Time-asymptotic propagation of approximate solutions of Schrödinger equations with both potential and initial condition in Fourier-frequency bands

Florent Dewez

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Abstract

In this paper, we consider the Schrödinger equation in one space-dimension with potential. We start by proving that this equation is well-posed in $H^1(\mathbb{R})$ if the potential belongs to $W^{1,\infty}(\mathbb{R})$, and we provide a representation of the solution as a series, called *Dyson-Phillips series*, by using semigroup theory. We focus our attention on the two first terms of this series: the first term is actually the free wave packet while the second term corresponds to the wave packet resulting from the first interaction between the free solution and the potential. To exhibit propagation features, we suppose that both potential and initial datum are in bounded Fourier-frequency bands; in particular a family of potentials satisfying this hypothesis is constructed for illustration. By representing the two first terms of the series as oscillatory integrals and by applying carefully a stationary phase method, we show that they are time-asymptotically localized in space-time cones depending explicitly on the Fourier-frequency bands. This permits to exhibit dynamic interaction phenomena produced by the potential.

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0 Introduction

The Schrödinger equation on the line with potential, *i.e.*

$$\begin{cases} i\,\partial_t u(t) = -\partial_{xx} u(t) + V(x) u(t) \\ u(0) = u_0 \end{cases} \tag{1}$$

where $t \geqslant 0$, describes the evolution of the wave function u of a non-relativistic particle moving on a line in an electric field V. Studying dispersive effects of the equation or

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describing propagation features of the solution may be helpful to understand the effects of the potential and to highlight certain quantum phenomena.

Estimates of the type $L^1 - L^\infty$ permit to show the dispersive nature of the equation and are especially essential to prove Strichartz' estimates; see [13, Section IX.7] for the $L^1 - L^\infty$ estimate of the solution in the free case, *i.e.* $V = 0$. On the other hand, propagation features are highlighted by considering initial data in bounded frequency bands and by expanding to one term and with respect to time the associated solutions, via stationary phase methods for example; see [3] for time-asymptotic expansions in space-time cones of the free solution for initial data in bounded Fourier-frequency bands. Estimates of type $L^1 - L^\infty$ and time-asymptotic expansions to one term are easier to obtain in the free case than in the general case thanks to the explicit solution formula given by an oscillatory integral.

Considering potentials having a particular form allows explicit calculations and may lead to an explicit representation of the solution as an oscillatory integral via spectral theory. In such cases, $L^1 - L^\infty$ estimates or time-asymptotic expansions to one term can be established; for instance see [5] for $L^1 - L^\infty$ estimates for the Schrödinger equation with delta-potentials or [4] for time-asymptotic expansions for the Klein-Gordon equation with step-potentials.

The case of general real-valued potentials can be treated thanks to a careful analysis of Born series and Jost functions under sufficient decay assumptions on the potential, leading to $L^1 - L^\infty$ estimates; see [16], [12], [8]. Further, this approach leads also to high frequency perturbation estimates for the solution of equation (1), which may be useful to derive approximate spatial information; see [2].

Nevertheless, it seems to be complicated to describe precisely propagation features for the solution of equation (1) in the general case. To give a partial answer to this problem, we propose in the present paper an approach which consists in studying the time-asymptotic motion of approximate solutions of equation (1). For this purpose, we consider the two first terms of the Dyson-Phillips series, which represents the exact solution, and we describe their time-asymptotic motions by means of expansions to one term under Fourier-frequency band hypotheses on the initial datum and the potential. The procedure which consists in studying only of the two first terms of a series is inspired by the physical literature; see for example [14, Section 5.3]. We hope that our approach, which does not require the potential to be real-valued, may help to derive precise information on the behaviour of the exact solution for long times and to bring more precision to some physical principles, as for example reflection and transmission type notions.

1 Survey of the literature and main results of the present paper

In this section, we start by listing some types of estimates of the solution of the Schrödinger equation and by commenting on their interpretation in terms of dispersion and propagation; let us emphasize that this list is non-exhaustive. We shall then explain the origin and the motivation for our approach, which allows the study of the time-asymptotic behaviour of approximate solutions of equation (1), and we shall introduce the main results of the present paper.

In the rest of the present section, the operator H is defined by $H := -\partial_{xx} + V$ with domain $H^2(\mathbb{R})$ and a given potential V. For $\sigma \in \mathbb{R}$, the weighted Lebesgue space $L^p_\sigma(\mathbb{R})$ is defined by

$$L^p_\sigma(\mathbb{R}) := \left\{ f : \mathbb{R} \longrightarrow \mathbb{C} \text{ measurable} \,\middle|\, \int_{\mathbb{R}} |f(x)|^p \left(1+|x|\right)^{p\sigma} dx < +\infty \right\}$$

if $1 \leqslant p < \infty$, and

$$L^\infty_\sigma(\mathbb{R}) := \left\{ f : \mathbb{R} \longrightarrow \mathbb{C} \text{ measurable} \,\middle|\, \sup_{x\in\mathbb{R}} |f(x)| \left(1+|x|\right)^{\sigma} < +\infty \right\} .$$

Finally if there is a bounded solution f_0 of $Hf_0 = 0$ (in the distributional sense), then we say that zero energy is a *resonance*; see for instance [15] for detailed explanations on this notion.

As explained in the preceding section, $L^1 - L^\infty$ estimates permit to measure the dispersion of the solution of equation (1). Such estimates in the case of a real-valued potential were established first in [16]: suppose either that zero energy is a resonance and $V \in L^1_\sigma(\mathbb{R})$ with $\sigma > \frac{5}{2}$, or zero energy is not a resonance and $V \in L^1_\sigma(\mathbb{R})$ with $\sigma > \frac{3}{2}$, then one has in both situations

$$\forall\, t > 0 \qquad \left\| e^{-itH} P_{ac}\, u_0 \right\|_{L^\infty(\mathbb{R})} \leqslant C \left\| u_0 \right\|_{L^1(\mathbb{R})} t^{-\frac{1}{2}} , \tag{2}$$

where $C > 0$ is a constant and P_{ac} is the projection onto the absolutely continuous spectral subspace of H. The presence of this projection is necessary to remove the bound states contained in the solution.
Let us remark that the hypotheses required for the proof of the above estimate have been successively weakened in [12] and [8]; in particular, it has been proved in [8] that the assumption $V \in L^1_1(\mathbb{R})$ is sufficient to obtain estimate (2) including the case that zero energy is a resonance.

In the literature (see [15], [8]), there exist estimates in weighted Lebesgue spaces:

$$\forall\, t > 0 \qquad \left\| e^{-itH} P_{ac}\, u_0 \right\|_{L^\infty_{-1}(\mathbb{R})} \leqslant C \left\| u_0 \right\|_{L^1_1(\mathbb{R})} t^{-\frac{3}{2}} . \tag{3}$$

The weight $(1+|x|)^{-1}$ acts as a spatial filter around the origin and hence, the above estimate combined with estimate (2) indicates that the solution tends to move away from 0 as the time tends to infinity, which represents some qualitative information on spatial dynamics.
Estimate (3) for the solution of equation (1) with a real-valued potential has been proved in [15] under the hypotheses that $V \in L^1_4(\mathbb{R})$ and zero energy is not a resonance, and in [8] under the weaker hypotheses $V \in L^1_2(\mathbb{R})$ and zero energy is not a resonance.

A similar estimate taking into the account the presence of a resonance has been established in [11]:

$$\forall\, t > 0 \qquad \left\| e^{-itH} P_{ac}\, u_0 - (4\pi i t)^{-\frac{1}{2}} P_0\, u_0 \right\|_{L^\infty_{-2}(\mathbb{R})} \leqslant C \left\| u_0 \right\|_{L^1_2(\mathbb{R})} t^{-\frac{3}{2}} , \tag{4}$$

provided $V \in L^1_4(\mathbb{R})$, where $P_0 : L^1_2 \longrightarrow L^\infty_{-2}$ is formally given by $P_0 f := \langle f, f_0\rangle_{L^2} f_0$; see [11] for a complete statement of this result. This last estimate shows that the solution in a neighbourhood of the origin is determined by the bound state f_0 with zero energy. The appearance of this term in the present case can be explained by the fact that the hypotheses on the potential allow the initial data to have a zero energy component which, therefore, does not move over time.
Let us mention that the authors of [7] have recently proved estimate (4) under weaker decay assumptions on the potential.

Another approach to obtain approximate spatial information on the propagation of the solution of equation (1) consists in comparing this perturbed solution with the free one, whose time-asymptotic behaviour is known. Hence the task consists in finding a setting in which perturbed and free solutions are sufficiently close, in other words finding conditions which diminish the influence of the potential. For example, this can be achieved by considering only initial data having sufficiently high frequencies, that is to say particles having sufficiently high initial energies. Following this idea, a high frequency perturbation estimate has been established for the solution of equation (1) in [2]:

$$\forall\, t > 0 \qquad \Big\| e^{-itH} \chi_{\lambda_0}(H) u_0 - e^{-itH_0} \chi_{\lambda_0}(H_0) u_0 \Big\|_{L^\infty(\mathbb{R})} \leqslant C(u_0)\, \frac{\|V\|_{L^1(\mathbb{R})}}{\sqrt{\lambda_0}}\, t^{-\frac{1}{2}} \;, \tag{5}$$

where χ_{λ_0} cuts smoothly the frequencies below $\lambda_0 \in \mathbb{R}$, the operator H_0 being the self-adjoint realization of $-\partial_{xx}$ on $L^2(\mathbb{R})$.
We mention that estimate (5) has been actually proved in [2] in the more general setting of the Schrödinger equation on an infinite star-shaped network with transmission conditions; an estimate of type (2) for the operator H defined on the network has been also established.

Following the papers [3] and [6], precise propagation features for the solution of the free Schrödinger equation can be obtained by considering initial data u_0 in a bounded Fourier-frequency band, that is to say initial data whose Fourier transform $\widehat{u_0}$ has a support contained in a bounded interval $[p_1, p_2]$. In terms of Quantum Mechanics, such an assumption means that the particle has a momentum localized in $[2p_1, 2p_2]$ (the factor 2 comes from the fact that we consider the operator $-\partial_{xx}$ and not $-\frac{1}{2}\partial_{xx}$). In this case, the solution is given by a *wave packet* whose frequency-components are localized between p_1 and p_2; according to the physical principle of *group velocity*, such a wave packet will travel in space at different speeds between $2p_1$ and $2p_2$ over time. Hence one expects that a free wave packet which is in the Fourier-frequency band $[p_1, p_2]$ is mainly spatially localized in the interval $\big[2p_1\, t, 2p_2\, t\big]$ for sufficiently large $t \geqslant 0$, describing the motion of the associated particle; see [6, Section 1] for more detailed explanations.
One way to highlight such a phenomenon consists in establishing the following asymptotic expansion in a certain space-time domain $\mathcal{D}$ for the solution of the free Schrödinger equation:

$$\forall\, (t,x) \in \mathcal{D} \qquad \big(e^{-itH_0} u_0\big)(x) = \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, \widehat{u_0}\Big(\frac{x}{2t}\Big)\, t^{-\frac{1}{2}} \;+\; R(t,x,u_0)\, t^{-\delta} \;, \tag{6}$$

where the remainder term $R(.,.,u_0) : \mathcal{D} \longrightarrow \mathbb{C}$ is a bounded function and $\delta > \frac{1}{2}$. For the sake of clarity, we suppose that expansion (6) is true on $(0,+\infty) \times \mathbb{R}$ (this is actually the case when $\widehat{u_0}$ is a $\mathcal{C}^1$-function on $\mathbb{R}$; see Remark 5.4). Since u_0 is supposed to be in the bounded Fourier-frequency band $[p_1, p_2]$, *i.e.* $\widehat{u_0} = 0$ outside $[p_1, p_2]$, we observe that the first term of the expansion in (6) is equal to 0 outside the *space-time cone*

$$\mathfrak{C}(p_1, p_2) := \left\{ (t, x) \in (0, +\infty) \times \mathbb{R} \,\middle|\, 2\,p_1 \leqslant \frac{x}{t} \leqslant 2\,p_2 \right\} ,$$

showing that the time-decay rate is faster outside $\mathfrak{C}(p_1, p_2)$ than inside. This shows that the free solution tends to be mainly localized inside this cone as the time tends to infinity, permitting to describe its time-asymptotic motion.
Asymptotic expansions with uniform remainder estimates have been established in [4], for the solution of the Klein-Gordon equation on a star-shaped network with different but constant coefficients, and in [3], for the solution of the free Schrödinger equation on the line with initial data having a L^1-singular frequency. The method consists mainly in representing the solution as an oscillatory integral and then applying carefully a stationary phase method. Let us remark that the asymptotic expansions given in these two papers are true only in domains $\mathcal{D}$ which do not contain certain space-time half-lines (or *space-time directions*): this is due to the existence of singular frequencies which produce a blow-up of the first term as well as of the remainder when approaching the critical space-time directions associated with the singularities. In the paper [6], uniform estimates of the free solution of the Schrödinger equation are established by means of careful applications of van der Corput-type estimates; this permits to cover the regions containing the above mentioned critical directions. Finally let us mention that expansion (6) is provided in the book [13] but in the L^2-sense, in other words the remainder is uniformly bounded in $L^2(\mathbb{R})$; this expansion covers especially the case of initial data having L^2-singular frequencies but not L^1-singular frequencies.

In the present paper, we propose an approach which describes precise propagation features for approximate solutions of the Schrödinger equation (1) and which is inspired by the papers [4] and [3]. More precisely, we aim at providing expansions to one term with uniform remainder estimates of these approximate solutions to show their localization in explicit space-time cones, which will permit in particular to exhibit the influence of a potential on the motion of wave packets.

To employ the method developped in the papers [4] and [3], we need that the formulas of the approximate solutions are given by oscillatory integrals. To achieve this, we shall represent the exact solution as a series, called *Dyson-Phillips series*: the terms of this series are recursively defined via a Volterra-type operator involving the free evolution and therefore, they inherit the oscillatory nature from the free solution.
Technically speaking, we shall use the theory of semigroups which affirms that the solution of equation (1) in $H^1(\mathbb{R})$ can be represented by the Dyson-Phillips series under the assumption that the potential V belongs to the Sobolev space $W^{1,\infty}(\mathbb{R})$; see Theorem 2.2 for a complete statement.

The approximate solutions we consider are actually given by the two first terms of the series; if u_0 is the initial datum, we note respectively $S_1(t)u_0$ and $S_2(t)u_0$ the first and the second term. Let us remark that the study of all the terms of the series falls out of the scope of this paper due to the intrinsic complexity appearing in this case. Our choice to consider the two first terms of the Dyson-Phillips series is originated from physical literature; indeed, one can mention that Fermi's golden rule in Quantum Mechanics, which gives the rate of transition from one eigenstate of a quantum system into another eigenstate in a continuum under the effect of a perturbation, can be derived via a precise study of the two first terms of the Dyson-Phillips series (we refer to the book [14, Section 5.3] for more explanations). In addition, this approach allows the comparison between the time-asymptotic behaviour of the first term, which is actually the wave packet giving the free solution, and of the second one, which can be interpreted as the wave packet resulting from the first interaction between the free wave packet and the potential, and thus as the influence of the potential on the motion for long times.

In the following, we shall consider potentials which are in bounded Fourier-frequency bands:

Condition $(\mathcal{P}_{l,[a,b]})$. Let $l \in \mathbb{N}$ and let $a < b$ be two finite real numbers.
An element V of $L^2(\mathbb{R})$ satisfies Condition $(\mathcal{P}_{l,[a,b]})$ if and only if $\widehat{V}$ is a $\mathcal{C}^l$-function on $\mathbb{R}$ which verifies $supp\,\widehat{V} \subseteq [a,b]$.

We emphasize that, as in [3] and [6], our method does not require any self-adjointness argument, contrary to the above mentioned papers. Thus complex-valued potentials can be considered and no hypothesis on the position of the bounded Fourier-frequency band of the potential is needed. In Theorem 3.4, we provide a large family of potentials which satisfy the above condition and which are approximately localized in space around an arbitrary point.

As explained previously, precise propagation features for wave packets can be exhibited by considering initial data in bounded frequency bands. Since the terms of the Dyson-Phillips series are defined via the free evolution, which makes appear the Fourier transform in the formulas of $S_1(t)u_0$ and $S_2(t)u_0$, we shall consider initial data in bounded Fourier-frequency bands :

Condition $(\mathcal{I}_{k,[p_1,p_2]})$. Let $k \in \mathbb{N}$ and let $p_1 < p_2$ be two finite real numbers.
An element u_0 of $H^3(\mathbb{R})$ satisfies Condition $(\mathcal{I}_{k,[p_1,p_2]})$ if and only if $\widehat{u_0}$ is a $\mathcal{C}^k$-function on $\mathbb{R}$ which verifies $supp\,\widehat{u_0} \subseteq [p_1,p_2]$.

Under this hypothesis on the initial datum u_0, we shall provide the following asymptotic expansion to one term of $S_1(t)u_0$ with uniform remainder estimate inside the cone $\mathfrak{C}(p_1,p_2)$ as well as a uniform estimate outside this cone:

1.1 Theorem. *Suppose that $u_0 \in H^3(\mathbb{R})$ satisfies Condition $(\mathcal{I}_{k,[p_1,p_2]})$, where $k \geqslant 1$, and fix $\delta_1 \in \left(\frac{1}{2},1\right)$. Then we have*

i) for all $(t,x) \in \mathfrak{C}(p_1,p_2)$,

$$\left| \big(S_1(t)u_0\big)(x) - \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, \widehat{u_0}\Big(\frac{x}{2t}\Big)\, t^{-\frac{1}{2}} \right| \leqslant C_1(\delta_1,p_1,p_2)\, \big\|\widehat{u_0}'\big\|_{L^\infty(\mathbb{R})}\, t^{-\delta_1} \ ;$$

ii) for all $(t,x) \in \mathfrak{C}(p_1,p_2)^c$,

$$\left| \big(S_1(t)u_0\big)(x) \right| \leqslant C_1(\delta_1,p_1,p_2)\, \big\|\widehat{u_0}'\big\|_{L^\infty(\mathbb{R})}\, t^{-\delta_1} \ .$$

See Theorem 5.3 for a more detailed statement; here we have defined

$$\mathfrak{C}(p_1,p_2)^c := \big((0,+\infty)\times\mathbb{R}\big) \setminus \mathfrak{C}(p_1,p_2) \ .$$

In particular, we deduce that the time-decay rate $t^{-\frac{1}{2}}$ is attained along the space-time direction

$$\mathfrak{D}(\overline{p}) := \left\{ (t,x) \in (0,+\infty) \times \mathbb{R} \,\middle|\, \frac{x}{t} = 2\,\overline{p} \right\} ,$$

for any $\overline{p} \in [p_1, p_2]$ satisfying $\left|\widehat{u_0}(\overline{p})\right| > 0$. This shows that the term $S_1(t)u_0$ is time-asymptotically localized in the union of all these directions, which are included in the cone $\mathfrak{C}(p_1, p_2)$; see Corollary 5.5 and Remark 5.6 for more details.
We provide similar estimates for the second term $S_2(t)u_0$. Our method requires $\widehat{u_0}$ to be more regular than in the preceding theorem and zero to be outside the Fourier-frequency band of u_0; this last hypothesis means that the associated particle is not allowed to have a momentum equal to 0 at $t = 0$.

