\frac{1}{\mu}\left(\frac{b}{2(a-1)^2}\right)\left(\frac{1}{\sqrt{a}}\sin\left(\frac{\mu}{\sqrt{a}}\right)+\sin(\mu)\right)+\frac{1}{\mu}\epsilon\left(\frac{1}{\mu}\right)
\end{aligned} \tag{70}$$

where $\epsilon(x)$ tends to 0 as x tends to 0.

Proof. Since $\delta(p,\mu) := (ap^2+1)(p^2+1) - b/\mu^2$, it holds, for μ such that $\mu^2 > b$ and p, such that $\Re(p) \geq 2$:

$$\frac{1}{\delta(p,\mu)} = \sum_{n\geq 0} \frac{b^n}{\mu^{2n}} \frac{1}{(ap^2+1)^{n+1}(p^2+1)^{n+1}}. \tag{71}$$

For any $n \in \mathbb{N}$, the decomposition is, for any $p \in \mathbb{C} - \{\pm i; \pm \frac{i}{\sqrt{a}}\}$:

$$\begin{aligned}\frac{1}{(ap^2+1)^{n+1}(p^2+1)^{n+1}} = \sum_{m=1}^{n+1} \Big(&\frac{B_{m,n}}{(p-i)^m} + \frac{\overline{B_{m,n}}}{(p+i)^m} \\ &+ \frac{C_{m,n}}{\left(p-\frac{i}{\sqrt{a}}\right)^m} + \frac{\overline{C_{m,n}}}{\left(p+\frac{i}{\sqrt{a}}\right)^m}\Big).\end{aligned} \tag{72}$$

Using the same arguments as in the proof of Lemma 4.3, it holds:

$$\begin{aligned}J_0(\mu) = \sum_{n\geq 0}\sum_{m=1}^{n+1} \frac{b^n}{(m-1)!\mu^{2n-m+1}} \big[&B_{m,n}e^{i\mu} + \overline{B_{m,n}}e^{-i\mu} \\ &+ C_{m,n}e^{\frac{i}{\sqrt{a}}\mu} + \overline{C_{m,n}}e^{-\frac{i}{\sqrt{a}}\mu}\big].\end{aligned} \tag{73}$$

To get (67)-(70), it is enough to compute $B_{1,0}$, $B_{2,1}$, $C_{1,0}$ and $C_{2,1}$, since the only interesting terms in the sum are those corresponding to $n = 0$, $m = 1$ and $n = 1$, $m = 2$. The other terms are of the form $\frac{1}{\mu}\epsilon\left(\frac{1}{\mu}\right)$.

The first two rational fractions of the sum (71) are decomposed in $\mathbb{R}$ for the sake of simplicity:

$$\frac{1}{(ap^2+1)(p^2+1)} = \frac{1}{a-1}\left(\frac{1}{p^2+1/a} - \frac{1}{p^2+1}\right) \tag{74}$$

$$\begin{aligned}\frac{1}{(ap^2+1)^2(p^2+1)^2} = &\frac{1}{(a-1)^2}\Big(\frac{1}{(p^2+1/a)^2} + \frac{1}{(p^2+1)^2} \\ &- \frac{2a}{a-1}\frac{1}{p^2+1/a} + \frac{2a}{a-1}\frac{1}{p^2+1}\Big).\end{aligned} \tag{75}$$

Now, for $n = 0$, the decomposition in $\mathbb{C}$ is, for any $p \in \mathbb{C} - \{\pm i; \pm \frac{i}{\sqrt{a}}\}$:

$$\begin{aligned}\frac{1}{(ap^2+1)(p^2+1)} &= \frac{B_{1,0}}{p-i} + \frac{\overline{B_{1,0}}}{p+i} + \frac{C_{1,0}}{p-\frac{i}{\sqrt{a}}} + \frac{\overline{C_{1,0}}}{p+\frac{i}{\sqrt{a}}} \\ &= \frac{i(B_{1,0}-\overline{B_{1,0}})}{p^2+1} + \frac{1}{\sqrt{a}}\frac{i(C_{1,0}-\overline{C_{1,0}})}{p^2+1/a}.\end{aligned} \tag{76}$$

Thus, identifying with the decomposition (74), $B_{1,0} = \overline{B_{1,0}}$ and $-2\Im(B_{1,0}) = \dfrac{1}{1-a}$.

Likewise $C_{1,0} = \overline{C_{1,0}}$ and $-2\Im(C_{1,0}) = \dfrac{\sqrt{a}}{a-1}$.

If $n = 1$, the decomposition in $\mathbb{C}$ is, for any $p \in \mathbb{C} - \{\pm i; \pm \frac{i}{\sqrt{a}}\}$:

$$\begin{aligned}\frac{1}{(ap^2+1)^2(p^2+1)^2} &= \frac{B_{2,1}}{(p-i)^2}+\frac{\overline{B_{2,1}}}{(p+i)^2}+\frac{C_{2,1}}{\left(p-\frac{i}{\sqrt{a}}\right)^2}+\frac{\overline{C_{2,1}}}{\left(p+\frac{i}{\sqrt{a}}\right)^2}\\ &+\frac{B_{1,1}}{p-i}+\frac{\overline{B_{1,1}}}{p+i}+\frac{C_{1,1}}{p-\frac{i}{\sqrt{a}}}+\frac{\overline{C_{1,1}}}{p+\frac{i}{\sqrt{a}}}.\end{aligned} \tag{77}$$

After calculations and identification with the decomposition (75):

$$B_{2,1}=\overline{B_{2,1}}=-\frac{1}{4(a-1)^2}.$$

Likewise $C_{2,1}=\overline{C_{2,1}}=-\dfrac{a}{4(a-1)^2}$.

Hence the asymptotic behaviour of J_0. Then, the following functions are derived iteratively to obtain the behaviour of J_1, J_2 and J_3: $\sqrt{a}\sin\left(\frac{\cdot}{\sqrt{a}}\right)-\sin(\cdot)$ and $\dfrac{b}{2(a-1)^2}\left(a\cos\left(\frac{\cdot}{\sqrt{a}}\right)\right)+\cos(\cdot)$. □

Proposition 3. *(Asymptotic behaviour of the characteristic "polynomial" Φ if $a=1$)*

Assume that $a=1$. Then $\exists C>0$, such that $\exists M>0, \forall\mu\geq M$:

$$|\Phi(\mu)|\geq C\mu^2. \tag{78}$$

where Φ is defined by (48).

Proof. Using (48) with $a=1$ leads to:

$$\left\{\begin{array}{l}\Re\Phi(\mu)=\mu^2R^2(\mu)\cdot[b^{-1}\mu^2(J_1+J_3)^2-J_1^2]=4\mu^2(J_1+J_3)^2-4bJ_1^2\\ \Im\Phi(\mu)=\beta\mu^2R^2(\mu)\cdot[-\mu^2(J_1+J_3)(J_0+J_2)+bJ_0J_1]\end{array}\right.$$

where $R^2(\mu)=4b/\mu^2$ (cf. (45)). Then Lemma 4.3 leads to:

$$\frac{\Re\Phi(\mu)}{4\mu^2}=4s_1^2\left(\cos^2(\mu)-\frac{b}{4}\sin^2(\mu)\right)+O\left(\frac{1}{\mu}\right) \tag{79}$$

and

$$\frac{\Im\Phi(\mu)}{4\mu^2}=-b(4+b)\beta s_1^2\cos(\mu)\sin(\mu)+O\left(\frac{1}{\mu}\right). \tag{80}$$

Thus

$$\frac{|\Phi(\mu)|^2}{16\mu^4s_1^4}=16\left(\cos^2(\mu)-\frac{b}{4}\sin^2(\mu)\right)^2+(4+b)^2b^2\beta^2\cos^2(\mu)\sin^2(\mu)+O\left(\frac{1}{\mu}\right). \tag{81}$$

$$\begin{aligned}\frac{|\Phi(\mu)|^2}{16\mu^4s_1^4} &= \left(16\left(1+\frac{b}{4}\right)^2-(4+b)^2b^2\beta^2\right)\cos^4(\mu)\\ &+\left((4+b)^2b^2\beta^2-8b\left(1+\frac{b}{4}\right)\right)\cos^2(\mu)+b^2+O\left(\frac{1}{\mu}\right)\end{aligned} \tag{82}$$

Let us denote by f the function defined on $\mathbb{R}$, by:

$$f(x) := \left(1+\frac{b}{4}\right)^2(1-b^2\beta^2)x^2 + \left(1+\frac{b}{4}\right)\left(b^2\beta^2\left(1+\frac{b}{4}\right)-\frac{b}{2}\right)x + \frac{b^2}{16}. \quad (83)$$

Our aim is to prove that there exists $C > 0$ such that $f(x) \geq C > 0$, for any $x \in [0;1]$.

First notice that $f(0) = \dfrac{b^2}{16} > 0$ and $f(1) = 1 > 0$.
In the following, β is chosen to be equal to 1. If it is not, changing the value of b so that (5) still holds is always possible.

Now we separate the different cases:

- Case $b = 1$: in that case, f is an affine function and f can not take negative values on the interval $[0;1]$, since it is strictly positive at 0 and at 1.
- Case $b > 1$: the coefficient of x^2 is strictly negative. Thus $f(x)$ is larger than the minimum of $f(0)$ and $f(1)$.
- Case $0 < b < 1$: the coefficient of x^2 is strictly positive.
  The sign of the discriminant is also the sign of:

$$\begin{aligned}\frac{\Delta(b)}{\left(1+\frac{b}{4}\right)^2} &:= \left(b^2\left(1+\frac{b}{4}\right)-\frac{b}{2}\right)^2 - (1-b^2)\frac{b^2}{4} = b^4\left(1+\frac{b}{4}\right)^2 - b^3 \\ &= b^3\left[b\left(1+\frac{b}{4}\right)^2 - 1\right].\end{aligned} \quad (84)$$

  Now, the sign of $\Delta(b)$ is that of the function g defined on $[0;1]$ by the expression $g(t) := t^3 + 4t - 4$.