1.2 Theorem. *Suppose that $u_0 \in H^3(\mathbb{R})$ and $V \in L^2(\mathbb{R})$ satisfy respectively Condition $(\mathcal{I}_{k,[p_1,p_2]})$ and Condition $(\mathcal{P}_{l,[a,b]})$, where $k \geqslant 5$ and $l \geqslant 4$, and suppose in addition that $0 \notin [p_1, p_2]$. Fix $\delta_1 \in \left(\frac{1}{2}, 1\right)$. For all $t \geqslant 0$, define the function $W(t,.) : \mathbb{R} \longrightarrow \mathbb{C}$ by*

$$W(t,p) = -i \int_0^t \int_a^b \widehat{V}(y)\, \widehat{u_0}(p-y)\, e^{-i\tau(p-y)^2}\, dy\, e^{i\tau p^2}\, d\tau \ ;$$

in particular, its support is contained in $[p_1 + a, p_2 + b]$ for all $t \geqslant 0$. Then we have

i) for all $(t,x) \in \mathfrak{C}\big(p_1 + a, p_2 + b\big)$,

$$\left| \big(S_2(t)u_0\big)(x) - \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, W\Big(t, \frac{x}{2t}\Big)\, t^{-\frac{1}{2}} \right| \leqslant C_2(\delta_1, a, b, p_1, p_2)\, \big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})} \big\|\widehat{u_0}\big\|_{W^{5,\infty}(\mathbb{R})}\, t^{-\delta_1} \ ;$$

ii) for all $(t,x) \in \mathfrak{C}\big(p_1 + a, p_2 + b\big)^c$,

$$\left| \big(S_2(t)u_0\big)(x) \right| \leqslant C_2(\delta_1, a, b, p_1, p_2)\, \big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})} \big\|\widehat{u_0}\big\|_{W^{5,\infty}(\mathbb{R})}\, t^{-\delta_1} \ .$$

See Theorem 6.3 for a more detailed statement. We observe that the term $S_2(t)u_0$ seems at the first sight to travel time-asymptotically in the cone $\mathfrak{C}\big(p_1 + a, p_2 + b\big)$, which depends explicitly on the frequency bands of u_0 and of V. Nevertheless the time-dependence of $W(t,.)$ prevents us from concluding that $t^{-\frac{1}{2}}$ is attained in $\mathfrak{C}\big(p_1 + a, p_2 + b\big)$ and from deriving any time-asymptotic localization of $S_2(t)u_0$. However, under additional hypotheses on the potential, we shall prove that

$$\forall\, p \in \mathbb{R} \setminus \left[\frac{a}{2}, \frac{b}{2}\right] \qquad W(t,p) = W_1(p) + W_2(t,p)\, t^{-1} \ ,$$

where $W_1, W_2(t,.) : \mathbb{R}\setminus\left[\frac{a}{2}, \frac{b}{2}\right] \longrightarrow \mathbb{C}$ are two regular functions and $W_2(t,.)$ is uniformly bounded with respect to time; see Propositions 6.5 and 6.7 for complete statements. Inserting this decomposition of $W(t,.)$ into the expansion given in the point i) of Theorem 1.1 leads to

1.3 Theorem. *Suppose that* $u_0 \in H^3(\mathbb{R})$ *and* $V \in L^2(\mathbb{R})$ *satisfy respectively Condition* $(\mathcal{I}_{k,[p_1,p_2]})$ *and Condition* $(\mathcal{P}_{l,[a,b]})$ *where* $k \geqslant 5$ *and* $l \geqslant 4$*; suppose in addition that* $0 \notin [p_1, p_2]$ *and that the function* $y \in \mathbb{R}\backslash\{0\} \longmapsto y^{-1}\widehat{V}(y) \in \mathbb{C}$ *can be extended to a* $\mathcal{C}^1$*-function* $\widetilde{V} : \mathbb{R} \longrightarrow \mathbb{C}$*. Fix* $\delta_1 \in \left(\frac{1}{2}, 1\right)$ *and* $\varepsilon \in \left(0, \frac{p_2-p_1}{2}\right)$*. Define the function* $W_1 : \mathbb{R}\backslash\left[\frac{a}{2}, \frac{b}{2}\right] \longrightarrow \mathbb{C}$ *by*

$$W_1(p) := -\int_a^b \frac{\widetilde{V}(y)\, \widehat{u_0}(p-y)}{y-2p}\, dy\ ;$$

in particular, its support is contained in $[p_1 + a, p_2 + b]$*. Then we have*

$$\begin{aligned}\Big|\big(S_2(t)u_0\big)(x) - \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, W_1\Big(\frac{x}{2t}\Big)\, t^{-\frac{1}{2}}\Big| \\ \leqslant C_3(\delta_1, a, b, p_1, p_2)\, \big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})} \big\|\widehat{u_0}\big\|_{W^{5,\infty}(\mathbb{R})}\, t^{-\delta_1} \\ + C_4(a, b, p_1, p_2, \varepsilon)\, \big\|\widetilde{V}\big\|_{W^{1,\infty}(\mathbb{R})} \big\|\widehat{u_0}\big\|_{W^{1,\infty}(\mathbb{R})}\, t^{-\frac{3}{2}}\ ,\end{aligned}$$

for all $(t, x) \in \mathfrak{C}\big(p_1 + a, p_2 + b\big) \cap \mathfrak{C}\big(\frac{a}{2} - \varepsilon, \frac{b}{2} + \varepsilon\big)^c$*.*

See Theorem 6.9 for a more detailed statement. The preceding theorem implies that the time-decay rate $t^{-\frac{1}{2}}$ is attained along the space-time direction $\mathfrak{D}(\overline{p})$, for any $\overline{p}$ belonging to $[p_1 + a, p_2 + b] \cap \big(\mathbb{R}\backslash\left[\frac{a}{2}, \frac{b}{2}\right]\big)$ and satisfying $\big|W_1(\overline{p})\big| > 0$. In Proposition 6.12, we shall show that, under extra hypotheses on the potential and the initial datum, intervals of the form $(p_1 + a, p_1 + \eta_1)$ or $(p_2 + \eta_2, p_2 + b)$ are included in

$$\left\{p \in \mathbb{R} \setminus \left[\frac{a}{2}, \frac{b}{2}\right]\ \Big|\ \big|W_1(p)\big| > 0\right\}\ ,$$

for certain real numbers η_1, η_2.

To explain the influence of the Fourier-frequencies of the potential on the motion of the wave packet $S_2(t)u_0$, let us suppose that

$$\left\{p \in \mathbb{R}\ \Big|\ \big|\widehat{u_0}(p)\big| > 0\right\} = (p_1, p_2) \subset (0, +\infty)\ ,$$

until the end of the present section; in particular, we have $supp\, \widehat{u_0} = [p_1, p_2]$. This means that the initial momentum of the associated particle is positive and, according to Theorem 1.1, the free wave packet tends to move to the right in space, the minimal and maximal speeds being respectively given by $2p_1$ and $2p_2$. Now let us distinguish two cases:

- If $a > 0$, then the cone $\mathfrak{C}\big(p_1 + a, p_2 + b\big)$ is more inclined to the right in space-time than $\mathfrak{C}\big(p_1, p_2\big)$, showing that the front of the wave packet $S_2(t)u_0$ may travel faster to the right than the front of $S_1(t)u_0$ as the time tends to infinity. We deduce that positive Fourier-frequencies of the potential may accelerate wave packets.

- If $b < 0$, then the cone $\mathfrak{C}\big(p_1 + a, p_2 + b\big)$ is more inclined to the left than $\mathfrak{C}\big(p_1, p_2\big)$, indicating that negative Fourier-frequencies of the potential may diminish the speed of wave packets, even change totally their direction of propagation if the largest initial Fourier-frequency p_2 is not sufficiently high.

In the physical case, that is to say when V is real-valued (implying in particular that $\widehat{V}$ is an even real-valued function on $\mathbb{R}$), dynamic interaction phenomena produced by the potential are highlighted. In this situation, we have especially $a = -b$, implying

$$\mathfrak{C}\big(p_1 + a, p_2 + b\big) = \mathfrak{C}\big(p_1 - b, p_2 + b\big) \; .$$

Then we observe that the front of the wave packet $S_2(t)u_0$ may travel faster than the front of the free wave packet $S_1(t)u_0$ and in the same direction; this can be interpreted as the existence of an advanced transmission. Concerning the back of the wave packet $S_2(t)u_0$, we distinguish two cases again:

- If $p_1 > b$ then the back of $S_2(t)u_0$ may travel slower than the one of $S_1(t)u_0$ but in the same direction; this can be interpreted as the existence of a retarded transmission.
- If $p_1 < b$ then the back of $S_2(t)u_0$ may travel in the opposite propagation direction of $S_1(t)u_0$, expressing the existence of a reflection.

To finish, let us note that the hypotheses we make on u_0 and V do not furnish information on existence or absence of bound states. Since our results prove that the two first terms of the Dyson-Phillips series tend to 0 as the time tends to infinity under our hypotheses, a bound state (if it exists) must appear in the rest of the series. More generally, the relation of spectral theory for the free Hamiltonian and for the Hamiltonian with a potential remains unclear.
Nevertheless there is hope that the contribution of the two first terms of the series might be visible in physical experiences at a certain distance of the potential, thanks to the time-asymptotic spatial dissociation of these terms which tend to be localized in different space-time cones.

2 Dyson-Phillips series for the solution of the Schrödinger equation with potential

Let us consider the Schrödinger equation with potential on the line, *i.e.*

$$\begin{cases} i\,\partial_t u(t) = -\partial_{xx} u(t) + V(x)u(t) \\ u(0) = u_0 \end{cases} \tag{7}$$

for all $t \geqslant 0$, where the potential V is an element of $W^{1,\infty}(\mathbb{R})$ and not necessarily real-valued. The aim of the present section is to assure existence and uniqueness of a solution for the Schrödinger equation (7) in $H^1(\mathbb{R})$ by exploiting the theory of semigroups, and to provide a representation of this solution as a series, called Dyson-Phillips series.

Before stating the above mentioned results, let us define some objects that will be used throughout the rest of this paper:

2.1 Definition. i) Let $\mathcal{F}_{x\to p}$ be the Fourier transform on $L^2(\mathbb{R})$ and $\mathcal{F}^{-1}_{p\to x}$ its inverse. For f in the Schwartz space $\mathcal{S}(\mathbb{R})$, $\mathcal{F}_{x\to p} f$ is a complex-valued function on $\mathbb{R}$ given by

$$\forall\, p \in \mathbb{R} \qquad \big(\mathcal{F}_{x\to p} f\big)(p) = \int_{\mathbb{R}} f(x)\, e^{-ixp}\, dx \; .$$

If there is no risk of confusion, we shall note $\widehat{f} := \mathcal{F}_{x\to p} f$ in favour of readability.

ii) Let A be the operator given by $A := i\frac{d^2}{dx^2}$ with domain $D(A) := H^3(\mathbb{R}) \subset H^1(R)$.

iii) For $V \in W^{1,\infty}(\mathbb{R})$, let B be the operator defined on $H^1(\mathbb{R})$ by

$$\forall\, f \in H^1(\mathbb{R}) \qquad (Bf)(x) := -i\,V(x)\,f(x) \quad a.e\ .$$

In this case, we have $Bf \in H^1(\mathbb{R})$ (see [9, Chapter VI, Lemma 5.20]).

2.2 Theorem. *Suppose that u_0 belongs to $H^3(\mathbb{R})$ and that V belongs to $W^{1,\infty}(\mathbb{R})$. Then there exists a unique function $u : \mathbb{R}_+ \longrightarrow H^1(\mathbb{R})$ which is continuously differentiable with respect to the H^1-norm, $u(t) \in H^3(\mathbb{R})$ for all $t \geqslant 0$, and u satisfies the Schrödinger equation (1).*
Moreover the function $u : \mathbb{R}_+ \longrightarrow H^1(\mathbb{R})$ can be represented as follows:

$$\forall\, t \geqslant 0 \qquad \lim_{N\to+\infty} \left\| u(t) - \sum_{n=1}^{N} S_n(t)u_0 \right\|_{H^1(\mathbb{R})} = 0\ ,$$

where

$$\begin{cases} S_1(t)u_0 := \mathcal{F}^{-1}_{p\to x}\Big(e^{-itp^2}\,\widehat{u_0}(p)\Big) \\ S_{n+1}(t)u_0 := \displaystyle\int_0^t S_n(t-\tau)\,B\,S_1(\tau)u_0\,d\tau \quad , \quad \forall\, n \in \mathbb{N}\setminus\{0\} \end{cases} .$$

2.3 Remark. i) The function $u : \mathbb{R}_+ \longrightarrow H^1(\mathbb{R})$ is called the *classical solution* of the Schrödinger equation (7) (see [9, Chapter II, Proposition 6.2]) and the series $\sum_{n\geqslant 1} S_n(t)u_0$ is called the *Dyson-Phillips series* for the solution u (see [9, Chapter III, Theorem 1.10]).

ii) For each $n \in \mathbb{N}\setminus\{0\}$, the term $S_{n+1}(t)u_0$ belongs at least to $H^1(\mathbb{R})$ for all fixed $t \geqslant 0$ if $u_0 \in H^3(\mathbb{R})$, and thus it defines a continuous function on $\mathbb{R}$. To evaluate it at any point $x \in \mathbb{R}$, let us define the operator $E_x : H^1(\mathbb{R}) \longrightarrow \mathbb{C}$ by

$$\forall\, f \in H^1(\mathbb{R}) \qquad E_x f := f(x)\ .$$

Then E_x is a bounded operator from $H^1(\mathbb{R})$ into $\mathbb{C}$ thanks to the continuous embedding of $H^1(\mathbb{R})$ into $\mathcal{C}^0_0(\mathbb{R}) := \left\{ f \in \mathcal{C}^0(\mathbb{R}) \,\middle|\, \lim_{|x|\to 0} f(x) = 0 \right\}$. Hence Proposition 7.4 is applicable and provides[1]

$$\begin{aligned} \forall\, x \in \mathbb{R} \qquad \big(S_{n+1}(t)u_0\big)(x) &= E_x\big(S_{n+1}(t)u_0\big) \\ &= E_x\bigg(\int_0^t S_n(t-\tau)\,B\,S_1(\tau)u_0\,d\tau\bigg) \\ &= \int_0^t \Big(S_n(t-\tau)\,B\,S_1(\tau)u_0\Big)(x)\,d\tau\ . \end{aligned} \tag{8}$$

This result will be employed in Section 6.

[1] To apply Proposition 7.4, the integral defining $S_{n+1}(t)u_0$ and the integral given in (8) are here interpreted as Bochner-integrals. Especially, the integrand of (8) is complex-valued and, due to the construction of the Bochner-integral, it is actually an integral of Lebesgue-type.

Proof of Theorem 2.2. In order to apply results from semigroup theory, we start by rewriting the Schrödinger equation (7) as an evolution equation of the form

$$\begin{cases} \dot{u}(t) = \big(A+B\big)u(t) \\ u(0) = u_0 \end{cases} ,$$

where A and B are given in Definition 2.1.
Now let us recall that the operator $\big(A, D(A)\big)$ is the generator of the strongly continuous semigroup $\big(T(t)\big)_{t\geqslant 0}$ on $H^1(\mathbb{R})$ represented by

$$\forall\, t \geqslant 0 \qquad T(t)f = \mathcal{F}^{-1}_{p\to x}\Big(e^{-itp^2}\,\widehat{f}(p)\Big) , \tag{9}$$

for $f \in H^1(\mathbb{R})$. Moreover the operator B belongs to $\mathcal{L}\big(H^1(\mathbb{R})\big)$, the space of bounded operators from $H^1(\mathbb{R})$ into itself. Indeed we have

$$\begin{aligned} \big\|Bf\big\|^2_{H^1(\mathbb{R})} &= \int_{\mathbb{R}} \big|V(x)f(x)\big|^2\,dx + \int_{\mathbb{R}} \big|(Vf)'(x)\big|^2\,dx \\ &= \int_{\mathbb{R}} \big|V(x)f(x)\big|^2\,dx + \int_{\mathbb{R}} \big|V'(x)f(x)\big|^2\,dx + \int_{\mathbb{R}} \big|V(x)f'(x)\big|^2\,dx \\ &\leqslant 2\,\big\|V\big\|^2_{W^{1,\infty}(\mathbb{R})}\big\|f\big\|^2_{H^1(\mathbb{R})} , \end{aligned}$$

since $V \in W^{1,\infty}(\mathbb{R})$. According to Theorems 7.1 and 7.2, if $u_0 \in H^3(\mathbb{R}) = D(A)$ then the Schrödinger equation (7) has a unique classical solution belonging to $\mathcal{C}^1\big(\mathbb{R}^+, H^1(\mathbb{R})\big)$. More precisely, the solution $u : \mathbb{R}^+ \longrightarrow H^1(\mathbb{R})$ is given by

$$\forall\, t \geqslant 0 \qquad u(t) = S(t)u_0 ,$$

where $\big(S(t)\big)_{t\geqslant 0}$ is the semigroup generated by the operator $\big(A+B, D(A)\big)$.
Employing Theorem 7.3, the solution of equation (7) can be represented as follows,

$$\forall\, t \geqslant 0 \qquad \lim_{N\to+\infty} \left\| u(t) - \sum_{n=1}^{N} S_n(t)u_0 \right\|_{H^1(\mathbb{R})} = 0 ,$$

where $S_1(t) := T(t)$ and

$$S_{n+1}(t)u_0 := \int_0^t S_n(t-\tau)\, B\, T(\tau)u_0\, ds .$$

According to equality (9), we have $S_1(t)u_0 = T(t)u_0 = \mathcal{F}^{-1}_{p\to x}\Big(e^{-itp^2}\,\widehat{u_0}(p)\Big)$, which ends the proof. □

3 Potentials in bounded Fourier-frequency bands

The main goal of this short section is to introduce the hypotheses that we shall make on the potential: roughly speaking, the potential is supposed to be in a bounded Fourier-frequency band, permitting in particular to study the influence of the Fourier-frequencies of the potential on the time-asymptotic motion of perturbed wave packets. To illustrate our hypotheses, we shall construct a class of potentials satisfying them.