  The only real root of g is $t_0 := \sqrt[3]{2\left(1+\dfrac{\sqrt{40}}{3\sqrt{3}}\right)} + \sqrt[3]{2\left(1-\dfrac{\sqrt{40}}{3\sqrt{3}}\right)} < 1$. Then

$$\Delta(b) < 0, \forall b \in (0;t_0^2),\ \Delta(b) \geq 0, \forall b \in [t_0^2;1). \quad (85)$$

  If $b \in (0;t_0^2)$, the function f has no root in $[0;1]$ and the required property is satisfied since the coefficient of x^2 is strictly positive.
  If $b \in (t_0^2;1)$, the function f has two real roots. The product of these roots is $P := \dfrac{b^2}{(b^2+4)(1-b^2)} > 0$ and their sum is
  $S := \dfrac{1}{(1+b/4)(1-b^2)}\left(\dfrac{b}{2} - b^2\left(1+\dfrac{b}{4}\right)\right)$.
  The function $b \mapsto (b^2+4b-2)$ is non-negative on $(x_0^2;1)$. Thus both roots of f are non-positive and the minimum value of f on $[0;1]$ is $b^2/16$.
  At last, if $b = t_0^2$, the function f has a unique root x_0 which satisfies $2x_0 = S < 0$. Thus the minimum value of f on $[0;1]$ is also $b^2/16$.

To conclude, we have proved that there exists $C > 0$ such that $f(x) \geq C > 0$, for any $x \in [0;1]$. □

Proposition 4. *(Lower bound for the characteristic "polynomial" Φ, $a \neq 1$, $\sqrt{a}$ in $\mathbb{Q}$)*

Assume that $\sqrt{a} \in \mathbb{Q}^+ - \{0;1\}$. *There exists* $(c;d) \in (\mathbb{N}^*)^2$ *such that* c *and* d *are coprime and* $\sqrt{a} = c/d$.

- *Assume that* c *or* d *is even. Then* $\exists C > 0$, *such that* $\exists M > 0, \forall \mu \geq M$:

$$|\Phi(\mu)| \geq C\mu^4. \tag{86}$$

- *Assume that* c *and* d *are both odd numbers.*
  Then $\exists C > 0$, *such that* $\exists M > 0, \forall \mu \geq M$:

$$|\Phi(\mu)| \geq C\mu^3. \tag{87}$$

Proof. Let us denote by Ψ the function defined on $\mathbb{R}$ by:

$$\Psi(\mu) := \frac{\Phi(\mu)}{\mu^4 R^2(\mu)}. \tag{88}$$

- The contrapositive is proved: assume that, there exists a sequence $(\mu_n)_n$ which tends to $+\infty$ with n and which satisfies $\Psi(\mu_n) = o(1)$, that is to say that $\Psi(\mu_n)$ tends to 0 as n tends to $+\infty$.
  Then $b^{-1}a(J_1(\mu_n)+aJ_3(\mu_n))(J_1(\mu_n)+J_3(\mu_n)) = o(1)$ as well (cf. (48)). Now the symmetric matrix A defined by

$$A := \begin{pmatrix} 1 & \dfrac{a+1}{2} \\ \dfrac{a+1}{2} & a \end{pmatrix}$$

is orthogonally diagonalizable. Denoting by $\lambda_2 > 0 > \lambda_1$ the two eigenvalues of A (the determinant of A is $-(1/4)(a-1)^2 < 0$), it holds:

$$\min(|\lambda_1|,|\lambda_2|)\cdot(J_1^2+J_3^2) \leq |(J_1+aJ_3)(J_1+J_3)| \leq \max(|\lambda_1|,|\lambda_2|)\cdot(J_1^2+J_3^2). \tag{89}$$

where J_1 and J_3 depend on μ and (89) holds for all μ in $\mathbb{R}$.
Thus $\Psi(\mu_n) = o(1) \Rightarrow J_1(\mu_n) = o(1);\ J_3(\mu_n) = o(1)$.

Using (68) and (70), $\Psi(\mu_n) = o(1) \Rightarrow \cos(\mu_n) = o(1);\ \cos\left(\frac{\mu_n}{\sqrt{a}}\right) = o(1)$.
There exists $(k_n;l_n)$ in $\mathbb{Z}^2$ such that

$$\mu_n = k_n\pi + \frac{\pi}{2} + o(1) \text{ and } \frac{\mu_n}{\sqrt{a}} = l_n\pi + \frac{\pi}{2} + o(1).$$

Here again, $o(1)$ is a function which tends to 0 when n tends to ∞. Then, writing $\sqrt{a}$ as the irreducible fraction c/d:

$$k_n\pi + \frac{\pi}{2} + o(1) = \frac{c}{d}\left(l_n\pi + \frac{\pi}{2} + o(1)\right) \Rightarrow (2k_n+1)d = (2l_n+1)c + o(1).$$

Now, since $(2k_n+1)d$ and $(2l_n+1)c$ are both integer numbers, $o(1)$ must vanish and there exists $(k;l) \in \mathbb{N}^2$, such that

$$\sqrt{a} = \frac{c}{d} = \frac{2k+1}{2l+1}. \tag{90}$$

Note that there exists an infinity of such couples $(k;l) \in \mathbb{Z}^2$. Indeed, (90) implies

$$k = \frac{c - d + 2cl}{2d}.$$

Since c and d are both supposed to be odd integer numbers, $\exists p \in \mathbb{Z}, c - d = 2p$ thus $k = \dfrac{p + cl}{d}$ which is equivalent to $dk - cl = p$. Bezout's Theorem gives an infinity of solutions to this equation.

The conclusion is: if c or d is even, $\exists K > 0$, such that $\exists L > 0, \forall \mu \geq L$, $|\Psi(\mu)| \geq K$.
Since $r_1^2 := q_1^2/\mu^2 = \dfrac{1}{2a}\left(a + 1 + \dfrac{1}{\mu}\sqrt{(a-1)^2\mu^2 + 4ab}\right)$ and since r_2^2 has the same expression except for the sign before the square root, the asymptotic behaviour of $R(\mu)$ for large values of μ is given by:

$$R(\mu) = a(r_2^2 - r_1^2) = -\frac{1}{\mu}\sqrt{(a-1)^2\mu^2 + 4ab} = -|a - 1| + o(1)$$

Hence the first result with $C = (a-1)^2 K$.

- Assume that c and d are both odd numbers and that there exists a sequence $(\mu_n)_n$ which tends to $+\infty$ with n and which satisfies $\Psi(\mu_n) = o\left(\dfrac{1}{\mu_n}\right)$, that is to say that $\mu_n \Psi(\mu_n)$ tends to 0 as n tends to $+\infty$.
In particular $\Psi(\mu_n) = o(1)$ and that still implies $\cos(\mu_n) = o(1)$ and $\cos\left(\dfrac{\mu_n}{\sqrt{a}}\right) = o(1)$ as in the first part of the proof. Thus the limit of $\sin\left(\dfrac{\mu_n}{\sqrt{a}}\right)$ is ± 1.

Then, using the imaginary part of $\Phi(\mu_n)$ (cf. (48)) and (67) to (70):

$$\begin{aligned}
&\beta(J_1 + aJ_3)(J_0 + J_2) = o\left(\frac{1}{\mu_n}\right) = \\
&\beta\left(\cos(\mu_n) + \frac{b}{2(1-a)}\frac{\sin(\mu_n)}{\mu_n} + o\left(\frac{1}{\mu_n^2}\right)\right) \qquad (91)\\
&\times\left(\frac{1}{\sqrt{a}}\sin\left(\frac{\mu_n}{\sqrt{a}}\right) + o\left(\frac{1}{\mu_n}\right)\right)
\end{aligned}$$

$$\Rightarrow \cos(\mu_n) = -\frac{b}{2(1-a)}\frac{\sin(\mu_n)}{\mu_n} + o\left(\frac{1}{\mu_n}\right). \qquad (92)$$

Now $\cos(\mu_n) = o(1) \Rightarrow \exists k_n \in \mathbb{Z}, \mu_n = k_n\pi + \dfrac{\pi}{2} + e(\mu_n)$, with $e(\mu_n) = o(1)$. Note that the sequence (k_n) has nothing to do with that of the first part of the proof. Inserting this expression into (92) and using classical trigonometric formulas for the cosine and sine of a sum, as well as $\sin(k\pi + \pi/2) = (-1)^k$, lead to:

$$-(-1)^{k_n}(e(\mu_n) + o(e(\mu_n))) = (-1)^{k_n}\frac{b}{2(-1+a)\pi k_n} + o\left(\frac{1}{k_n}\right) \qquad (93)$$

$$\mu_n = k_n\pi + \frac{\pi}{2} - \frac{b}{2(-1+a)k_n\pi} + o\left(\frac{1}{k_n}\right). \qquad (94)$$

Likewise, starting from the real part of $\Phi(\mu_n)$ (cf. (48)), $\exists l_n \in \mathbb{Z}$:

$$\frac{\mu_n}{\sqrt{a}} = l_n\pi + \frac{\pi}{2} - \frac{\sqrt{a}b}{2(-1+a)\pi l_n} + o\left(\frac{1}{l_n}\right). \tag{95}$$