In favour of readability, the expression *frequency* will refer to *Fourier-frequency* throughout the rest of the present paper.

Prior to this, we establish the following lemma which provides a family of functions localized in a bounded frequency band $[a,b]$ and, at the same time, approximately localized in space in an interval centered at a point $x_0 \in \mathbb{R}$ with arbitrary precision if $(b-a)$ is sufficiently large. This result, whose proof lies essentially on Chebyshev's inequality, is originated from [1] and will allow us to construct the above mentioned class of potentials.

3.1 Lemma. *Let $l \geqslant 1$ be an integer, let a, b, x_0 be three finite real numbers such that $a < b$, and let $\varphi : \mathbb{R} \longrightarrow \mathbb{C}$ be a $\mathcal{C}^l$-function such that $supp\,\varphi \subseteq [-1,1]$.*
Let f be the element of $L^2(\mathbb{R})$ whose Fourier transform $\widehat{f}$ is the complex-valued function given by

$$\forall\, p \in \mathbb{R} \qquad \widehat{f}(p) := \varphi\bigg(\frac{2p-(a+b)}{b-a}\bigg)\, e^{-ix_0 p} \ .$$

Then $\widehat{f}$ is a $\mathcal{C}^l$-function on $\mathbb{R}$ supported on the interval $[a,b]$ and f is an analytic function on $\mathbb{R}$ satisfying

$$\forall\, c > 0 \qquad \int_{|x-x_0|\geqslant c} \big|f(x)\big|^2 dx \leqslant \frac{2}{c^2}\,\frac{1}{b-a}\,\big\|\varphi'\big\|^2_{L^2(\mathbb{R})} \ . \tag{10}$$

Proof. Since φ is a $\mathcal{C}^l$-function on $\mathbb{R}$ supported on $[-1,1]$, where $l \geqslant 1$, the Fourier transform $\widehat{f}$ of f is clearly a $\mathcal{C}^l$-function on $\mathbb{R}$ such that

$$supp\,\widehat{f} \subseteq [a,b] \ ;$$

the boundedness of the support of $\widehat{f}$ implies in particular that f is analytic on $\mathbb{R}$.
Now let us prove inequality (10). For this purpose, we apply Chebyshev's inequality to the function f:

$$\int_{|x-x_0|\geqslant c} \big|f(x)\big|^2 dx \leqslant \frac{1}{c^2}\int_{\mathbb{R}} (x-x_0)^2\,\big|f(x)\big|^2 dx = \frac{1}{c^2}\int_{\mathbb{R}} (x-x_0)^2\,\Big|\Big(\mathcal{F}^{-1}_{p\to x}\widehat{f}\Big)(x)\Big|^2 dx \ , \tag{11}$$

for all $c > 0$. Then by a simple substitution, we have for all $x \in \mathbb{R}$,

$$\begin{aligned}\Big(\mathcal{F}^{-1}_{p\to x}\widehat{f}\Big)(x) &= \frac{1}{2\pi}\int_{\mathbb{R}} \varphi\bigg(\frac{2p-(a+b)}{b-a}\bigg)\, e^{i(x-x_0)p}\, dp \\ &= \frac{b-a}{4\pi}\, e^{i\frac{a+b}{2}(x-x_0)}\,\widehat{\varphi}\bigg(\frac{b-a}{2}\,(x_0-x)\bigg) \ .\end{aligned} \tag{12}$$

Putting this into inequality (11) provides for all $c > 0$,

$$\int_{|x-x_0|\geqslant c} \left|f(x)\right|^2 dx \leqslant \frac{1}{c^2} \int_{\mathbb{R}} (x-x_0)^2 \left|\left(\mathcal{F}^{-1}_{p\to x}\widehat{f}\right)(x)\right|^2 dx$$

$$= \frac{1}{c^2}\,\frac{(b-a)^2}{16\pi^2} \int_{\mathbb{R}} (x-x_0)^2 \left|\widehat{\varphi}\left(\frac{b-a}{2}\,(x_0-x)\right)\right|^2 dx \tag{13}$$

$$= \frac{1}{c^2}\,\frac{1}{2\pi^2(b-a)} \int_{\mathbb{R}} \left|y\,\widehat{\varphi}(y)\right|^2 dy \tag{14}$$

$$= \frac{2}{c^2}\,\frac{1}{b-a}\,\left\|\varphi'\right\|^2_{L^2(\mathbb{R})} \tag{15}$$

- (13): use equality (12) ;
- (14): use the substitution $y = \dfrac{b-a}{2}\,(x_0 - x)$;
- (15): use the relation $\widehat{(\varphi')}(y) = i\,y\,\widehat{\varphi}(y)$ and Plancherel's theorem.

The proof is now complete. □

3.2 Remark. For fixed $c > 0$, one can increase the localization of f in the interval $[x_0 - c, x_0 + c]$ by enlarging the support of $\widehat{f}$.

Let us now state the condition that the potential will verify throughout the rest of the paper.

Condition $(\mathcal{P}_{l,[a,b]})$. Let $l \in \mathbb{N}$ and let $a < b$ be two finite real numbers.
An element V of $L^2(\mathbb{R})$ satisfies Condition $(\mathcal{P}_{l,[a,b]})$ if and only if $\widehat{V}$ is a $\mathcal{C}^l$-function on $\mathbb{R}$ which verifies *supp* $\widehat{V} \subseteq [a,b]$.

3.3 Remark.

i) If U is a $\mathcal{C}^l$-function supported on $[a,b]$ then it belongs to $L^2(\mathbb{R})$. Hence, thanks to the fact that $\mathcal{F}_{x\to p} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is bijective, there exists $V \in L^2(\mathbb{R})$ such that $U = \widehat{V}$. In this case, V satisfies Condition $(\mathcal{P}_{l,[a,b]})$ and so the set of functions satisfying Condition $(\mathcal{P}_{l,[a,b]})$ is non-empty.

ii) Under Condition $(\mathcal{P}_{l,[a,b]})$, the function V is actually analytic due to the boundedness of the support of $\widehat{V}$.

iii) If $V \in L^2(\mathbb{R})$ satisfies Condition $(\mathcal{P}_{l,[a,b]})$ then it is bounded on $\mathbb{R}$ as well as its first derivative, thanks to the fact that $\widehat{V}$ is continuous and has a compact support. In particular, the function V belongs to $W^{1,\infty}(\mathbb{R})$ and so the associated operator B defined in Section 2 belongs to $\mathcal{L}\big(H^1(\mathbb{R})\big)$.

iv) An element V of $L^2(\mathbb{R})$ satisfying Condition $(\mathcal{P}_{l,[a,b]})$ is not necessarily real-valued: it is real-valued if and only if $\widehat{V}$ is an even real-valued function. In the real-valued case, the support of $\widehat{V}$ is contained in a symmetric interval centered at the origin; hence we have especially $a = -b$.

To illustrate the above Condition ($\mathcal{P}_{l,[a,b]}$), we construct a family of admissible potentials which are approximately localized in space in an interval centered at a point $x_0 \in \mathbb{R}$ with arbitrary precision if $(b-a)$ is sufficiently large. This is a direct consequence of Lemma 3.1.

3.4 Theorem. *Let $l \geqslant 1$ be an integer, let a, b, x_0 be three finite real numbers such that $a < b$, and let $v : \mathbb{R} \longrightarrow \mathbb{C}$ be a $\mathcal{C}^l$-function such that $\operatorname{supp} v \subseteq [-1, 1]$.*
Let V be the element of $L^2(\mathbb{R})$ whose Fourier transform $\widehat{V}$ is the complex-valued function given by

$$\forall\, p \in \mathbb{R} \qquad \widehat{V}(p) := v\bigg(\frac{2p-(a+b)}{b-a}\bigg)\, e^{-ix_0 p} \,.$$

Then V verifies Condition ($\mathcal{P}_{l,[a,b]}$) and it satisfies

$$\forall\, c > 0 \qquad \int_{|x-x_0|\geqslant c} \big|V(x)\big|^2 dx \leqslant \frac{2}{c^2}\,\frac{1}{b-a}\,\big\|v'\big\|^2_{L^2(\mathbb{R})} \,.$$

Proof. Simple application of Lemma 3.1 to the function V. ☐

4 A stationary phase method with explicit remainder estimates for oscillatory integrals having a quadratic phase

In this section, we aim at providing asymptotic expansions with explicit and uniform remainder estimates of oscillatory integrals of the form

$$\int_{p_1}^{p_2} U(p)\, e^{-i\omega(p-p_0)^2}\, dp \tag{16}$$

where $\omega > 0$ and $p_0 \in \mathbb{R}$. These results are motivated by the fact that such integrals appear in the formulas of the terms of the Dyson-Philipps series, which represents the solution of the Schrödinger equation with potential (7). The study of the time-asymptotic motion of the two first terms $S_1(t)u_0$ and $S_2(t)u_0$ of this series will be based on asymptotic expansions of oscillatory integrals of type (16).
To provide expansions of the oscillatory integral (16) coupled with explicit remainder estimates, we shall adapt the version of the stationary phase method of Erdélyi (see [10, Section 2.9]) to our simple case. We mention that the stationary phase method of Erdélyi covers more general amplitude functions and phase functions; see the paper [3] for a modern formulation of this result and for a complete proof. Moreover we have chosen to restrict the study of the present paper to quadratic phase functions in view of the applications; in particular, the proofs are shorter than those in the general case and we shall deduce the values of the coefficients and the constants more easily.

Before expanding oscillatory integrals of the form (16) in Theorems 4.3 and 4.5, let us establish the following lemma in which useful integral representations of the successive primitives of the function $s \in [0, s_0] \longmapsto e^{-i\omega s} \in \mathbb{C}$, for fixed $s_0, \omega > 0$, are furnished. Moreover their values at the origin and estimates are also provided. These results will be

substantially employed in the proof of Theorems 4.3 and 4.5.
To prove Lemma 4.1, we refer to the paper [3] which gives the successive primitives of more general functions by using essentially complex analysis; see [3, Theorems 6.4, 6.5 and Corollary 6.6].

4.1 Lemma. *Let $\omega, s_0 > 0$ be two real numbers and let $n \geqslant 1$ be an integer. Define the function $\phi_n(.,\omega) : [0, s_0] \longrightarrow \mathbb{C}$ by*

$$\phi_n(s,\omega) := \frac{(-1)^n}{(n-1)!} \int_{\Lambda(s)} (z-s)^{n-1}\, e^{-i\omega z^2}\, dz \ ,$$

where $\Lambda(s)$ is the half-line in the complex plane given by

$$\Lambda(s) := \left\{ s + t\, e^{-i\frac{\pi}{4}} \,\middle|\, t \geqslant 0 \right\} \ .$$

Then

i) the function $\phi_n(.,\omega)$ is the n-th primitive of the function $s \in [0, s_0] \longmapsto e^{-i\omega s^2} \in \mathbb{C}$;

ii) we have

$$\phi_n(0,\omega) = \frac{(-1)^n}{(n-1)!}\, \frac{1}{2}\, \Gamma\Big(\frac{n}{2}\Big)\, e^{-i\frac{\pi}{4}n}\, \omega^{-\frac{n}{2}} \ ,$$

where Γ is the Gamma function;

iii) the function $\phi_n(.,\omega)$ satisfies

$$\big|\phi_n(s,\omega)\big| \leqslant L_n(\delta)\, s^{n-2\delta}\, \omega^{-\delta} \ ,$$

where the real number δ is arbitrarily chosen in $\left(\frac{n}{2}, \frac{n+1}{2}\right)$ and the constant $L_n(\delta) > 0$ depends only on n and δ.

Proof. The function $\phi_n(.,\omega)$ of the present paper corresponds actually to the function $\phi_n^{(2)}(.,\omega,2,1)$ defined in Theorem 2.3 of [3]. Hence we apply the results established in [3] to the present situation:

i) One proves this first point by applying Corollary 6.6 of [3], which is a consequence of Theorems 6.4 and 6.5 of [3], in the case $j = 2$, $\rho_j = 2$ and $\mu_j = 1$.

ii) The proof of this point lies only on basic computations which are carried out in the fourth step of the proof of Theorem 2.3 in [3].

iii) The combination of Lemmas 2.4 and 2.6 of [3] assures this last point.

□

4.2 Remark. Let us note that Lemma 2.6 of [3] claims that $L_n(\delta)$ is actually of the form $L_n(\delta) = a_n\, K_n^{2\delta-n}$, where K_n is the positive solution of the quadratic equation

$$a_n X^2 - b_n X - c_n = 0 \ ,$$

in other words $K_n = \dfrac{b_n + \sqrt{b_n^2 + 4\,a_n c_n}}{2\,a_n}$. Moreover Lemma 2.4 of [3] provides the values of the coefficients, which are recalled just below for the sake of completeness:

$$\begin{cases} a_1 := \dfrac{\sqrt{\pi}}{2} & ; \; b_1 := \dfrac{1}{2} & ; \; c_1 := \dfrac{\sqrt{\pi}}{4} \,, \\ a_n := \dfrac{1}{2}\,\dfrac{1}{(n-1)!}\,\Gamma\Big(\dfrac{n}{2}\Big) & ; \; b_n := \dfrac{1}{4}\,\dfrac{1}{(n-2)!}\,\Gamma\Big(\dfrac{n-1}{2}\Big) & ; \; c_n := \dfrac{1}{4}\,\dfrac{1}{(n-1)!}\,\Gamma\Big(\dfrac{n}{2}\Big) \,, \\ \text{for } n \geqslant 2 \,. \end{cases}$$

Let us now provide asymptotic expansions to one term of oscillatory integrals of the form (16) with explicit and uniform (with respect to p_0) remainder estimates.
To prove the following theorem, we follow the lines of the proof of Erdélyi's stationary phase method and we adapt each step to the present setting in order to provide a shorter and simpler proof. In favour of readability, we distinguish two cases: $p_0 \in (p_1, p_2)$ and $p_0 \notin (p_1, p_2)$. In the first case, we split the integral at the stationary point p_0 and we study separately the two resulting integrals; in the other case, we make a distinction between the cases $p_0 \leqslant p_1$ and $p_2 \leqslant p_0$. In each situation, the method consists firstly in making a substitution to change the phase function into $s \longmapsto -s^2$, secondly in integrating by parts to create the expansion by using Lemma 4.1 i) and ii), and finally in estimating the remainder term thanks to Lemma 4.1 iii).

4.3 Theorem. *Let p_0, p_1 and p_2 be three finite real numbers such that $p_1 < p_2$. Let $U \in \mathcal{C}^1(\mathbb{R}, \mathbb{C})$ such that*

$$supp\, U \subseteq [p_1, p_2] \,.$$

Then we have for all $\omega > 0$,

$$\int_{p_1}^{p_2} U(p)\, e^{-i\omega(p-p_0)^2}\, dp = \sqrt{\pi}\, e^{-i\frac{\pi}{4}}\, U(p_0)\, \omega^{-\frac{1}{2}} \; + \; R_1(\omega, p_0, U) \,,$$

where the remainder term $R_1(\omega, p_0, U)$ is uniformly bounded with respect to p_0 as follows,

$$\big| R_1(\omega, p_0, U) \big| \leqslant C_1(\delta_1)\, (p_2 - p_1)^{2-2\delta_1}\, \big\| U' \big\|_{L^\infty(\mathbb{R})}\, \omega^{-\delta_1} \;;$$

the real number δ_1 is arbitrarily chosen in $\big(\frac{1}{2}, 1\big)$ and the constant $C_1(\delta_1) > 0$ is given in (18).

Proof. In favour of readability, we distinguish two cases.

Case 1: $p_0 \in (p_1, p_2)$
We start this case by splitting the integral at the stationary point p_0:

$$\begin{aligned} \int_{p_1}^{p_2} U(p)\, e^{-i\omega(p-p_0)^2}\, dp \; &= \; \int_{p_1}^{p_0} U(p)\, e^{-i\omega(p-p_0)^2}\, dp \; + \; \int_{p_0}^{p_2} U(p)\, e^{-i\omega(p-p_0)^2}\, dp \\ &= \int_0^{p_0-p_1} U(p_0 - s)\, e^{-i\omega s^2}\, ds \; + \; \int_0^{p_2-p_0} U(p_0 + s)\, e^{-i\omega s^2}\, ds \;; \quad (17) \end{aligned}$$

note that we have made the substitutions $s = p_0 - p$ in the first integral and $s = p - p_0$ in the second one to obtain (17). Let us now integrate by parts by exploiting Lemma 4.1 i):

- $\displaystyle\int_0^{p_0-p_1} U(p_0 - s)\, e^{-i\omega s^2}\, ds \;=\; -U(p_0)\, \phi_1(0,\omega) \;+\; \int_0^{p_0-p_1} U'(p_0 - s)\, \phi_1(s,\omega)\, ds\,;$
- $\displaystyle\int_0^{p_2-p_0} U(p_0 + s)\, e^{-i\omega s^2}\, ds \;=\; -U(p_0)\, \phi_1(0,\omega) \;-\; \int_0^{p_2-p_0} U'(p_0 + s)\, \phi_1(s,\omega)\, ds\,;$

we have used the fact that U vanishes at p_1 and p_2 since $supp\, U \subseteq [p_1, p_2]$ and U is continuous on $\mathbb{R}$. By adding the two last equalities up, it follows

$$\begin{aligned}\int_{p_1}^{p_2} U(p)\, e^{-i\omega(p-p_0)^2}\, dp \;=\; -2\, U(p_0)\, \phi_1(0,\omega) \;+\; \int_0^{p_0-p_1} U'(p_0 - s)\, \phi_1(s,\omega)\, ds \\ -\; \int_0^{p_2-p_0} U'(p_0 + s)\, \phi_1(s,\omega)\, ds\,.\end{aligned}$$

We obtain then, by employing Lemma 4.1 ii),

$$\int_{p_1}^{p_2} U(p)\, e^{-i\omega(p-p_0)^2}\, dp \;=\; \sqrt{\pi}\, e^{-i\frac{\pi}{4}}\, U(p_0)\, \omega^{-\frac{1}{2}} \;+\; R_1(\omega, p_0, U)\,,$$

where we have defined

$$R_1(\omega, p_0, U) := \int_0^{p_0-p_1} U'(p_0 - s)\, \phi_1(s,\omega)\, ds \;-\; \int_0^{p_2-p_0} U'(p_0 + s)\, \phi_1(s,\omega)\, ds\,.$$