Here again the sequence (l_n) has nothing to do with that of the first part of the proof. The comparison between (94) and (95) allows to write

$$k_n + \frac{1}{2} - \sqrt{a}\left(l_n + \frac{1}{2}\right) = o(1) \text{ i.e. } k_n = -\frac{1}{2} + \frac{1}{2\sqrt{a}} + \frac{l_n}{\sqrt{a}} + o(1).$$

In fact, since $\sqrt{a}$ is supposed to be a rational number, $o(1) = 0$ (cf. first part of the proof) and

$$k_n = -\frac{1}{2} + \frac{1}{2\sqrt{a}} + \frac{l_n}{\sqrt{a}} \text{ which implies } k_n\pi + \frac{\pi}{2} = \sqrt{a}\left(l_n\pi + \frac{\pi}{2}\right) \text{ or } k_n = \sqrt{a}l_n.$$

At last (94)-(95) is

$$\frac{(1+a)b}{2(-1+a)k_n\pi} = o\left(\frac{1}{k_n}\right)$$

which is impossible.
The conclusion is: if c and d are both odd, $\exists K > 0$, such that $\exists L > 0, \forall \mu \geq L$, $|\Psi(\mu)/\mu| \geq K$.
The end of the proof is identical with the end of the proof of the first part. □

5. Resolvent estimate.

Theorem 5.1. *(Estimate for the resolvent operator norm)*

1. *Assume that $a = 1$. Then*

$$\sup_{\mu\in\mathbb{R}} \|(i\mu I - \mathcal{A})^{-1}\| < \infty. \tag{96}$$

2. *Assume that $\sqrt{a} \in \mathbb{Q}^+ - \{0; 1\}$. There exists $(c; d) \in (\mathbb{N}^*)^2$ such that c and d are coprime and $\sqrt{a} = c/d$.*
   - *Assume that c or d is even. Then*

$$\sup_{\mu\in\mathbb{R}} \|(i\mu I - \mathcal{A})^{-1}\| < \infty. \tag{97}$$

   - *Assume that c and d are both odd numbers. Then*

$$\sup_{\mu\in\mathbb{R}} \frac{1}{|\mu|}\|(i\mu I - \mathcal{A})^{-1}\| < \infty. \tag{98}$$

Proving Theorem 5.1 requires the estimation of $\|U\|_{\mathcal{H}}$ with respect to $\|U_1\|_{\mathcal{H}}$, where U and U_1 are defined by (14). The explicit espression for U is given by Proposition 2.

First, let us explicit the functions introduced in that proposition, in particular those defined by (21). It holds, for $x \geq 0$:

$$\left\{\begin{array}{rcl} w_{11}(x) &=& \dfrac{\left(b+aq_1^2-\mu^2\right)}{q_1}\sin(q_1x)-\dfrac{\left(b+aq_2^2-\mu^2\right)}{q_2}\sin(q_2x),\\ w_{12}(x) &=& -\cos(q_1x)+\cos(q_2x),\\ w_{21}(x) &=& b\left(\cos(q_1x)-\cos(q_2x)\right),\\ w_{22}(x) &=& \dfrac{q_1^2-\mu^2}{q_1}\sin(q_1x)-\dfrac{q_2^2-\mu^2}{q_2}\sin(q_2x). \end{array}\right. \tag{99}$$

Now $q_1^2 := \dfrac{\mu^2}{2a}\left(a+1+\dfrac{1}{\mu}\sqrt{(a-1)^2\mu^2+4ab}\right)$ (cf. (19)).

Then $q_1^2 = \left(\dfrac{a+1+|a-1|}{2a}\right)\mu^2+\dfrac{b}{|a-1|}+o(1)$ if $a\neq 1$ and $q_1^2=\mu^2+\sqrt{b}\mu$ if $a=1$.

Likewise $q_2^2 = \left(\dfrac{a+1-|a-1|}{2a}\right)\mu^2-\dfrac{b}{|a-1|}+o(1)$ if $a\neq 1$ and $q_2^2=\mu^2-\sqrt{b}\mu$ if $a=1$.

5.1. **Proof of Theorem 5.1 in the case a=1.** If $a=1$

$\dfrac{b+q_1^2-\mu^2}{q_1} = (b+\sqrt{b}\mu)\left(\dfrac{1}{\mu}\left(1-\dfrac{\sqrt{b}}{2\mu}+o\left(\dfrac{1}{\mu}\right)\right)\right) = \sqrt{b}+O\left(\dfrac{1}{\mu}\right)$. Likewise

$\dfrac{b+q_2^2-\mu^2}{q_2} = (b-\sqrt{b}\mu)\left(\dfrac{1}{\mu}\left(1+\dfrac{\sqrt{b}}{2\mu}+o\left(\dfrac{1}{\mu}\right)\right)\right) = -\sqrt{b}+O\left(\dfrac{1}{\mu}\right)$.

Step 1: Estimate of u_p and y_p given by (25).

Thus, if $a=1$, $w_{11}(x)=\sqrt{b}(\sin(q_1x)+\sin(q_2x))+r_{11}(x)$
where $\max_{x\in[0,1]}|r_{11}(x)| = O\left(\dfrac{1}{\mu}\right)$.

Then, using (24), $w_{11}*f_1=\sqrt{b}(\sin(q_1.)+\sin(q_2.))*(-v_1-i\mu u_1)+r_{11}*(-v_1-i\mu u_1)$. Moreover, for $x\in[0;1]$:

$$\begin{array}{rcl} |(\sin(q_1.)*(-i\mu u_1))(x)| &=& \left|\displaystyle\int_0^x \sin(q_1(x-s))(-i\mu u_1)(x)ds\right|\\ &=& \left|\dfrac{(-i\mu u_1)(x)}{q_1}-\displaystyle\int_0^x \dfrac{\cos(q_1(x-s))}{q_1}(-i\mu u_{1,x})(s)ds\right|\\ &\leq& \dfrac{|\mu|}{q_1}(|u_1(x)|+\|u_{1,x}\|)\leq O(1)\cdot\|U_1\|_{\mathcal{H}} \end{array}$$

where $O(1)$ is a bounded function of μ, $\|\cdot\|$ is the $L^2(\Omega)$-norm and $\|\cdot\|_{\mathcal{H}}$ is the norm introduced in Section 2. Since, for any $x\in(0;1)$
$|(r_{11}*(-i\mu u_1))(x)|\leq|\mu|\cdot\max_{x\in[0,1]}|r_{11}(x)|\cdot\|u_1\|\leq O(1)\cdot\|U_1\|_{\mathcal{H}}$, it holds, for any $x\in[0;1]$:

$$|(w_{11}*f_1)(x)|\leq O(1)\cdot\|U_1\|_{\mathcal{H}}. \tag{100}$$

If $a=1$, $\dfrac{q_1^2-\mu^2}{q_1}=\sqrt{b}+O\left(\dfrac{1}{\mu}\right)$. Likewise $\dfrac{q_2^2-\mu^2}{q_2}=-\sqrt{b}+O\left(\dfrac{1}{\mu}\right)$. Thus, similarly, it holds, for any $x\in[0;1]$:

$$|(w_{22}*g_1)(x)|\leq O(1)\cdot\|U_1\|_{\mathcal{H}}. \tag{101}$$

The other two functions w_{12} and w_{21} have even simpler expressions: analogously

$$\begin{cases} \max_{x\in[0,1]} |(w_{12} * g_1)(x)| = O(1) \cdot \|U_1\|_{\mathcal{H}} \\ \max_{x\in[0,1]} |(w_{21} * f_1)(x)| = O(1) \cdot \|U_1\|_{\mathcal{H}} \end{cases}$$

Since $a(q_1^2 - q_2^2) = 2\sqrt{b}\mu$ (for $a = 1$), it holds, for $x \in [0;1]$:

$$|u_p(x)| \leq O\left(\frac{1}{\mu}\right) \cdot \|U_1\|_{\mathcal{H}} \text{ and } |y_p(x)| \leq O\left(\frac{1}{\mu}\right) \cdot \|U_1\|_{\mathcal{H}}. \tag{102}$$

By definition (cf. (20)), the function ϕ vanishes at 0 as well as its first and second derivatives. Then w_{11}, w_{12}, w_{21} and w_{22} also vanish at 0. Thus

$$\begin{cases} u_{p,x}(x) &= \dfrac{1}{q_1^2 - q_2^2}\left[w_{11,x} * f_1 + w_{12,x} * g_1\right] \\ y_{p,x}(x) &= \dfrac{1}{q_1^2 - q_2^2}\left[w_{21,x} * f_1 + w_{22,x} * g_1\right] \end{cases} \tag{103}$$

Now

$$\begin{aligned} w_{11,x}(x) &= \left(b + q_1^2 - \mu^2\right)\cos(q_1 x) - \left(b + q_2^2 - \mu^2\right)\cos(q_2 x) \\ &= \sqrt{b}(q_1\cos(q_1 x) + q_2\cos(q_2 x)) + r_{11,x}(x) \end{aligned} \tag{104}$$

where $\max_{x\in[0,1]} |r_{11,x}(x)| = O(1)$. Thus, for $x \in [0;1]$:

$$|u_{p,x}(x)| \leq O(1) \cdot \|U_1\|_{\mathcal{H}} \text{ and } |y_{p,x}(x)| \leq O(1) \cdot \|U_1\|_{\mathcal{H}}. \tag{105}$$