We use now Lemma 4.1 iii) to estimate the remainder term:

$$\begin{aligned}\big|R_1(\omega, p_0, U)\big| &\leqslant L_1(\delta_1) \int_0^{p_0-p_1} s^{1-2\delta_1}\, ds\, \big\|U'\big\|_{L^\infty(p_1,p_0)}\, \omega^{-\delta_1} \\ &\quad +\; L_1(\delta_1) \int_0^{p_2-p_0} s^{1-2\delta_1}\, ds\, \big\|U'\big\|_{L^\infty(p_0,p_2)}\, \omega^{-\delta_1} \\ &\leqslant \frac{L_1(\delta_1)}{2-2\delta_1}\, \big\|U'\big\|_{L^\infty(\mathbb{R})} \Big((p_0 - p_1)^{2-2\delta_1} + (p_2 - p_0)^{2-2\delta_1} \Big)\, \omega^{-\delta_1} \\ &\leqslant \frac{L_1(\delta_1)}{1-\delta_1}\, \big\|U'\big\|_{L^\infty(\mathbb{R})}\, (p_2 - p_1)^{2-2\delta_1}\, \omega^{-\delta_1}\,,\end{aligned}$$

with $\delta_1 \in \big(\frac{1}{2}, 1\big)$. We conclude this case by providing the expression of the constant $C_1(\delta_1) > 0$ appearing in the statement of Theorem 4.3. Remark 4.2 assures that $L_1(\delta_1)$ is given by

$$L_1(\delta_1) = a_1\, K_1^{2\delta_1-1} = \frac{\sqrt{\pi}}{2} \left(\frac{1}{2\sqrt{\pi}} + \sqrt{\frac{1}{4\pi} + \frac{1}{2}} \right)^{2\delta_1-1}\,,$$

leading to

$$C_1(\delta_1) := \frac{L_1(\delta_1)}{1-\delta_1} = \frac{\sqrt{\pi}}{2-2\delta_1} \left(\frac{1}{2\sqrt{\pi}} + \sqrt{\frac{1}{4\pi} + \frac{1}{2}} \right)^{2\delta_1-1}\,. \tag{18}$$

Case 2: $p_0 \notin (p_1, p_2)$
In the present situation, we do not split the integral as previously but we make the same changes of variables: $s = p - p_0$ in the case $p_0 \leqslant p_1$ and $s = p_0 - p$ in the other case, leading to

- $$\forall\, p_0 \leqslant p_1 \qquad \int_{p_1}^{p_2} U(p)\, e^{-i\omega(p-p_0)^2}\, dp \;=\; \int_{p_1-p_0}^{p_2-p_0} U(p_0+s)\, e^{-i\omega s^2}\, ds\;;$$
- $$\forall\, p_0 \geqslant p_2 \qquad \int_{p_1}^{p_2} U(p)\, e^{-i\omega(p-p_0)^2}\, dp \;=\; \int_{p_0-p_2}^{p_0-p_1} U(p_0-s)\, e^{-i\omega s^2}\, ds\;.$$

Integrating by parts as above, we obtain

- $$\forall\, p_0 \leqslant p_1 \qquad \int_{p_1-p_0}^{p_2-p_0} U(p_0+s)\, e^{-i\omega s^2}\, ds \;=\; -\int_{p_1-p_0}^{p_2-p_0} U'(p_0+s)\, \phi_1(s,\omega)\, ds =: R_1(\omega, p_0, U)\;;$$
- $$\forall\, p_0 \geqslant p_2 \qquad \int_{p_0-p_2}^{p_0-p_1} U(p_0-s)\, e^{-i\omega s^2}\, ds \;=\; \int_{p_0-p_2}^{p_0-p_1} U'(p_0-s)\, \phi_1(s,\omega)\, ds =: R_1(\omega, p_0, U)\;;$$

in the present situation, the boundary terms are equal to 0 due to the fact that $U(p_1) = U(p_2) = 0$. In particular, the first term of the asymptotic expansion is equal to 0.
To finish, we estimate the remainder $R_1(\omega, p_0, U)$ by employing the same method as above:

- $\forall\, p_0 \leqslant p_1$
$$\begin{aligned} \big|R_1(\omega,p_0,U)\big| &\leqslant L_1(\delta_1) \int_{p_1-p_0}^{p_2-p_0} s^{1-2\delta_1}\, ds\, \big\|U'\big\|_{L^\infty(\mathbb{R})}\, \omega^{-\delta_1} \\ &= \frac{L_1(\delta_1)}{2-2\delta_1} \Big((p_2-p_0)^{2-2\delta_1} - (p_1-p_0)^{2-2\delta_1} \Big) \big\|U'\big\|_{L^\infty(\mathbb{R})}\, \omega^{-\delta_1} \\ &\leqslant \frac{L_1(\delta_1)}{2-2\delta_1}\, (p_2-p_1)^{2-2\delta_1}\, \big\|U'\big\|_{L^\infty(\mathbb{R})}\, \omega^{-\delta_1} \qquad (19) \\ &\leqslant C_1(\delta_1)\, (p_2-p_1)^{2-2\delta_1}\, \big\|U'\big\|_{L^\infty(\mathbb{R})}\, \omega^{-\delta_1}\;; \qquad (20) \end{aligned}$$
- $\forall\, p_0 \geqslant p_2$
$$\begin{aligned} \big|R_1(\omega,p_0,U)\big| &\leqslant L_1(\delta_1) \int_{p_0-p_2}^{p_0-p_1} s^{1-2\delta}\, ds\, \big\|U'\big\|_{L^\infty(\mathbb{R})}\, \omega^{-\delta_1} \\ &= \frac{L_1(\delta_1)}{2-2\delta_1} \Big((p_0-p_1)^{2-2\delta_1} - (p_0-p_2)^{2-2\delta_1} \Big) \big\|U'\big\|_{L^\infty(\mathbb{R})}\, \omega^{-\delta_1} \\ &\leqslant \frac{L_1(\delta_1)}{2-2\delta_1}\, (p_2-p_1)^{2-2\delta_1}\, \big\|U'\big\|_{L^\infty(\mathbb{R})}\, \omega^{-\delta_1} \qquad (21) \\ &\leqslant C_1(\delta_1)\, (p_2-p_1)^{2-2\delta_1}\, \big\|U'\big\|_{L^\infty(\mathbb{R})}\, \omega^{-\delta_1}\;; \qquad (22) \end{aligned}$$

note that we have used Lemma 4.4 (see below) to obtain inequalities (19) and (21), and the fact that $\dfrac{L_1(\delta_1)}{2-2\delta} = \dfrac{1}{2}\, C_1(\delta_1)$ to obtain inequalites (20) and (22). □

4.4 Lemma. *Let $\mu \in (0,1]$ and let $x \geqslant y$ be two non-negative real numbers. Then we have*
$$x^\mu - y^\mu \leqslant (x-y)^\mu\;.$$

Proof. Since the case $\mu = 1$ and the case $y = 0$ are trivial, we assume $\mu < 1$ and $y > 0$. Then the above inequality is equivalent to

$$\left(\frac{x}{y}\right)^{\mu} - 1 \leqslant \left(\frac{x}{y} - 1\right)^{\mu} .$$

We define the function $h : [1, +\infty) \longrightarrow \mathbb{R}$ by $h(t) := (t-1)^{\mu} - t^{\mu} + 1$, which satisfies

$$\forall\, t > 1 \qquad h'(t) = \mu\left((t-1)^{\mu-1} - t^{\mu-1}\right) \geqslant 0 ,$$

since $\mu - 1 < 0$. Hence $h(t) \geqslant h(1) = 0$ for all $t \geqslant 1$, proving the lemma. □

In the following result, we furnish an asymptotic expansion to two terms of oscillatory integrals of type (16): this provides in particular a remainder term having a higher decay rate than previously. This fact will be actually useful to control uniformly an oscillatory integral of type (16) in the proof of Proposition 6.4.
The proof of the following theorem follows the same steps as those of the proof of Theorem 4.3 but, here, we integrate by parts four times.

4.5 Theorem. *Let p_0, p_1 and p_2 be three finite real numbers such that $p_1 < p_2$. Let $U \in \mathcal{C}^4(\mathbb{R}, \mathbb{C})$ such that*

$$supp\, U \subseteq [p_1, p_2] .$$

Then we have for all $\omega > 0$,

$$\int_{p_1}^{p_2} U(p)\, e^{-i\omega(p-p_0)^2}\, dp = \sqrt{\pi}\, e^{-i\frac{\pi}{4}}\, U(p_0)\, \omega^{-\frac{1}{2}} \; + \; \frac{\sqrt{\pi}}{4}\, e^{-i\frac{3\pi}{4}}\, U''(p_0)\, \omega^{-\frac{3}{2}} \; + \; R_2(\omega, p_0, U) ,$$

where the remainder term $R_2(\omega, p_0, U)$ is uniformly bounded with respect to p_0 as follows,

$$\left|R_2(\omega, p_0, U)\right| \leqslant C_2(\delta_2)\, (p_2 - p_1)^{5-2\delta_2}\, \left\|U^{(4)}\right\|_{L^\infty(\mathbb{R})} \omega^{-\delta_2} ;$$

the real number δ_2 is arbitrarily chosen in $\left(2, \frac{5}{2}\right)$ and the constant $C_2(\delta_2) > 0$ is given in (23)*.*

Proof. As in the proof of Theorem 4.3, we distinguish two cases.

Case 1: $p_0 \in (p_1, p_2)$
We consider equality (17) from the proof of Theorem 4.3, which holds also in the present situation. Here we integrate by parts four times each integral in (17) by exploiting Lemma 4.1 i):

- $$\int_0^{p_0-p_1} U(p_0 - s)\, e^{-i\omega s^2}\, ds \;=\; -U(p_0)\, \phi_1(0,\omega) \;-\; U'(p_0)\, \phi_2(0,\omega) \;-\; U''(p_0)\, \phi_3(0,\omega) \;-\; U^{(3)}(p_0)\, \phi_4(0,\omega) \;+\; \int_0^{p_0-p_1} U^{(4)}(p_0 - s)\, \phi_4(s,\omega)\, ds ;$$

- $$\int_0^{p_2-p_0} U(p_0 + s)\, e^{-i\omega s^2}\, ds \;=\; -U(p_0)\, \phi_1(0,\omega) \;+\; U'(p_0)\, \phi_2(0,\omega) \;-\; U''(p_0)\, \phi_3(0,\omega) \;+\; U^{(3)}(p_0)\, \phi_4(0,\omega) \;+\; \int_0^{p_2-p_0} U^{(4)}(p_0 + s)\, \phi_4(s,\omega)\, ds ;$$

in the present situation, the fact that $U^{(k)}(p_j) = 0$, for $k \in \{0,1,2,3\}$ and $j \in \{1,2\}$, has been used. Then adding the two preceding equalities up and using Lemma 4.1 ii) gives

$$\int_{p_1}^{p_2} U(p)\, e^{-i\omega(p-p_0)^2}\, dp \;=\; \sqrt{\pi}\, e^{-i\frac{\pi}{4}}\, U(p_0)\, \omega^{-\frac{1}{2}} \;+\; \frac{\sqrt{\pi}}{4}\, e^{-i\frac{3\pi}{4}}\, U''(p_0)\, \omega^{-\frac{3}{2}} \;+\; R_2(\omega, p_0, U)\ ,$$

where we have defined

$$R_2(\omega, p_0, U) := \int_0^{p_0-p_1} U^{(4)}(p_0 - s)\, \phi_4(s,\omega)\, ds \;+\; \int_0^{p_2-p_0} U^{(4)}(p_0 + s)\, \phi_4(s,\omega)\, ds\ .$$

To estimate this remainder term, we follow the same steps as those used to estimate $R_1(\omega, p_0, U)$ in the proof of Theorem 4.3, case $p_0 \in (p_1, p_2)$, leading to

$$\big|R_2(\omega, p_0, U)\big| \leqslant \frac{2\, L_4(\delta_2)}{5 - 2\delta_2}\, \big\|U^{(4)}\big\|_{L^\infty(\mathbb{R})}\, (p_2 - p_1)^{5-2\delta_2}\, \omega^{-\delta_2}\ ,$$

where $\delta_2 \in \left(2, \frac{5}{2}\right)$; here we have used Lemma 4.1 iii) in the case $n = 4$. Let us now give the expression of the constant $C_2(\delta_2) > 0$; by Remark 4.2, we have

$$L_4(\delta_2) = a_4\, K_4^{2\delta_2-4} = \frac{1}{12}\left(\frac{3\sqrt{\pi}}{8} + \sqrt{\frac{9\pi}{64} + \frac{1}{2}}\right)^{2\delta_2-4}\ ,$$

which leads to

$$C_2(\delta_2) := \frac{2\, L_4(\delta_2)}{5 - 2\delta_2} = \frac{1}{30 - 12\delta_2}\left(\frac{3\sqrt{\pi}}{8} + \sqrt{\frac{9\pi}{64} + \frac{1}{2}}\right)^{2\delta_2-4}\ . \tag{23}$$

Case 2: $p_0 \notin (p_1, p_2)$
Let us recall that we have

- $\forall\, p_0 \leqslant p_1 \qquad \displaystyle\int_{p_1}^{p_2} U(p)\, e^{-i\omega(p-p_0)^2}\, dp \;=\; \int_{p_1-p_0}^{p_2-p_0} U(p_0 + s)\, e^{-i\omega s^2}\, ds\ ;$
- $\forall\, p_0 \geqslant p_2 \qquad \displaystyle\int_{p_1}^{p_2} U(p)\, e^{-i\omega(p-p_0)^2}\, dp \;=\; \int_{p_0-p_2}^{p_0-p_1} U(p_0 - s)\, e^{-i\omega s^2}\, ds\ .$

Integrating by parts four times as above and noting that the boundary terms are equal to 0, we obtain

- $\forall\, p_0 \leqslant p_1 \qquad \displaystyle\int_{p_1-p_0}^{p_2-p_0} U(p_0 + s)\, e^{-i\omega s^2}\, ds \;=\; \int_{p_1-p_0}^{p_2-p_0} U^{(4)}(p_0 + s)\, \phi_4(s,\omega)\, ds \;=:\; R_2(\omega, p_0, U)\ ;$
- $\forall\, p_0 \geqslant p_2 \qquad \displaystyle\int_{p_0-p_2}^{p_0-p_1} U(p_0 - s)\, e^{-i\omega s^2}\, ds \;=\; \int_{p_0-p_2}^{p_0-p_1} U^{(4)}(p_0 - s)\, \phi_4(s,\omega)\, ds \;=:\; R_2(\omega, p_0, U)\ .$

To conclude the proof, we estimate $R_2(\omega, p_0, U)$ by employing the same method as the one used to estimate $R_1(\omega, p_0, U)$ in the proof of Theorem 4.3, case $p_0 \notin (p_1, p_2)$, and Lemma 4.1 iii), case $n = 4$; hence we obtain

- $\forall\, p_0 \leqslant p_1$
$$\begin{aligned}\big|R_2(\omega, p_0, U)\big| &\leqslant \frac{L_4(\delta_2)}{5-2\delta_2}\,(p_2-p_1)^{5-2\delta_2}\,\big\|U^{(4)}\big\|_{L^\infty(\mathbb{R})}\,\omega^{-\delta_2}\\ &\leqslant C_2(\delta_2)\,(p_2-p_1)^{5-2\delta_2}\,\big\|U^{(4)}\big\|_{L^\infty(\mathbb{R})}\,\omega^{-\delta_2}\ ;\end{aligned} \tag{24}$$

- $\forall\, p_0 \geqslant p_2$
$$\begin{aligned}\big|R_2(\omega, p_0, U)\big| &\leqslant \frac{L_4(\delta_2)}{5-2\delta_2}\,(p_2-p_1)^{5-2\delta_2}\,\big\|U^{(4)}\big\|_{L^\infty(\mathbb{R})}\,\omega^{-\delta_2}\\ &\leqslant C_2(\delta_2)\,(p_2-p_1)^{5-2\delta_2}\,\big\|U^{(4)}\big\|_{L^\infty(\mathbb{R})}\,\omega^{-\delta_2}\ ;\end{aligned} \tag{25}$$

note that we have used the fact that $\dfrac{L_4(\delta_2)}{5-2\delta_2} = \dfrac{1}{2}\,C_2(\delta_2)$ to obtain inequalites (24) and (25). □

In order to have convenient estimates which will be easily applicable in the next sections, we propose the following corollary whose estimates are direct consequences of Theorems 4.3 and 4.5.

4.6 Corollary. *Suppose that the hypotheses of Theorem 4.3 are satisfied, and fix $\delta_1 \in \big(\frac{1}{2}, 1\big)$ and $\delta_2 \in \big(2, \frac{5}{2}\big)$. Then*

i) we have for all $\omega \geqslant 1$ and $p_0 \in \mathbb{R}$,
$$\left|\int_{p_1}^{p_2} U(p)\, e^{-i\omega(p-p_0)^2}\, dp\right| \leqslant C_1(\delta_1, p_1, p_2)\,\Big(\big\|U\big\|_{L^\infty(\mathbb{R})} + \big\|U'\big\|_{L^\infty(\mathbb{R})}\Big)\,\omega^{-\frac{1}{2}}\ ;$$

ii) we have for all $\omega > 0$ and $p_0 \in [p_1, p_2]$,
$$\begin{aligned}&\left|\int_{p_1}^{p_2} U(p)\, e^{-i\omega(p-p_0)^2}\, dp\ -\ \sqrt{\pi}\, e^{-i\frac{\pi}{4}}\, U(p_0)\,\omega^{-\frac{1}{2}}\right|\\ &\qquad\leqslant C_2(\delta_1, p_1, p_2)\,\big\|U'\big\|_{L^\infty(\mathbb{R})}\,\omega^{-\delta_1}\ ;\end{aligned}$$

iii) we have for all $\omega > 0$ and $p_0 \notin [p_1, p_2]$,
$$\left|\int_{p_1}^{p_2} U(p)\, e^{-i\omega(p-p_0)^2}\, dp\right| \leqslant C_2(\delta_1, p_1, p_2)\,\big\|U'\big\|_{L^\infty(\mathbb{R})}\,\omega^{-\delta_1}\ ;$$

iv) if we suppose in addition that $U \in \mathcal{C}^4(\mathbb{R}, \mathbb{C})$, then we have for all $\omega > 0$ and $p_0 \notin [p_1, p_2]$,
$$\left|\int_{p_1}^{p_2} U(p)\, e^{-i\omega(p-p_0)^2}\, dp\right| \leqslant C_3(\delta_2, p_1, p_2)\,\big\|U^{(4)}\big\|_{L^\infty(\mathbb{R})}\,\omega^{-\delta_2}\ ;$$

the constants $C_1(\delta_1, p_1, p_2)$, $C_2(\delta_1, p_1, p_2)$, $C_3(\delta_2, p_1, p_2) > 0$ are respectively given in (26), (27) and (28).