Step 2: Estimate of u_r and y_r given by (26).
By definition of γ_1 and γ_2 (cf. (27)), $\gamma_1 = O\left(\frac{1}{\mu}\right) \cdot \|U_1\|_{\mathcal{H}}$, $\gamma_2 = O(1) \cdot \|U_1\|_{\mathcal{H}}$.
Now, since $w_{22}(1) = O(1)$, $w_{22,x}(1) = O(\mu)$, $w_{12}(1) = O(1)$, $w_{11}(1) = O(1)$, $w_{21,x}(1) = O(\mu)$, $w_{21,xx}(1) = O(\mu^2)$, using (28) leads to: $\alpha_1 = \dfrac{O(1)}{\Phi(\mu)} \cdot \|U_1\|_{\mathcal{H}}$ and $\alpha_2 = \dfrac{O(\mu)}{\Phi(\mu)} \cdot \|U_1\|_{\mathcal{H}}$.

It follows from (26) and (99) that, for $x \in [0;1]$:

$$\begin{cases} |u_r(x)| &= |\alpha_1 w_{11}'(x) + \alpha_2 w_{12}(x)| \leq \dfrac{O(\mu)}{|\Phi(\mu)|} \cdot \|U_1\|_{\mathcal{H}} \\ |y_r(x)| &= |\alpha_1 w_{21}'(x) + \alpha_2 w_{22}(x)| \leq \dfrac{O(\mu)}{|\Phi(\mu)|} \cdot \|U_1\|_{\mathcal{H}} \end{cases} \tag{106}$$

and

$$|u_{r,x}(x)| \leq \frac{O(\mu^2)}{|\Phi(\mu)|} \cdot \|U_1\|_{\mathcal{H}} \text{ and } |y_{r,x}(x)| \leq \frac{O(\mu^2)}{|\Phi(\mu)|} \cdot \|U_1\|_{\mathcal{H}}. \tag{107}$$

Step 3: Conclusion for the resolvent operator norm.

Using the usual norm on $\mathcal{H}$ (which is equivalent to that defined by (9)) as well as the above estimates (102), (105), (106) and (107), there exists a constant C such that:

$$\|U\|_{\mathcal{H}} \leq C\cdot(\|u_p\| + \|u_{p,x}\| + \|v\| + \|y_p\| + \|y_{p,x}\| + \|z\|) \leq \left(O(1) + \frac{O(\mu^2)}{\Phi(\mu)}\right)\cdot\|U_1\|_{\mathcal{H}}. \tag{108}$$

Recall that v and z are defined by (23). The result follows from Proposition 3.

5.2. **Proof of Theorem 5.1 in the case $a \neq 1$:** The proof is similar to the previous one for $a = 1$. We give less details. The results, for any $x \in [0;1]$, are:

$$\left\{\begin{array}{lcl} |(w_{11} * f_1)(x)| & \leq & O(\mu)\cdot\|U_1\|_{\mathcal{H}} \\ |(w_{22} * g_1)(x)| & \leq & O(\mu)\cdot\|U_1\|_{\mathcal{H}} \\ |(w_{12} * g_1)(x)| & \leq & O(1)\cdot\|U_1\|_{\mathcal{H}} \\ |(w_{21} * f_1)(x)| & \leq & O(1)\cdot\|U_1\|_{\mathcal{H}}. \end{array}\right. \tag{109}$$

Since $a(q_1^2 - q_2^2) = |a-1|\mu^2 + O(1)$, both (102) and (105) still hold, for $a \neq 1$. The behaviour of γ_1 and γ_2 are still given by $\gamma_1 = O\left(\frac{1}{\mu}\right)\cdot\|U_1\|_{\mathcal{H}}$, $\gamma_2 = O(1)\cdot\|U_1\|_{\mathcal{H}}$. Now, since $w_{22}(1) = O(\mu)$, $w_{22,x}(1) = O(\mu^2)$, $w_{12}(1) = O(1)$, $w_{11}(1) = O(\mu)$, $w_{21,x}(1) = O(\mu)$, $w_{21,xx}(1) = O(\mu^2)$, using (28) leads to: $\alpha_1 = \frac{O(\mu)}{\Phi(\mu)}\cdot\|U_1\|_{\mathcal{H}}$ and $\alpha_2 = \frac{O(\mu)}{\Phi(\mu)}\cdot\|U_1\|_{\mathcal{H}}$.

It follows from (26) and (99) that, for $x \in [0;1]$:

$$\left\{\begin{array}{lcl} |u_r(x)| & = & |\alpha_1 w_{11}'(x) + \alpha_2 w_{12}(x)| \leq \frac{O(\mu^3)}{|\Phi(\mu)|}\cdot\|U_1\|_{\mathcal{H}} \\ |y_r(x)| & = & |\alpha_1 w_{21}'(x) + \alpha_2 w_{22}(x)| \leq \frac{O(\mu^2)}{|\Phi(\mu)|}\cdot\|U_1\|_{\mathcal{H}} \end{array}\right. \tag{110}$$

and

$$|u_{r,x}(x)| \leq \frac{O(\mu^4)}{|\Phi(\mu)|}\cdot\|U_1\|_{\mathcal{H}} \text{ and } |y_{r,x}(x)| \leq \frac{O(\mu^3)}{|\Phi(\mu)|}\cdot\|U_1\|_{\mathcal{H}}. \tag{111}$$

To conclude:

$$\|U\|_{\mathcal{H}} \leq \frac{O(\mu^4)}{\Phi(\mu)}\cdot\|U_1\|_{\mathcal{H}}. \tag{112}$$

The results follow from Proposition 4.