Proof. The points i), ii) and iii) are direct consequences of Theorem 4.3 while the point iv) is derived from Theorem 4.5. Let us provide the values of the constants:

- $$C_1(\delta_1, p_1, p_2) := \sqrt{\pi} + C_1(\delta_1)\,(p_2 - p_1)^{2-2\delta_1}$$
$$= \sqrt{\pi} + \frac{\sqrt{\pi}}{2-2\delta_1}\left(\frac{1}{2\sqrt{\pi}} + \sqrt{\frac{1}{4\pi} + \frac{1}{2}}\right)^{2\delta_1 - 1}(p_2 - p_1)^{2-2\delta_1}\ ; \qquad (26)$$
- $$C_2(\delta_1, p_1, p_2) := C_1(\delta_1)\,(p_2 - p_1)^{2-2\delta_1}$$
$$= \frac{\sqrt{\pi}}{2-2\delta_1}\left(\frac{1}{2\sqrt{\pi}} + \sqrt{\frac{1}{4\pi} + \frac{1}{2}}\right)^{2\delta_1 - 1}(p_2 - p_1)^{2-2\delta_1}\ ; \qquad (27)$$
- $$C_3(\delta_2, p_1, p_2) := C_2(\delta_2)\,(p_2 - p_1)^{5-2\delta_2}$$
$$= \frac{1}{30-12\delta_2}\left(\frac{3\sqrt{\pi}}{8} + \sqrt{\frac{9\pi}{64} + \frac{1}{2}}\right)^{2\delta_2 - 4}(p_2 - p_1)^{5-2\delta_2}\ . \qquad (28)$$

□

5 Time-asymptotic behaviour of the free solution

The aim of this section is to expand and to estimate in space-time cones the term $S_1(t)u_0$ of the Dyson-Phillips series given in Theorem 2.2, when the initial datum u_0 is supposed to be in a bounded frequency band (we recall that *frequency* refers to *Fourier-frequency* throughout the rest of the paper). This will allow a precise description of the time-asymptotic motion of $S_1(t)u_0$.

We start by recalling the definition of a space-time cone and a space-time direction:

5.1 Definition. *Let $\overline{p}, p_1, p_2$ be three finite real numbers such that $p_1 < p_2$.*

i) We define the space-time cone $\mathfrak{C}(p_1, p_2)$ as follows:

$$\mathfrak{C}(p_1, p_2) := \left\{ (t, x) \in (0, +\infty) \times \mathbb{R} \,\middle|\, 2\,p_1 \leqslant \frac{x}{t} \leqslant 2\,p_2 \right\}\ .$$

Let $\mathfrak{C}(p_1, p_2)^c$ be the complement of the cone $\mathfrak{C}(p_1, p_2)$ in $(0, +\infty) \times \mathbb{R}$.

ii) We define the space-time direction $\mathfrak{D}(\overline{p})$ as follows:

$$\mathfrak{D}(\overline{p}) := \left\{ (t, x) \in (0, +\infty) \times \mathbb{R} \,\middle|\, \frac{x}{t} = 2\,\overline{p} \right\}\ .$$

As explained in Section 0, the initial datum is supposed to satisfy the following condition:

Condition $(\mathcal{I}_{k,[p_1,p_2]})$. Let $k \in \mathbb{N}$ and let $p_1 < p_2$ be two finite real numbers.
An element u_0 of $H^3(\mathbb{R})$ satisfies Condition $(\mathcal{I}_{k,[p_1,p_2]})$ if and only if $\widehat{u_0}$ is a $\mathcal{C}^k$-function on $\mathbb{R}$ which verifies *supp* $\widehat{u_0} \subseteq [p_1, p_2]$.

5.2 Remark. i) One proves that the set of elements of $H^3(\mathbb{R})$ satisfying Condition $(\mathcal{I}_{k,[p_1,p_2]})$ is non-empty by using the following argument:
If $U : \mathbb{R} \longrightarrow \mathbb{C}$ is a $\mathcal{C}^\infty$-function supported on $[p_1, p_2]$ then there exists $u_0 \in \mathcal{S}(\mathbb{R})$, where $\mathcal{S}(\mathbb{R})$ is the Schwartz space, such that $\widehat{u_0} = U$ since the Fourier transform is a bijection on $\mathcal{S}(\mathbb{R})$. In particular the function u_0 belongs to $H^3(\mathbb{R})$ and it satisfies Condition $(\mathcal{I}_{k,[p_1,p_2]})$.

ii) Since the support of $\widehat{u_0}$ is contained in a bounded interval under Condition $(\mathcal{I}_{k,[p_1,p_2]})$, u_0 is an analytic function on $\mathbb{R}$.

iii) If $u_0 \in H^3(\mathbb{R})$ satisfies Condition $(\mathcal{I}_{k,[p_1,p_2]})$ then its Fourier transform is an integrable function on $\mathbb{R}$. Hence for $t \geqslant 0$, the term $S_1(t)u_0$, defined in Theorem 2.2, is actually a complex-valued function on $\mathbb{R}$ given by

$$\forall\, x \in \mathbb{R} \qquad \big(S_1(t)u_0\big)(x) = \frac{1}{2\pi} \int_{p_1}^{p_2} \widehat{u_0}(p)\, e^{-itp^2+ixp}\, dp\,. \tag{29}$$

In Theorem 5.3, we provide a time-asymptotic expansion to one term of $S_1(t)u_0$ with uniform remainder estimate inside the space-time cone $\mathfrak{C}(p_1, p_2)$ and a uniform estimate outside this cone. A L^∞-norm estimate is provided to show that our method permits also to establish estimates in the whole space-time.
The method lies on a combination of a rewriting of the solution as an oscillatory integral of type (16) and an application of Corollary 4.6.

5.3 Theorem. *Suppose that $u_0 \in H^3(\mathbb{R})$ satisfies Condition $(\mathcal{I}_{k,[p_1,p_2]})$, where $k \geqslant 1$, and fix $\delta_1 \in \left(\frac{1}{2}, 1\right)$. Then we have*

i) for all $t \geqslant 1$,

$$\big\|S_1(t)u_0\big\|_{L^\infty(\mathbb{R})} \leqslant \tilde{C}_1(\delta_1, p_1, p_2) \left(\big\|\widehat{u_0}\big\|_{L^\infty(\mathbb{R})} + \big\|\widehat{u_0}'\big\|_{L^\infty(\mathbb{R})}\right) t^{-\frac{1}{2}}\ ;$$

ii) for all $(t, x) \in \mathfrak{C}\big(p_1, p_2\big)$,

$$\left|\big(S_1(t)u_0\big)(x) - \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, \widehat{u_0}\Big(\frac{x}{2t}\Big)\, t^{-\frac{1}{2}}\right| \leqslant \tilde{C}_2(\delta_1, p_1, p_2)\, \big\|\widehat{u_0}'\big\|_{L^\infty(\mathbb{R})}\, t^{-\delta_1}\ ;$$

iii) for all $(t, x) \in \mathfrak{C}\big(p_1, p_2\big)^c$,

$$\Big|\big(S_1(t)u_0\big)(x)\Big| \leqslant \tilde{C}_2(\delta_1, p_1, p_2)\, \big\|\widehat{u_0}'\big\|_{L^\infty(\mathbb{R})}\, t^{-\delta_1}\ ;$$

the constants $\tilde{C}_1(\delta_1, p_1, p_2)$, $\tilde{C}_2(\delta_1, p_1, p_2) > 0$ are respectively given in (30) *and* (31).

Proof. We start the proof by factorizing the phase function $p \longmapsto -tp^2 + xp$ by t in the formula (29), giving

$$\big(S_1(t)u_0\big)(x) = \frac{1}{2\pi} \int_{p_1}^{p_2} \widehat{u_0}(p)\, e^{it\left(-p^2 + \frac{x}{t}\,p\right)}\, dp = \int_{p_1}^{p_2} \frac{1}{2\pi}\, \widehat{u_0}(p)\, e^{-it\left(p - \frac{x}{2t}\right)^2}\, dp\, e^{i\frac{x^2}{4t}}\ ,$$

for all $(t,x) \in (0,+\infty) \times \mathbb{R}$. Hence we observe that the term $\big(S_1(t)u_0\big)(x)$ is actually an oscillatory integral of the form

$$\int_{p_1}^{p_2} U(p)\, e^{-i\omega(p-p_0)^2}\, dp\, e^{i\omega p_0^2} ,$$

where $\omega = t$, $p_0 = \frac{x}{2t}$ and $U = \frac{1}{2\pi}\,\widehat{u_0}$; in particular, one remarks that $U \in \mathcal{C}^1(\mathbb{R},\mathbb{C})$ and

$$p_0 \in [p_1,p_2] \qquad \Longleftrightarrow \qquad (t,x) \in \mathfrak{C}\big(p_1,p_2\big) .$$

We are now in position to prove the three points given in the statement of the theorem.

i) if $t \geqslant 1$ and $x \in \mathbb{R}$, then $p_0 \in \mathbb{R}$ and hence the hypotheses of Corollary 4.6 i) are satisfied. Then it follows

$$\begin{aligned}\Big|\big(S_1(t)u_0\big)(x)\Big| &\leqslant C_1(\delta_1,p_1,p_2)\left(\Big\|\frac{1}{2\pi}\,\widehat{u_0}\Big\|_{L^\infty(\mathbb{R})} + \Big\|\frac{1}{2\pi}\,\widehat{u_0}'\Big\|_{L^\infty(\mathbb{R})}\right) t^{-\frac{1}{2}} \\ &= \tilde{C}_1(\delta_1,p_1,p_2)\left(\big\|\widehat{u_0}\big\|_{L^\infty(\mathbb{R})} + \big\|\widehat{u_0}'\big\|_{L^\infty(\mathbb{R})}\right) t^{-\frac{1}{2}} ,\end{aligned}$$

where δ_1 is arbitrarily chosen in $\big(\frac{1}{2},1\big)$ and

$$\begin{aligned}\tilde{C}_1(\delta_1,p_1,p_2) &:= \frac{C_1(\delta_1,p_1,p_2)}{2\pi} \\ &= \frac{1}{2\sqrt{\pi}} + \frac{1}{(4-4\delta_1)\sqrt{\pi}}\left(\frac{1}{2\sqrt{\pi}} + \sqrt{\frac{1}{4\pi} + \frac{1}{2}}\right)^{2\delta_1-1} (p_2-p_1)^{2-2\delta_1} .\end{aligned} \tag{30}$$

ii) if $(t,x) \in \mathfrak{C}(p_1,p_2)$ then $p_0 \in [p_1,p_2]$. Therefore Corollary 4.6 ii) is applicable:

$$\begin{aligned}&\left|\big(S_1(t)u_0\big)(x) \;-\; \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, \widehat{u_0}\Big(\frac{x}{2t}\Big)\, t^{-\frac{1}{2}}\right| \\ &\qquad = \left|\int_{p_1}^{p_2} \frac{1}{2\pi}\,\widehat{u_0}(p)\, e^{-it\left(p-\frac{x}{2t}\right)^2}\, dp \;-\; \sqrt{\pi}\, e^{-i\frac{\pi}{4}}\, \frac{1}{2\pi}\,\widehat{u_0}\Big(\frac{x}{2t}\Big)\, t^{-\frac{1}{2}}\right| \\ &\qquad \leqslant C_2(\delta_1,p_1,p_2)\, \Big\|\frac{1}{2\pi}\,\widehat{u_0}'\Big\|_{L^\infty(\mathbb{R})} t^{-\delta_1} ;\end{aligned}$$

the real number δ_1 is arbitrarily chosen in $\big(\frac{1}{2},1\big)$ and we define

$$\begin{aligned}\tilde{C}_2(\delta_1,p_1,p_2) &:= \frac{C_2(\delta_1,p_1,p_2)}{2\pi} \\ &= \frac{1}{(4-4\delta_1)\sqrt{\pi}}\left(\frac{1}{2\sqrt{\pi}} + \sqrt{\frac{1}{4\pi} + \frac{1}{2}}\right)^{2\delta_1-1} (p_2-p_1)^{2-2\delta_1} .\end{aligned} \tag{31}$$

iii) if $(t,x) \in \mathfrak{C}(p_1,p_2)^c$ then $p_0 \notin [p_1,p_2]$. Here we apply Corollary 4.6 iii), leading to

$$\begin{aligned}\Big|\big(S_1(t)u_0\big)(x)\Big| &\leqslant C_2(\delta_1,p_1,p_2)\, \Big\|\frac{1}{2\pi}\,\widehat{u_0}'\Big\|_{L^\infty(\mathbb{R})} t^{-\delta_1} \\ &= \tilde{C}_2(\delta_1,p_1,p_2)\, \big\|\widehat{u_0}'\big\|_{L^\infty(\mathbb{R})}\, t^{-\delta_1} ,\end{aligned}$$

where δ_1 and $\tilde{C}_2(\delta_1,p_1,p_2)$ are defined just above.

□

5.4 Remark. The following time-asymptotic expansion to one term of $S_1(t)u_0$,

$$\big(S_1(t)u_0\big)(x) = \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, \widehat{u_0}\Big(\frac{x}{2t}\Big)\, t^{-\frac{1}{2}} \;+\; R(t,x,u_0)\, t^{-\delta} \ ,$$

for all $(t,x) \in (0,+\infty) \times \mathbb{R}$, with uniform remainder estimate,

$$\forall\, (t,x) \in (0,+\infty) \times \mathbb{R} \qquad \big|R(t,x,u_0)\big| \leqslant \tilde{C}_2(\delta_1,p_1,p_2)\, \big\|\widehat{u_0}'\big\|_{L^\infty(\mathbb{R})} \ ,$$

is actually equivalent to the points ii) and iii) of Theorem 5.3.

A direct consequence of the preceding theorem is the fact that the time-decay rate $t^{-\frac{1}{2}}$ is attained along certain space-time directions inside the cone $\mathfrak{C}(p_1,p_2)$.

5.5 Corollary. *Suppose that the hypotheses of Theorem 5.3 are satisfied. Then the time-decay rate $t^{-\frac{1}{2}}$ for the term $S_1(t)u_0$ is attained along any space-time direction $\mathfrak{D}(\overline{p})$ such that $\overline{p} \in \mathbb{R}$ satisfies $\big|\widehat{u_0}(\overline{p})\big| > 0$. For such a real number $\overline{p}$, we have especially $\mathfrak{D}(\overline{p}) \subset \mathfrak{C}(p_1,p_2)$.*

Proof. Choose $\overline{p} \in \mathbb{R}$ such that $\big|\widehat{u_0}(\overline{p})\big| > 0$; in particular, it belongs to *supp* $\widehat{u_0} \subseteq [p_1,p_2]$, which assures $\mathfrak{D}(\overline{p}) \subset \mathfrak{C}(p_1,p_2)$. Then for all $(t,x) \in \mathfrak{D}(\overline{p})$, the modulus of the coefficient of the expansion given in Theorem 5.3 ii) is equal to

$$\left|\frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{it\overline{p}^2}\, \widehat{u_0}(\overline{p})\right| = \frac{1}{2\sqrt{\pi}}\big|\widehat{u_0}(\overline{p})\big| \ .$$

This quantity is independent from t and different from 0, assuring that $t^{-\frac{1}{2}}$ is attained along $\mathfrak{D}(\overline{p})$. □

5.6 Remark. i) For an initial datum $u_0 \in H^3(\mathbb{R})$ satisfying Condition *($\mathcal{I}_{k,[p_1,p_2]}$)*, the existence of a real number $\overline{p}$ satisfying $\big|\widehat{u_0}(\overline{p})\big| > 0$ is assured if $u_0 \neq 0$.

ii) The preceding corollary implies especially that the time-decay rate $t^{-\frac{1}{2}}$ is optimal in the union of all $\mathfrak{D}(\overline{p})$ such that $\overline{p}$ verifies $\big|\widehat{u_0}(\overline{p})\big| > 0$. Since $S_1(t)u_0$ tends faster to 0 in the other space-time regions, we deduce that it is time-asymptotically localized in the above union of space-time directions, which is contained in the cone $\mathfrak{C}(p_1,p_2)$. According to this observation, the explicit knowledge of the set

$$\Big\{p \in \mathbb{R} \,\Big|\, \big|\widehat{u_0}(p)\big| > 0\Big\}$$

permits to describe precisely the time-asymptotic localization of the term $S_1(t)u_0$. For example, under the hypotheses of Theorem 5.3, if we suppose in addition that

$$\forall\, p \in (p_1,p_2) \qquad \big|\widehat{u_0}(p)\big| > 0 \ ,$$

then the term $S_1(t)u_0$ is localized in the whole cone $\mathfrak{C}(p_1,p_2)$, excluding its boundaries, as the time tends to infinity.

6 Time-asymptotic behaviour of the wave packet issued by the first interaction with the potential

The aim of this section is to obtain similar results to those of the preceding section for the second term $S_2(t)u_0$ of the Dyson-Phillips series representing the solution of the Schrödinger equation with potential (7). The time-asymptotic propagation features that we shall derive for this term permit in particular to deduce information on the influence of the frequencies of the potential on time-asymptotic motions of wave packets.
We shall begin with a rewriting of $S_2(t)u_0$ as an oscillatory integral with respect to time in order to apply the method employed in the preceding section. It will turn out that the resulting amplitude depends on time, preventing us from concluding on the optimality of the time-decay rate $t^{-\frac{1}{2}}$. Consequently we shall carry out a precise study of this amplitude to extract its time-independent part, permitting to prove the optimality of $t^{-\frac{1}{2}}$.

In the first proposition of the present section, we rewrite the integral representing the second term $S_2(t)u_0$ as an oscillatory integral of type (16); let us note that the resulting amplitude depends on time. In Theorem 6.3, we shall use this rewriting to expand time-asymptotically $S_2(t)u_0$ in space-time cones by following the lines of the proof of Theorem 5.3; we note that additional calculations will be necessary to control uniformly the time-dependent amplitude.
The approach to prove Proposition 6.1 is based on the explicit formula of $S_2(t)u_0$, which is given in Theorem 2.2 and which involves the Fourier transform. Applications of basic properties of the Fourier transform and of Fubini's theorem lead to the result. We remark that the amplitude of the resulting oscillatory integral is defined through a convolution between $\widehat{V}(.)$ and $\widehat{u_0}(.)\, e^{-i\tau .^2}$. Since these two functions are compactly supported, the amplitude has also a bounded support.

6.1 Proposition. *Suppose that $u_0 \in H^3(\mathbb{R})$ and $V \in L^2(\mathbb{R})$ satisfy respectively Condition $(\mathcal{I}_{k,[p_1,p_2]})$ and Condition $(\mathcal{P}_{l,[a,b]})$ where $k, l \geqslant 0$. Let $t \geqslant 0$ and let $W(t,.) : \mathbb{R} \longrightarrow \mathbb{C}$ be the function defined by*

$$W(t,p) := -i \int_0^t \widetilde{W}(\tau,p)\, e^{i\tau p^2}\, d\tau \ ,$$

where the function $\widetilde{W}(.,.) : [0,t] \times \mathbb{R} \longrightarrow \mathbb{C}$ is defined by

$$\widetilde{W}(\tau,p) := \Big(\widehat{V} * \big(e^{-i\tau .^2}\, \widehat{u_0}(.) \big) \Big)(p) = \int_a^b \widehat{V}(y)\, \widehat{u_0}(p-y)\, e^{-i\tau(p-y)^2}\, dy \ .$$

Then we have $supp\, W(t,.) \subseteq [p_1 + a, p_2 + b]$ for all $t \geqslant 0$, and

$$\forall\, (t,x) \in \mathbb{R}_+ \times \mathbb{R} \qquad \big(S_2(t)u_0\big)(x) = \frac{1}{2\pi} \int_{p_1+a}^{p_2+b} W(t,p)\, e^{-itp^2 + ixp}\, dp \ .$$

Proof. Let $t \geqslant 0$ and $x \in \mathbb{R}$. By using equality (8) from Remark 2.3 (ii) and the Fourier representation of $S_1(\tau) = T(\tau)$, we obtain

$$\begin{aligned}
\big(S_2(t)u_0\big)(x) &= \int_0^t \big(T(t-\tau)\, B\, T(\tau)u_0\big)(x)\, d\tau \\
&= -i \int_0^t \mathcal{F}^{-1}_{p\to x}\Big(e^{-i(t-\tau)\,p^2} \mathcal{F}_{x\to p}\Big(V(x)\mathcal{F}^{-1}_{p\to x}\big(e^{-i\tau\,p^2}\, \widehat{u_0}(p)\big)(x)\Big)(p)\Big)(x)\, d\tau \\
&= -i \int_0^t \mathcal{F}^{-1}_{p\to x}\Big(e^{-i(t-\tau)\,p^2}\Big(\widehat{V} * \big(e^{-i\tau\,.^2}\, \widehat{u_0}(.)\big)\Big)(p)\Big)(x)\, d\tau \\
&= -i \int_0^t \mathcal{F}^{-1}_{p\to x}\Big(e^{-i(t-\tau)\,p^2}\, \widetilde{W}(\tau,\, p)\Big)(x)\, d\tau\ . \qquad (32)
\end{aligned}$$

Now let us remark that, for any $\tau \in [0,t]$, $\widetilde{W}(\tau,.) : \mathbb{R} \longrightarrow \mathbb{C}$ is defined by the convolution of two compactly supported and continuous functions on $\mathbb{R}$. Therefore $\widetilde{W}(\tau,.)$ is also a continuous function on $\mathbb{R}$ such that

$$supp\, \widetilde{W}(\tau,.) = supp\, \Big(\widehat{V} * \big(e^{-i\tau\,.^2}\, \widehat{u_0}(.)\big)\Big) \subseteq \overline{[a,b] + [p_1,p_2]} = [p_1 + a, p_2 + b]\ ,$$

for any $\tau \in [0,t]$. It follows that $\widetilde{W}(\tau,.)$ is an integrable function and so the quantity $\mathcal{F}^{-1}_{p\to x}\big(e^{-i(t-\tau)\,p^2}\, \widetilde{W}(\tau,\, p)\big)(x)$ can be given by the integral representation of the inverse Fourier transform for integrable functions with respect to the variable p:

$$\mathcal{F}^{-1}_{p\to x}\Big(e^{-i(t-\tau)\,p^2}\, \widetilde{W}(\tau,p)\Big)(x) = \frac{1}{2\pi}\int_{p_1+a}^{p_2+b} \widetilde{W}(\tau,p)\, e^{-i(t-\tau)p^2 + ixp}\, dp\ .$$

Combining this with equality (32) leads to

$$\big(S_2(t)u_0\big)(x) = -i \int_0^t \left(\frac{1}{2\pi}\int_{p_1+a}^{p_2+b} \widetilde{W}(\tau,p)\, e^{-i(t-\tau)p^2 + ixp}\, dp\right) d\tau\ .$$

Since the integrand in the preceding double integral is a continuous function on the compact domain $[0,t] \times [p_1 + a, p_2 + b]$, we can apply Fubini's theorem to obtain the desired equality, namely,

$$\begin{aligned}
\big(S_2(t)u_0\big)(x) &= \frac{1}{2\pi}\int_{p_1+a}^{p_2+b} \left(-i\int_0^t \widetilde{W}(\tau,p)\, e^{i\tau p^2}\, d\tau\right) e^{-itp^2 + ixp}\, dp \\
&= \frac{1}{2\pi}\int_{p_1+a}^{p_2+b} W(t,p)\, e^{-itp^2 + ixp}\, dp\ .
\end{aligned}$$

□

6.2 Remark. For a fixed $p \in \mathbb{R}$, we precise that the function $y \in \mathbb{R} \longmapsto \widehat{V}(y)\, \widehat{u_0}(p-y) \in \mathbb{C}$ is actually equal to 0 outside the interval

$$I_p := \big\{y \in [a,b] \,\big|\, p - y \in [p_1,p_2]\big\} = [a,b] \cap [p - p_2, p - p_1]\ ,$$

since $supp\, \widehat{V} \subseteq [a,b]$ and $supp\, \widehat{u_0} \subseteq [p_1,p_2]$ by hypothesis. We shall exploit this property several times in the present section.

In the following result, we provide an asymptotic expansion to one term with uniform remainder estimate of $S_2(t)u_0$ inside the space-time cone $\mathfrak{C}\big(p_1+a, p_2+b\big)$ and a uniform estimate outside; an L^∞-norm estimate is also given. Here the result shows that the time-decay rate of $S_2(t)u_0$ seems to be faster outside the cone $\mathfrak{C}\big(p_1+a, p_2+b\big)$ than inside. Nevertheless we can not conclude at the moment on the optimality of the decay rate inside the cone due to the fact that the coefficient of the first term of the expansion has a modulus depending on time along the space-time directions inside the cone; this comes from the time-dependence of the amplitude $W(t,p)$.
Thanks to the rewriting of $S_2(t)u_0$ given in Proposition 6.1, we are allowed to employ the method which permitted to establish Theorem 5.3. Let us note that this method leads to a remainder estimate depending on $\big\|W(t,.)\big\|_{L^\infty(\mathbb{R})}$ and $\big\|\partial_p W(t,.)\big\|_{L^\infty(\mathbb{R})}$; in Proposition 6.4, these two last quantities are proved to be uniformly bounded with respect to time. We mention that our proof of Proposition 6.4 requires $\widehat{V}$ and $\widehat{u_0}$ to have a certain regularity and zero to be outside the frequency band of u_0.

6.3 Theorem. *Suppose that $u_0 \in H^3(\mathbb{R})$ and $V \in L^2(\mathbb{R})$ satisfy respectively Condition $(\mathcal{I}_{k,[p_1,p_2]})$ and Condition $(\mathcal{P}_{l,[a,b]})$, where $k \geqslant 5$ and $l \geqslant 4$, and suppose in addition that $0 \notin [p_1,p_2]$. Fix $\delta_1 \in \big(\frac{1}{2},1\big)$. Then we have*

i) for all $t \geqslant 1$,

$$\big\|S_2(t)u_0\big\|_{L^\infty(\mathbb{R})} \leqslant c_1(\delta_1,a,b,p_1,p_2)\, \big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})}\big\|\widehat{u_0}\big\|_{W^{5,\infty}(\mathbb{R})}\, t^{-\frac{1}{2}} \;;$$

ii) for all $(t,x) \in \mathfrak{C}\big(p_1+a, p_2+b\big)$,

$$\begin{aligned}\bigg|\big(S_2(t)u_0\big)(x) - \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, W\Big(t,\frac{x}{2t}\Big)\, t^{-\frac{1}{2}}\bigg| \\ \leqslant c_2(\delta_1,a,b,p_1,p_2)\, \big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})}\big\|\widehat{u_0}\big\|_{W^{5,\infty}(\mathbb{R})}\, t^{-\delta_1} \;;\end{aligned}$$

iii) for all $(t,x) \in \mathfrak{C}\big(p_1+a, p_2+b\big)^c$,

$$\Big|\big(S_2(t)u_0\big)(x)\Big| \leqslant c_2(\delta_1,a,b,p_1,p_2)\, \big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})}\big\|\widehat{u_0}\big\|_{W^{5,\infty}(\mathbb{R})}\, t^{-\delta_1} \;;$$

the constants $c_1(\delta_1,a,b,p_1,p_2)$, $c_2(\delta_1,a,b,p_1,p_2) > 0$ are respectively given in (33) *and* (34)*.*

Proof. We start by combining the result of Proposition 6.1 with the method of the proof of Theorem 5.3 in order to show that $\big(S_2(t)u_0\big)(x)$, where $(t,x) \in (0,+\infty)\times\mathbb{R}$, is actually an oscillatory integral of the form (16). This allows us to apply the three first points of Corollary 4.6, leading to

- for all $t \geqslant 1$,

$$\big\|S_2(t)u_0\big\|_{L^\infty(\mathbb{R})} \leqslant \tilde{C}_1(\delta_1, p_1+a, p_2+b)\, \Big(\big\|W(t,.)\big\|_{L^\infty(\mathbb{R})} + \big\|\partial_p\, W(t,.)\big\|_{L^\infty(\mathbb{R})}\Big)\, t^{-\frac{1}{2}} \;;$$

- for all $(t,x) \in \mathfrak{C}\big(p_1+a, p_2+b\big)$,

$$\begin{aligned}\bigg|\big(S_2(t)u_0\big)(x) - \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, W\Big(t,\frac{x}{2t}\Big)\, t^{-\frac{1}{2}}\bigg| \\ \leqslant \tilde{C}_2(\delta_1, p_1+a, p_2+b)\, \big\|\partial_p\, W(t,.)\big\|_{L^\infty(\mathbb{R})}\, t^{-\delta_1} \;;\end{aligned}$$

- for all $(t,x) \in \mathfrak{C}\big(p_1+a, p_2+b\big)^c$,

$$\Big|\big(S_2(t)u_0\big)(x)\Big| \leqslant \tilde{C}_2(\delta_1, p_1+a, p_2+b)\, \big\|\partial_p W(t,.)\big\|_{L^\infty(\mathbb{R})}\, t^{-\delta_1} \ ;$$

the real number δ_1 belongs to $\big(\frac{1}{2}, 1\big)$ and the constants are given by

- $$\tilde{C}_1(\delta_1, p_1+a, p_2+b) := \frac{1}{2\sqrt{\pi}} + \frac{1}{(4-4\delta_1)\sqrt{\pi}} \left(\frac{1}{2\sqrt{\pi}} + \sqrt{\frac{1}{4\pi} + \frac{1}{2}}\right)^{2\delta_1 - 1} \times \big((p_2-p_1)+(b-a)\big)^{2-2\delta_1} \ ;$$
- $$\tilde{C}_2(\delta_1, p_1+a, p_2+b) := \frac{1}{(4-4\delta_1)\sqrt{\pi}} \left(\frac{1}{2\sqrt{\pi}} + \sqrt{\frac{1}{4\pi} + \frac{1}{2}}\right)^{2\delta_1 - 1} \times \big((p_2-p_1)+(b-a)\big)^{2-2\delta_1} \ .$$

The second step of the proof consists in inserting the estimates of $\big\|W(t,.)\big\|_{L^\infty(\mathbb{R})}$ and $\big\|\partial_p W(t,.)\big\|_{L^\infty(\mathbb{R})}$ given in Proposition 6.4 (see below) into the above inequalities, providing the estimates of Theorem 6.3. To finish the proof, let us furnish the expressions of the constants:

- $$c_1(\delta_1, a, b, p_1, p_2) := \tilde{C}_1(\delta_1, p_1+a, p_2+b)\, \Big(M_1(a,b) + M_2(a,b)\Big) \ ; \tag{33}$$
- $$c_2(\delta_1, a, b, p_1, p_2) := \tilde{C}_2(\delta_1, p_1+a, p_2+b)\, M_2(a,b) \ ; \tag{34}$$

the constants $M_1(a,b)$ and $M_2(a,b)$ are respectively defined in (39) and (40), in the proof of Proposition 6.4. □

To estimate uniformly

$$W(t,p) = -i \int_0^t \int_a^b \widehat{V}(y)\, \widehat{u_0}(p-y)\, e^{-i\tau(p-y)^2}\, dy\, e^{i\tau p^2}\, d\tau \ ,$$

we distinguish two cases: $t \in [0,1]$ and $t > 1$. In the first case, we use the fact that $W(t,p)$ is given by the integral of a continuous function on a compact set. In the second case $t > 1$, we split the interval $[0,t]$ at the point 1 and we proceed as follows: on $[0,1]$, we employ the same method as above; on $[1,t]$, we exploit the fact that the y-integral on $[a,b]$ is actually an oscillatory integral with respect to τ, permitting to apply Corollary 4.6 iv). The application of this corollary to the y-integral provides an estimate which is τ-integrable on $[1,t] \subseteq [1,+\infty)$, leading to the desired result. The same method is employed to estimate $\partial_p W(t,p)$.

6.4 Proposition. *Suppose that $u_0 \in H^3(\mathbb{R})$ and $V \in L^2(\mathbb{R})$ satisfy respectively Condition $(\mathcal{I}_{k,[p_1,p_2]})$ and Condition $(\mathcal{P}_{l,[a,b]})$, where $k \geqslant 5$ and $l \geqslant 4$, and suppose in addition that $0 \notin [p_1,p_2]$. Then we have for all $t \geqslant 0$,*

- $$\big\|W(t,.)\big\|_{L^\infty(\mathbb{R})} \leqslant M_1(a,b)\, \big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})} \big\|\widehat{u_0}\big\|_{W^{4,\infty}(\mathbb{R})} \ ;$$
- $$\big\|\partial_p W(t,.)\big\|_{L^\infty(\mathbb{R})} \leqslant M_2(a,b)\, \big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})} \big\|\widehat{u_0}\big\|_{W^{5,\infty}(\mathbb{R})} \ ;$$

the constants $M_1(a,b), M_2(a,b) > 0$ are respectively given in (39) *and* (40).

Proof. Let $p \in \mathbb{R}$. We recall that

$$W(t,p) = -i \int_0^t \widetilde{W}(\tau,p)\, e^{i\tau p^2}\, d\tau = -i \int_0^t \int_a^b \widehat{V}(y)\, \widehat{u_0}(p-y)\, e^{-i\tau(p-y)^2}\, dy\, e^{i\tau p^2}\, d\tau \ .$$

For $t \in [0,1]$, we have clearly

$$\big|W(t,p)\big| \leqslant \int_0^t \int_a^b \Big|\widehat{V}(y)\, \widehat{u_0}(p-y)\Big|\, dy\, d\tau \leqslant \big\|\widehat{V}\big\|_{L^1(\mathbb{R})} \big\|\widehat{u_0}\big\|_{L^\infty(\mathbb{R})} \ .$$

Suppose now $t > 1$. Noting that $[0,t] = [0,1] \cup [1,t]$, we can split $W(t,p)$ into two terms and hence deduce the following estimate:

$$\begin{aligned}\big|W(t,p)\big| \leqslant &\left|\int_0^1 \int_a^b \widehat{V}(y)\, \widehat{u_0}(p-y)\, e^{-i\tau(p-y)^2}\, dy\, e^{i\tau p^2}\, d\tau\right| \\ &+ \left|\int_1^t \int_a^b \widehat{V}(y)\, \widehat{u_0}(p-y)\, e^{-i\tau(p-y)^2}\, dy\, e^{i\tau p^2}\, d\tau\right| \\ \leqslant &\ \big\|\widehat{V}\big\|_{L^1(\mathbb{R})} \big\|\widehat{u_0}\big\|_{L^\infty(\mathbb{R})} + \int_1^t \left|\int_a^b \widehat{V}(y)\, \widehat{u_0}(p-y)\, e^{-i\tau(p-y)^2}\, dy\right| d\tau \ . \end{aligned} \tag{35}$$

Thus the aim consists in controlling the behaviour of the oscillatory integral

$$\int_a^b \widehat{V}(y)\, \widehat{u_0}(p-y)\, e^{-i\tau(p-y)^2}\, dy \ , \tag{36}$$

for $\tau \geqslant 1$. To do so, we observe that the stationary point of the oscillatory integral (36) is given by the point $p \in \mathbb{R}$ which does not belong to the support of the function $y \longmapsto \widehat{V}(y)\, \widehat{u_0}(p-y)$. Indeed, p does not belong to $[p-p_2, p-p_1]$, since $0 \notin \ supp\, \widehat{u_0} \subseteq [p_1,p_2]$ by hypothesis; so it can not belong to the interval $I_p = [a,b] \cap [p-p_2, p-p_1]$, which contains the support of $y \longmapsto \widehat{V}(y)\, \widehat{u_0}(p-y)$ according to Remark 6.2. Hence Corollary 4.6 iv) is applicable to the oscillatory integral (36) where $\omega = \tau$, $p_0 = p$, $p_1 = a$, $p_2 = b$ and $U = \widehat{V}(.)\widehat{u_0}(p-.)$, leading to

$$\left|\int_a^b \widehat{V}(y)\, \widehat{u_0}(p-y)\, e^{-i\tau(p-y)^2}\, dy\right| \leqslant C_3(\delta_2,a,b) \left\|\partial_y^4 \Big[\widehat{V}(.)\, \widehat{u_0}(p-.)\Big]\right\|_{L^\infty(\mathbb{R})} \tau^{-\delta_2} \ , \tag{37}$$

where we have arbitrarily fixed the real number δ_2 in $\big(2, \frac{5}{2}\big)$. Further we have for all $y \in \mathbb{R}$,

$$\begin{aligned}\left|\partial_y^4 \Big[\widehat{V}(y)\, \widehat{u_0}(p-y)\Big]\right| &= \left|\sum_{k=0}^4 \binom{4}{k} \widehat{V}^{(4-k)}(y)\, \partial_y^4 \big[\widehat{u_0}(p-.)\big](y)\right| \\ &\leqslant 6 \sum_{k=0}^4 \left\|\widehat{V}^{(4-k)}\right\|_{L^\infty(\mathbb{R})} \left\|\widehat{u_0}^{(k)}\right\|_{L^\infty(\mathbb{R})} \\ &\leqslant 30 \,\big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})} \big\|\widehat{u_0}\big\|_{W^{4,\infty}(\mathbb{R})} \ . \end{aligned}$$

Putting (37) and the previous inequality into (35), and using the definition of the constant $C_3(\delta_2, a, b)$ given in (28) provides

$$\begin{aligned} \big|W(t,p)\big| &\leqslant \big\|\widehat{V}\big\|_{L^1(\mathbb{R})} \big\|\widehat{u_0}\big\|_{L^\infty(\mathbb{R})} + C_3(\delta_2,a,b) \left\|\partial_y^4\Big[\widehat{V}(.)\,\widehat{u_0}(p-.)\Big]\right\|_{L^\infty(\mathbb{R})} \int_1^t \tau^{-\delta_2}\,d\tau \\ &\leqslant \big\|\widehat{V}\big\|_{L^1(\mathbb{R})} \big\|\widehat{u_0}\big\|_{L^\infty(\mathbb{R})} + \frac{30\,C_3(\delta_2,a,b)}{\delta_2-1} \big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})} \big\|\widehat{u_0}\big\|_{W^{4,\infty}(\mathbb{R})} \\ &= \big\|\widehat{V}\big\|_{L^1(\mathbb{R})} \big\|\widehat{u_0}\big\|_{L^\infty(\mathbb{R})} + \frac{5}{(5-2\delta_2)(\delta_2-1)} \left(\frac{3\sqrt{\pi}}{8} + \sqrt{\frac{9\pi}{64}+\frac{1}{2}}\right)^{2\delta_2-4} \\ &\qquad \times\,(b-a)^{5-2\delta_2} \big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})} \big\|\widehat{u_0}\big\|_{W^{4,\infty}(\mathbb{R})} \\ &\leqslant M_1(a,b) \big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})} \big\|\widehat{u_0}\big\|_{W^{4,\infty}(\mathbb{R})} \,, \end{aligned} \tag{38}$$

where we have defined

$$M_1(a,b) := \left((b-a) + \frac{5}{(5-2\delta_2)(\delta_2-1)} \left(\frac{3\sqrt{\pi}}{8} + \sqrt{\frac{9\pi}{64}+\frac{1}{2}}\right)^{2\delta_2-4} (b-a)^{5-2\delta_2} \right). \tag{39}$$

Let us remark that we have used the fact that

$$\int_1^t \tau^{-\delta_2}\,d\tau = \frac{1}{1-\delta_2}\left(t^{1-\delta_2}-1\right) = \frac{1}{\delta_2-1}\left(1-t^{1-\delta_2}\right) \leqslant \frac{1}{\delta_2-1}\,,$$

since $\delta_2 > 2 > 1$, to obtain inequality (38).