6. Decay rate of the energy.

Theorem 6.1. *(Decay rate)*
Assume that $\sqrt{a} \in \mathbb{Q}$.

1. *Assume that $a = 1$ and b satisfies Condition (C_1) and (C_2). Then there exist positive constants $K_1 > 0$ and $K_2 > 0$, such that for all initial $U_0 = (u_0, u_1, y_0, y_1) \in \mathcal{D}(\mathcal{A})$, the energy of the system (1)-(5) satisfies the following decay rate:*

$$E(t) \leq K_1 e^{-K_2 t}\|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{113}$$

2. *Assume that $\sqrt{a} \in \mathbb{Q}^+ - \{0;1\}$ and b satisfies Condition (C_1). There exists $(c;d) \in (\mathbb{N}^*)^2$ such that c and d are coprime and $\sqrt{a} = c/d$.*

- *Assume that c or d is even. Then there exist positive constants $L_1 > 0$ and $L_2 > 0$, such that for all initial $U_0 = (u_0, u_1, y_0, y_1) \in \mathcal{D}(\mathcal{A})$ the energy of the system (1)-(5) satisfies the following decay rate:*

$$E(t) \leq L_1 e^{-L_2 t} \|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{114}$$

- *Assume that c and d are both odd numbers. Then there exists a positive constant $M > 0$ such that for all initial $U_0 = (u_0, u_1, y_0, y_1) \in \mathcal{D}(\mathcal{A})$ the energy of the system (1)-(5) satisfies the following decay rate:*

$$E(t) \leq \frac{M}{t^2} \|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0. \tag{115}$$

Proof. 1. To prove (113) and (114), we use the following frequency domain theorem for uniform stability of a C_0 semigroup of contractions on a Hilbert space (cf. [16] and [27]): a C_0 semigroup e^{tA} on a Hilbert space H satisfies $\|e^{tA}\| \leq Ce^{-\tau t}$ for some constant $C > 0$ and for $\tau > 0$ if and only if

$$i\mathbb{R} \subset \rho(A) \text{ and } \limsup_{|\mu| \longrightarrow \infty} \|(i\mu I - A)^{-1}\| < \infty. \tag{116}$$

where $\rho(A)$ denotes the resolvent set of the operator A.
Then, Theorems 2.2 and 5.1 imply (113) and (114).

2. To prove (115), we use Theorem 2.4 of [10] (see also [18]): a C_0-semigroup of contractions $e^{t\mathcal{A}}$ in a Hilbert space $\mathcal{H}$ satisfies

$$E(t) \leq \frac{c}{t^{2/l}} \|U_0\|^2_{D(\mathcal{A})}, \quad \forall t > 0 \tag{117}$$

if

$$i\mathbb{R} \subset \rho(\mathcal{A}) \text{ and } \sup_{\mu \in \mathbb{R}} \frac{1}{|\mu|^l} \|(i\mu I - \mathcal{A})^{-1}\| < \infty. \tag{118}$$

Then, Theorems 2.2 and 5.1 imply (115). □

7. **Conclusion.**

As announced in the introduction, the decay rate of the energy of the solution of (1)-(5) is given for any rational value of $\sqrt{a}$, using an innovative technique.
If $\sqrt{a}$ is irrational the problem is solvable if it is proved that there exists $C > 0$, such that, for any $k \in \mathbb{N}$, $|k \sin(k) + \cos(k)| \geq C$. To our knowledge, the answer to this question is not obvious.

From a result of Lambert, it is known that $k \sin(k) + \cos(k) \neq 0$ for all $k \in \mathbb{N}$. And with a formal calculation software we have obtained:

$$\min_{k=1,\ldots,10^6} |k \sin(k) + \cos(k)| = |3 \sin(3) + \cos(3)| \approx 0.566632.$$

But it also holds: $|k \sin(k) + \cos(k)| \approx 0.183953$ for $k = 80143857$ and
$|k \sin(k) + \cos(k)| \approx 0.0764463$ for $k = 6167950454$.

We do not know if there exists, in the literature, an answer to this problem. But from the last two examples we may conjecture that

$$\min_{k \in \mathbb{N}} |k \sin(k) + \cos(k)| = 0.$$

For that reason we think that the decay rate (for an irrational value of $\sqrt{a}$) is very small and not easy to study.

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