Let us now study the function $\partial_p W(t,.) : \mathbb{R} \longrightarrow \mathbb{C}$. Thanks to the regularity of $\widehat{u_0}$ and $\widehat{V}$, we are allowed to differentiate under the integral sign:

$$\begin{aligned} \partial_p W(t,p) = &-i \int_0^t \int_a^b \widehat{V}(y)\,\widehat{u_0}'(p-y)\,e^{-i\tau(p-y)^2}\,dy\,e^{i\tau p^2}\,d\tau \\ &+ 2\int_0^t \tau \int_a^b y\,\widehat{V}(y)\,\widehat{u_0}(p-y)\,e^{-i\tau(p-y)^2}\,dy\,e^{i\tau p^2}\,d\tau\,. \end{aligned}$$

As previously, it is possible to estimate uniformly $\partial_p W(t,.)$ with respect to t when $t \in [0,1]$. So let us assume $t > 1$ and let us follow the same method as above to provide the following estimate:

$$\begin{aligned} \big|\partial_p W(t,p)\big| \leqslant\ & \big\|\widehat{V}\big\|_{L^1(\mathbb{R})} \big\|\widehat{u_0}'\big\|_{L^\infty(\mathbb{R})} + \int_1^t \left|\int_a^b \widehat{V}(y)\,\widehat{u_0}'(p-y)\,e^{-i\tau(p-y)^2}\,dy\right| d\tau \\ &+ \big\|.\,\widehat{V}(.)\big\|_{L^1(\mathbb{R})} \big\|\widehat{u_0}\big\|_{L^\infty(\mathbb{R})} + 2\int_1^t \tau \left|\int_a^b y\,\widehat{V}(y)\,\widehat{u_0}(p-y)\,e^{-i\tau(p-y)^2}\,dy\right| d\tau\,. \end{aligned}$$

As above, we apply Corollary 4.6 iv), whose hypotheses are verified again, to the oscillatory integrals

$$\int_a^b \widehat{V}(y)\,\widehat{u_0}'(p-y)\,e^{-i\tau(p-y)^2}\,dy \qquad \text{and} \qquad \int_a^b y\,\widehat{V}(y)\,\widehat{u_0}(p-y)\,e^{-i\tau(p-y)^2}\,dy\,,$$

and we obtain then

- $$\left|\int_a^b \widehat{V}(y)\,\widehat{u_0}'(p-y)\,e^{-i\tau(p-y)^2}\,dy\right| \leqslant C_3(\delta_2,a,b)\,\Big\|\partial_y^4\Big[\widehat{V}(.)\,\widehat{u_0}'(p-.)\Big]\Big\|_{L^\infty(\mathbb{R})}\,\tau^{-\delta_2}$$
$$\leqslant 30\,C_3(\delta_2,a,b)\,\big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})}\big\|\widehat{u_0}\big\|_{W^{5,\infty}(\mathbb{R})}\,\tau^{-\delta_2}\ ;$$
- $$\left|\int_a^b y\,\widehat{V}(y)\,\widehat{u_0}(p-y)\,e^{-i\tau(p-y)^2}\,dy\right| \leqslant C_3(\delta_2,a,b)\,\Big\|\partial_y^4\Big[.\,\widehat{V}(.)\,\widehat{u_0}(p-.)\Big]\Big\|_{L^\infty(\mathbb{R})}\,\tau^{-\delta_2}$$
$$= C_3(\delta_2,a,b)\,\Big\|4\,\partial_y^3\Big[\widehat{V}(.)\,\widehat{u_0}(p-.)\Big] + .\,\partial_y^4\Big[\widehat{V}(.)\,\widehat{u_0}(p-.)\Big]\Big\|_{L^\infty(\mathbb{R})}\,\tau^{-\delta_2}$$
$$\leqslant C_3(\delta_2,a,b)\,\Big(12+30\max\big\{|a|,|b|\big\}\Big) \times \big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})}\big\|\widehat{u_0}\big\|_{W^{4,\infty}(\mathbb{R})}\,\tau^{-\delta_2}\ ,$$

where $\delta_2 \in \big(2,\frac{5}{2}\big)$. Therefore we have

$$\begin{aligned}
\big|\partial_p W(t,p)\big| &\leqslant \big\|\widehat{V}\big\|_{L^1(\mathbb{R})}\,\big\|\widehat{u_0}'\big\|_{L^\infty(\mathbb{R})} + 30\,C_3(\delta_2,a,b)\,\big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})}\big\|\widehat{u_0}\big\|_{W^{5,\infty}(\mathbb{R})}\int_1^t \tau^{-\delta_2}\,d\tau \\
&\qquad + \big\|.\,\widehat{V}(.)\big\|_{L^1(\mathbb{R})}\,\big\|\widehat{u_0}\big\|_{L^\infty(\mathbb{R})} + 2\,C_3(\delta_2,a,b)\,\Big(12+30\max\big\{|a|,|b|\big\}\Big) \\
&\qquad\quad \times\ \big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})}\big\|\widehat{u_0}\big\|_{W^{4,\infty}(\mathbb{R})}\int_1^t \tau^{1-\delta_2}\,d\tau \\
&\leqslant \big\|\widehat{V}\big\|_{L^1(\mathbb{R})}\,\big\|\widehat{u_0}'\big\|_{L^\infty(\mathbb{R})} + \frac{30\,C_3(\delta_2,a,b)}{\delta_2-1}\,\big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})}\big\|\widehat{u_0}\big\|_{W^{5,\infty}(\mathbb{R})} \\
&\qquad + \big\|.\,\widehat{V}(.)\big\|_{L^1(\mathbb{R})}\,\big\|\widehat{u_0}\big\|_{L^\infty(\mathbb{R})} + \frac{2\,C_3(\delta_2,a,b)}{\delta_2-2}\,\Big(12+30\max\big\{|a|,|b|\big\}\Big) \\
&\qquad\quad \times\ \big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})}\big\|\widehat{u_0}\big\|_{W^{4,\infty}(\mathbb{R})} \\
&\leqslant \big\|\widehat{V}\big\|_{L^1(\mathbb{R})}\,\big\|\widehat{u_0}'\big\|_{L^\infty(\mathbb{R})} + \frac{5}{(5-2\delta_2)(\delta_2-1)}\left(\frac{3\sqrt{\pi}}{8}+\sqrt{\frac{9\pi}{64}+\frac{1}{2}}\right)^{2\delta_2-4} \\
&\qquad\quad \times\ (b-a)^{5-2\delta_2}\,\big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})}\big\|\widehat{u_0}\big\|_{W^{5,\infty}(\mathbb{R})} \\
&\qquad + \big\|.\,\widehat{V}(.)\big\|_{L^1(\mathbb{R})}\,\big\|\widehat{u_0}\big\|_{L^\infty(\mathbb{R})} + \frac{1}{(15-6\delta_2)(\delta_2-2)} \\
&\qquad\quad \times\ \left(\frac{3\sqrt{\pi}}{8}+\sqrt{\frac{9\pi}{64}+\frac{1}{2}}\right)^{2\delta_2-4}\,(b-a)^{5-2\delta_2}\,\Big(12+30\max\big\{|a|,|b|\big\}\Big) \\
&\qquad\qquad \times\ \big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})}\big\|\widehat{u_0}\big\|_{W^{4,\infty}(\mathbb{R})} \\
&\leqslant M_2(a,b)\,\big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})}\big\|\widehat{u_0}\big\|_{W^{5,\infty}(\mathbb{R})}\ ;
\end{aligned}$$

as previously, we have used the fact that $\delta_2 > 2$ to obtain the second inequality and the definition of $C_3(\delta_2,a,b)$ to obtain the third one. Finally the constant $M_2(a,b)$ is defined by

$$M_2(a,b) := (b-a) + \frac{5}{(5-2\delta_2)(\delta_2-1)} \left(\frac{3\sqrt{\pi}}{8} + \sqrt{\frac{9\pi}{64} + \frac{1}{2}} \right)^{2\delta_2 - 4} (b-a)^{5-2\delta_2}$$

$$+ \max\big\{|a|,|b|\big\}(b-a) + \frac{1}{(15-6\delta_2)(\delta_2-2)} \left(\frac{3\sqrt{\pi}}{8} + \sqrt{\frac{9\pi}{64} + \frac{1}{2}} \right)^{2\delta_2 - 4}$$

$$\times\ (b-a)^{5-2\delta_2} \Big(12 + 30 \max\big\{|a|,|b|\big\} \Big)\ . \tag{40}$$

□

Our goal now is to assure that the time-decay rate $t^{-\frac{1}{2}}$ for the term $S_2(t)u_0$ is attained along certain space-time directions contained in the cone $\mathfrak{C}(p_1+a, p_2+b)$. To do so, we proceed step by step: the first step consists in decomposing the amplitude $W(t,p)$ into a time-independent part and another part which tends uniformly to 0 as t goes to infinity; this is stated in Proposition 6.5.
The proof of Proposition 6.5 lies on the application of Fubini's theorem to $W(t,p)$ in order to interchange the τ-integral on $[0,t]$ and the y-integral on $[a,b]$. This leads to the computation of the τ-integral on $[0,t]$ of a complex exponential whose argument is given by

$$q(y,p) := (p-y)^2 - p^2\ .$$

Since the argument vanishes at $y=0$, additional hypotheses on the behaviour of $\widehat{V}$ at 0 are made to allow the division by $q(y,p)$, for all $y \in [a,b]$, and hence to obtain a primitive of the complex exponential. The desired time-independent part of $W(t,p)$ is given by the boundary term at $t=0$ of the above mentioned τ-integral on $[0,t]$. Moreover the argument $q(y,p)$ vanishes also at $y=2p$. Consequently the variable p is not allowed to belong to the interval $\left[\frac{a}{2},\frac{b}{2}\right]$ so that the point $2p$ is not in the interval of y-integration $[a,b]$.

6.5 Proposition. *Suppose that $u_0 \in H^3(\mathbb{R})$ and $V \in L^2(\mathbb{R})$ satisfy respectively Condition $(\mathcal{I}_{k,[p_1,p_2]})$ and Condition $(\mathcal{P}_{l,[a,b]})$ where $k,l \geqslant 1$; suppose in addition that $0 \notin [p_1,p_2]$ and that the function $y \in \mathbb{R}\backslash\{0\} \longmapsto y^{-1}\widehat{V}(y) \in \mathbb{C}$ can be extended to a $\mathcal{C}^1$-function $\widetilde{V} : \mathbb{R} \longrightarrow \mathbb{C}$.*
Let $t>0$ and let $W_1, W_2(t,.) : \mathbb{R}\backslash\left[\frac{a}{2},\frac{b}{2}\right] \longrightarrow \mathbb{C}$ be the functions defined by

- $$W_1(p) := -\int_a^b \frac{\widetilde{V}(y)\,\widehat{u_0}(p-y)}{y-2p}\,dy\ ,$$
- $$W_2(t,p) := -i\int_a^b \partial_y \left[\frac{\widetilde{V}(.)\,\widehat{u_0}(p-.)}{\tilde{q}(.,p)} \right](y)\, e^{-it\,q(y,p)}\,dy\ ,$$

where $q(y,p) := (p-y)^2 - p^2$ and $\tilde{q}(y,p) := 2\,(y-2p)\,(y-p)$. Note that for fixed $t>0$, the functions W_1 and $W_2(t,.)$ satisfy

$$supp\, W_1 = supp\, W_2(t,.) \subseteq [p_1+a, p_2+b]\ .$$

Then we have

$$\forall\, p \in \mathbb{R} \setminus \left[\frac{a}{2},\frac{b}{2}\right] \qquad W(t,p) = W_1(p) + W_2(t,p)\,t^{-1}\ . \tag{41}$$

6.6 Remark. i) The intersection between the interval $[p_1+a, p_2+b]$ and $\mathbb{R}\backslash\left[\frac{a}{2}, \frac{b}{2}\right]$ is always non-empty. Indeed, if we suppose that $[p_1+a, p_2+b] \subset \left[\frac{a}{2}, \frac{b}{2}\right]$ then

$$\frac{a}{2} < p_1 + a < p_2 + b < \frac{b}{2} \quad \Longrightarrow \quad \begin{cases} -\dfrac{a}{2} < p_1 \\ p_2 < -\dfrac{b}{2} \end{cases} \quad \Longrightarrow \quad -\frac{a}{2} < p_1 < p_2 < -\frac{b}{2} \,,$$

implying in particular that $b < a$, which is impossible since $a < b$ according to Condition $(\mathcal{P}_{l,[a,b]})$.

ii) The intersection we consider can be explicitly described in certain cases. For example, if $p_1 > \frac{b}{2} - a$ or $p_2 < \frac{a}{2} - b$ then

$$[p_1+a, p_2+b] \cap \left(\mathbb{R} \setminus \left[\frac{a}{2}, \frac{b}{2}\right]\right) = [p_1+a, p_2+b] \ .$$

In these cases, equality (41) holds on the entire interval $[p_1+a, p_2+b]$.

iii) In the case $0 \notin [a,b]$ (implying especially that V is not real-valued), the additional hypothesis on the behaviour of $\widehat{V}$ at zero is unnecessary.

Proof of Proposition 6.5. Let $t > 0$ and let us recall that for all $p \in \mathbb{R}$,

$$W(t,p) = -i \int_0^t \int_a^b \widehat{V}(y)\, \widehat{u_0}(p-y)\, e^{-i\tau(p-y)^2}\, dy\, e^{i\tau p^2}\, d\tau \ .$$

Now we remark that the function

$$(\tau, y) \longmapsto \widehat{V}(y)\, \widehat{u_0}(p-y)\, e^{-i\tau(p-y)^2 + i\tau p^2} \ ,$$

is continuous on $\mathbb{R}_+ \times \mathbb{R}$ for any $p \in \mathbb{R}$, so it is integrable on the compact domain $[0,t] \times [a,b]$. Applying Fubini's theorem gives

$$\begin{aligned} W(t,p) &= -i \int_0^t \left(\int_a^b \widehat{V}(y)\, \widehat{u_0}(p-y)\, e^{-i\tau(p-y)^2}\, dy \right) e^{i\tau p^2}\, d\tau \\ &= -i \int_a^b \widehat{V}(y)\, \widehat{u_0}(p-y) \left(\int_0^t e^{-i\tau\left((p-y)^2 - p^2\right)}\, d\tau \right) dy \\ &= -i \int_a^b \widehat{V}(y)\, \widehat{u_0}(p-y) \left(\int_0^t e^{-i\tau\, q(y,p)}\, d\tau \right) dy \ . \end{aligned}$$

Moreover, for all $p \in \mathbb{R}\backslash\left[\frac{a}{2}, \frac{b}{2}\right]$, we have formally

$$\begin{aligned} \widehat{V}(y) \int_0^t e^{-i\tau\, q(y,p)}\, d\tau &= \frac{\widehat{V}(y)}{-i\, q(y,p)} \left(e^{-it\, q(y,p)} - 1\right) \\ &= i\, \frac{\widehat{V}(y)}{y\,(y-2p)} \left(e^{-it\, q(y,p)} - 1\right) \\ &= i\, \frac{\widetilde{V}(y)}{y - 2p} \left(e^{-it\, q(y,p)} - 1\right) \\ &=: f_t(y,p) \ ; \end{aligned}$$

we observe that the quantity $y-2p$ is never equal to 0 when $y \in [a,b]$ and $p \in \mathbb{R}\backslash\big[\frac{a}{2},\frac{b}{2}\big]$, and we recall that $\widetilde{V}$ is well-defined and continuously differentiable on $\mathbb{R}$ by hypothesis. Hence $f_t(.,p) : \mathbb{R} \longrightarrow \mathbb{C}$ defines a $\mathcal{C}^1$-function such that $supp\, f_t(.,p) \subseteq [a,b]$, for all $p \in \mathbb{R}\backslash\big[\frac{a}{2},\frac{b}{2}\big]$; in particular, it is an integrable function. It follows

$$\begin{aligned} W(t,p) &= \int_a^b \frac{\widetilde{V}(y)\,\widehat{u_0}(p-y)}{y-2p}\big(e^{-it\,q(y,p)}-1\big)\,dy \\ &= \int_a^b \frac{\widetilde{V}(y)\,\widehat{u_0}(p-y)}{y-2p}\,e^{-it\,q(y,p)}dy - \int_a^b \frac{\widetilde{V}(y)\,\widehat{u_0}(p-y)}{y-2p}\,dy \\ &=: \widetilde{W_2}(t,p) + W_1(p)\;. \end{aligned}$$

By following the argument of Remark 6.2, we observe that the integrand of the integral defining $\widetilde{W_2}(t,p)$ is actually equal to zero outside $I_p := \big\{y \in [a,b]\,\big|\,p-y \in [p_1,p_2]\big\}$, for fixed $p \in \mathbb{R}\backslash\big[\frac{a}{2},\frac{b}{2}\big]$. On the interval I_p, the function $\partial_y q(y,p) = -2(p-y)$ does not vanish thanks to the assumption $0 \notin [p_1,p_2]$. Furthermore the functions $\widetilde{V}, \widehat{u_0} : \mathbb{R} \longrightarrow \mathbb{C}$ are continuously differentiable by hypothesis. Hence we can integrate by parts, providing

$$\begin{aligned} \widetilde{W_2}(t,p) &= i\int_a^b \frac{\widetilde{V}(y)\,\widehat{u_0}(p-y)}{(y-2p)\partial_y q(y,p)}\,\partial_y\Big[e^{-it\,q(.,p)}\Big](y)\,dy\,t^{-1} \\ &= i\int_a^b \frac{\widetilde{V}(y)\,\widehat{u_0}(p-y)}{\tilde{q}(y,p)}\,\partial_y\Big[e^{-it\,q(.,p)}\Big](y)\,dy\,t^{-1} \\ &= -i\int_a^b \partial_y\left[\frac{\widetilde{V}(.)\,\widehat{u_0}(p-.)}{\tilde{q}(.,p)}\right](y)\,e^{-it\,q(y,p)}\,dy\,t^{-1} \\ &=: W_2(t,p)\,t^{-1}\;, \end{aligned}$$

for all $p \in \mathbb{R}\backslash\big[\frac{a}{2},\frac{b}{2}\big]$. Let us remark that the boundary terms in the third equality are equal to 0 because $\widetilde{V}(a) = \widetilde{V}(b) = 0$. Indeed if $a \neq 0$ and $b \neq 0$ then

$$\widetilde{V}(a) = a^{-1}\widehat{V}(a) = 0 \qquad \text{and} \qquad \widetilde{V}(b) = b^{-1}\widehat{V}(b) = 0\;,$$

because $\widehat{V} : \mathbb{R} \longrightarrow \mathbb{C}$ is a continuous function such that $supp\,\widehat{V} \subseteq [a,b]$. In the case $a=0$ or $b=0$, the hypothesis that $\widetilde{V} : \mathbb{R} \longrightarrow \mathbb{C}$ is a $\mathcal{C}^1$-extension of $y \in \mathbb{R}\backslash\{0\} \longmapsto y^{-1}\widehat{V}(y) \in \mathbb{C}$ implies that $\widetilde{V}(0) = 0$, since the support of $\widetilde{V}$ would be strictly larger than $[a,b]$ if $\widetilde{V}(0) \neq 0$. □

The second step to assure the optimality of the time-decay rate $t^{-\frac{1}{2}}$ for $S_2(t)u_0$ consists in proving that the term $W_2(t,p)$ defined in Proposition 6.5 is actually uniformly bounded with respect to time for all $p \in [p_1+a, p_2+b] \cap \big(\mathbb{R}\backslash\big[\frac{a}{2}-\varepsilon, \frac{b}{2}+\varepsilon\big]\big)$, where $\varepsilon > 0$ is sufficiently small; this is stated in the following Proposition 6.7.
The proof of Proposition 6.7 lies essentially on estimates of the quantity $\tilde{q}(y,p)$ introduced in the preceding proposition and on the boundedness of the functions $\widetilde{V}$ and $\widehat{u_0}$.

6.7 Proposition. *Suppose that the hypotheses of Proposition 6.5 are satisfied and fix* $\varepsilon \in \big(0, \frac{p_2-p_1}{2}\big)$. *Then we have for all* $t > 0$ *and* $p \in [p_1+a, p_2+b] \cap \big(\mathbb{R}\backslash\big[\frac{a}{2}-\varepsilon, \frac{b}{2}+\varepsilon\big]\big)$,

$$\big|W_2(t,p)\big| \leqslant \tilde{c}_3(a,b,p_1,p_2,\varepsilon)\,\big\|\widetilde{V}\big\|_{W^{1,\infty}(\mathbb{R})}\big\|\widehat{u_0}\big\|_{W^{1,\infty}(\mathbb{R})}\;;$$

the constant $\tilde{c}_3(a,b,p_1,p_2,\varepsilon) > 0$ *is given in* (42).

6.8 Remark. By following the same method as the one employed in Remark 6.6, one proves

$$[p_1 + a, p_2 + b] \cap \left(\mathbb{R} \setminus \left[\frac{a}{2} - \varepsilon, \frac{b}{2} + \varepsilon \right] \right) \neq \emptyset \; ,$$

if $p_1 + \varepsilon < p_2 - \varepsilon$.
In particular, if $p_1 > \frac{b}{2} - a + \varepsilon$ or $p_2 < \frac{a}{2} - b - \varepsilon$ then

$$[p_1 + a, p_2 + b] \cap \left(\mathbb{R} \setminus \left[\frac{a}{2} - \varepsilon, \frac{b}{2} + \varepsilon \right] \right) = [p_1 + a, p_2 + b] \; .$$

and thus Proposition 6.7 is true on the entire interval $[p_1 + a, p_2 + b]$.

Proof of Proposition 6.7. The following estimates will be useful to prove the result:

- $p_1 \leqslant p - y \leqslant p_2 \qquad \Longrightarrow \qquad |p - y| \geqslant \min \big\{ |p_1|, |p_2| \big\} > 0 \; ;$
- $\left\{ \begin{array}{l} a \leqslant y \leqslant b \\ p \leqslant \dfrac{a}{2} - \varepsilon \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} a \leqslant y \leqslant b \\ p \geqslant \dfrac{b}{2} + \varepsilon \end{array} \right. \qquad \Longrightarrow \qquad y - 2p \geqslant 2\varepsilon \quad \text{or} \quad y - 2p \leqslant -2\varepsilon \; ;$
- $\left\{ \begin{array}{l} a \leqslant y \leqslant b \\ p_1 + a \leqslant p \leqslant p_2 + b \end{array} \right. \qquad \Longrightarrow \qquad 2a - 3(p_2 + b) \leqslant 2y - 3p \leqslant 2b - 3(p_1 + a) \; .$

Now let $t > 0$ and $p \in [p_1 + a, p_2 + b] \cap \big(\mathbb{R} \setminus \big[\frac{a}{2} - \varepsilon, \frac{b}{2} + \varepsilon \big] \big)$, and recall the definition of $W_2(t,p)$:

$$W_2(t,p) := -\, i \int_a^b \partial_y \left[\frac{\widetilde{V}(.)\, \widehat{u_0}(p - .)}{\tilde{q}(., p)} \right] (y)\, e^{-it\, q(y,p)}\, dy \; .$$

According to the above inequalities, the function $\tilde{q}(., p) : \mathbb{R} \longrightarrow \mathbb{R}$ defined in Proposition 6.5 satisfies on $I_p := \big\{ y \in [a, b] \,\big|\, p - y \in [p_1, p_2] \big\}$ the following estimates:

- $\big| \widetilde{q}(y, p) \big| \geqslant 4 \, \min \big\{ |p_1|, |p_2| \big\}\, \varepsilon > 0 \; ;$
- $\big| \partial_y \widetilde{q}(y, p) \big| = \big| 2(2y - 3p) \big| \leqslant 2 \, \max \Big\{ \big| 2a - 3(p_2 + b) \big|, \big| 2b - 3(p_1 + a) \big| \Big\} \; .$

Moreover, thanks to the regularity of $\widetilde{V}$ and $\widehat{u_0}$, we have for all $y \in I_p$,

$$\partial_y \left[\frac{\widetilde{V}(.)\, \widehat{u_0}(p - .)}{\tilde{q}(., p)} \right] (y) = \frac{\widetilde{V}'(y) \widehat{u_0}(p - y) - \widetilde{V}(y) \widehat{u_0}'(p - y)}{\widetilde{q}(y, p)} - \frac{\widetilde{V}(y) \widehat{u_0}(p - y) \partial_y \widetilde{q}(y, p)}{\widetilde{q}(y, p)^2} \; .$$

Therefore, it follows

$$
\begin{aligned}
\big|W_2(t,p)\big| &\leqslant \int_a^b \left|\partial_y \left[\frac{\widehat{V}(.)\,\widehat{u_0}(p-.)}{q(.,p)\,\partial_y q(.,p)}\right](y)\right| dy \\
&\leqslant \int_a^b \left|\frac{\widetilde{V}'(y)\widehat{u_0}(p-y) - \widetilde{V}(y)\widehat{u_0}'(p-y)}{\widetilde{q}(y,p)}\right| dy + \int_a^b \left|\frac{\widetilde{V}(y)\widehat{u_0}(p-y)\partial_y\widetilde{q}(y,p)}{\widetilde{q}(y,p)^2}\right| dy \\
&\leqslant 2\,\big\|\widetilde{V}\big\|_{W^{1,\infty}(\mathbb{R})}\big\|\widehat{u_0}\big\|_{W^{1,\infty}(\mathbb{R})} \int_a^b \left|\frac{1}{\widetilde{q}(y,p)}\right| dy \\
&\qquad + \big\|\widetilde{V}\big\|_{L^\infty(\mathbb{R})}\big\|\widehat{u_0}\big\|_{L^\infty(\mathbb{R})} \int_a^b \left|\frac{\partial_y\widetilde{q}(y,p)}{\widetilde{q}(y,p)^2}\right| dy \\
&\leqslant \big\|\widetilde{V}\big\|_{W^{1,\infty}(\mathbb{R})}\big\|\widehat{u_0}\big\|_{W^{1,\infty}(\mathbb{R})} \left(\frac{b-a}{2\,\min\big\{|p_1|,|p_2|\big\}}\,\varepsilon^{-1}\right. \\
&\qquad \left. + \frac{\max\Big\{\big|2a-3(p_2+b)\big|,\big|2b-3(p_1+a)\big|\Big\}(b-a)}{8\,\min\big\{|p_1|,|p_2|\big\}^2}\,\varepsilon^{-2}\right).
\end{aligned}
$$

We define the constant $\tilde{c}_3(a,b,p_1,p_2,\varepsilon) > 0$ as follows

$$
\begin{aligned}
\tilde{c}_3(a,b,p_1,p_2,\varepsilon) := \frac{b-a}{2\,\min\big\{|p_1|,|p_2|\big\}}\,\varepsilon^{-1} \\
+ \frac{\max\Big\{\big|2a-3(p_2+b)\big|,\big|2b-3(p_1+a)\big|\Big\}(b-a)}{8\,\min\big\{|p_1|,|p_2|\big\}^2}\,\varepsilon^{-2}\,. \qquad (42)
\end{aligned}
$$

□

The aim of the third step is to exploit the results of the two preceding propositions to provide a new asymptotic expansion to one term with uniform remainder estimate of $S_2(t)u_0$ in certain space-time regions contained in the cone $\mathfrak{C}(p_1+a,p_2+b)$. This new asymptotic expansion, given in Theorem 6.9, has the advantage that the modulus of the coefficient of the first term is time-independent, contrary to the one given in Theorem 6.3 ii).

The idea of the proof consists in inserting the expansion of the amplitude $W(t,p)$ given in Proposition 6.5 into the first term of the expansion of $S_2(t)u_0$ given in Theorem 6.3 ii): the time-independent part $W_1(p)$ furnishes actually the new first term of the expansion of $S_2(t)u_0$ and the time-dependent part $W_2(t,p)\,t^{-1}$, which tends to 0 at least as t^{-1} according to Proposition 6.7, produces a new remainder term.

6.9 Theorem. *Suppose that $u_0 \in H^3(\mathbb{R})$ and $V \in L^2(\mathbb{R})$ satisfy respectively Condition $(\mathcal{I}_{k,[p_1,p_2]})$ and Condition $(\mathcal{P}_{l,[a,b]})$ where $k \geqslant 5$ and $l \geqslant 4$; suppose in addition that $0 \notin [p_1,p_2]$ and that the function $y \in \mathbb{R}\backslash\{0\} \longmapsto y^{-1}\widehat{V}(y) \in \mathbb{C}$ can be extended to a $\mathcal{C}^1$-function $\widetilde{V} : \mathbb{R} \longrightarrow \mathbb{C}$. Fix $\delta_1 \in \big(\frac{1}{2},1\big)$ and $\varepsilon \in \big(0,\frac{p_2-p_1}{2}\big)$.*

Then we have for all $(t,x) \in \mathfrak{C}\big(p_1+a, p_2+b\big) \cap \mathfrak{C}\big(\frac{a}{2}-\varepsilon, \frac{b}{2}+\varepsilon\big)^c$,

$$\left| \big(S_2(t)u_0\big)(x) - \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, W_1\Big(\frac{x}{2t}\Big)\ t^{-\frac{1}{2}} \right|$$
$$\leqslant c_2(\delta_1,a,b,p_1,p_2)\, \big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})} \big\|\widehat{u_0}\big\|_{W^{5,\infty}(\mathbb{R})}\, t^{-\delta_1}$$
$$+ c_3(a,b,p_1,p_2,\varepsilon)\, \big\|\widetilde{V}\big\|_{W^{1,\infty}(\mathbb{R})} \big\|\widehat{u_0}\big\|_{W^{1,\infty}(\mathbb{R})}\, t^{-\frac{3}{2}}\ ;$$

the constants $c_2(\delta_1,a,b,p_1,p_2)$*,* $c_3(a,b,p_1,p_2,\varepsilon) > 0$ *are respectively given in* (34) *and* (43)*.*

6.10 Remark. Let us note that we have

$$\mathfrak{C}(p_1+a, p_2+b) \cap \mathfrak{C}\left(\frac{a}{2}-\varepsilon, \frac{b}{2}+\varepsilon\right)^c \neq \emptyset\ ,$$

since this property is equivalent to

$$\big[p_1+a, p_2+b\big] \cap \left(\mathbb{R} \setminus \left[\frac{a}{2}-\varepsilon, \frac{b}{2}+\varepsilon\right]\right) \neq \emptyset\ ,$$

which is true according to Remark 6.8. For example, if we suppose that $p_1 > \frac{b}{2} - a + \varepsilon$ or $p_2 < \frac{a}{2} - b - \varepsilon$, then we have

$$\mathfrak{C}\big(p_1+a, p_2+b\big) \cap \mathfrak{C}\Big(\frac{a}{2}-\varepsilon, \frac{b}{2}+\varepsilon\Big)^c = \mathfrak{C}\big(p_1+a, p_2+b\big)\ ,$$

showing that the time-asymptotic expansion given in Theorem 6.9 is true in the whole cone $\mathfrak{C}(p_1+a, p_2+b)$ in these cases.

Proof of Theorem 6.9. Let $(t,x) \in \mathfrak{C}\big(p_1+a, p_2+b\big) \cap \mathfrak{C}\big(\frac{a}{2}-\varepsilon, \frac{b}{2}+\varepsilon\big)^c$; in particular, (t,x) satisfies

$$\frac{x}{2t} \in \big[p_1+a, p_2+b\big] \cap \left(\mathbb{R} \setminus \left[\frac{a}{2}-\varepsilon, \frac{b}{2}+\varepsilon\right]\right)\ .$$

Hence Proposition 6.5 is applicable:

$$\big(S_2(t)u_0\big)(x) - \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, W_1\Big(\frac{x}{2t}\Big)\ t^{-\frac{1}{2}}$$
$$= \big(S_2(t)u_0\big)(x) - \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, W\Big(t, \frac{x}{2t}\Big)\ t^{-\frac{1}{2}} + \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, W_2\Big(t, \frac{x}{2t}\Big)\ t^{-\frac{3}{2}}\ ,$$

providing

$$\left| \big(S_2(t)u_0\big)(x) - \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, W_1\Big(\frac{x}{2t}\Big)\ t^{-\frac{1}{2}} \right|$$
$$\leqslant \left| \big(S_2(t)u_0\big)(x) - \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, W\Big(t, \frac{x}{2t}\Big)\ t^{-\frac{1}{2}} \right| + \frac{1}{2\sqrt{\pi}} \left| W_2\Big(t, \frac{x}{2t}\Big) \right|\ t^{-\frac{3}{2}}\ .$$

According to Theorem 6.3 ii) and Proposition 6.7, we obtain then

$$\left| \big(S_2(t)u_0\big)(x) - \frac{1}{2\sqrt{\pi}}\, e^{-i\frac{\pi}{4}}\, e^{i\frac{x^2}{4t}}\, W_1\Big(\frac{x}{2t}\Big)\ t^{-\frac{1}{2}} \right|$$
$$\leqslant c_2(\delta_1,a,b,p_1,p_2)\, \big\|\widehat{V}\big\|_{W^{4,\infty}(\mathbb{R})} \big\|\widehat{u_0}\big\|_{W^{5,\infty}(\mathbb{R})}\, t^{-\delta_1}$$
$$+ \frac{1}{2\sqrt{\pi}}\, \tilde{c}_3(a,b,p_1,p_2,\varepsilon)\, \big\|\widetilde{V}\big\|_{W^{1,\infty}(\mathbb{R})} \big\|\widehat{u_0}\big\|_{W^{1,\infty}(\mathbb{R})}\, t^{-\frac{3}{2}}\ .$$

Finally we define the constant $c_3(a,b,p_1,p_2,\varepsilon) > 0$ as follows

$$\begin{aligned} c_3(a,b,p_1,p_2,\varepsilon) &:= \frac{1}{2\sqrt{\pi}}\,\tilde{c}_3(a,b,p_1,p_2,\varepsilon) \\ &= \frac{b-a}{4\sqrt{\pi}\,\min\big\{|p_1|,|p_2|\big\}}\,\varepsilon^{-1} \\ &\quad + \frac{\max\Big\{\big|2a-3(p_2+b)\big|,\big|2b-3(p_1+a)\big|\Big\}(b-a)}{16\sqrt{\pi}\,\min\big\{|p_1|,|p_2|\big\}^2}\,\varepsilon^{-2}\,. \end{aligned} \tag{43}$$

□

Thanks to the preceding steps, we are now in position to establish the following corollary which assures that the time-decay rate $t^{-\frac{1}{2}}$ for the term $S_2(t)u_0$ is attained along any space-time direction $\mathfrak{D}(\overline{p})$, where $\overline{p} \in \mathbb{R}\backslash\big[\frac{a}{2}-\varepsilon,\frac{b}{2}+\varepsilon\big]$ satisfies $\big|W_1(\overline{p})\big| > 0$: this is a consequence of the fact that the modulus of the coefficient of the first term of the expansion given in Theorem 6.9 is independent from time along space-time directions. In proposition 6.12, we shall give some examples of non-empty sets contained in

$$\left\{p \in \mathbb{R} \setminus \left[\frac{a}{2}-\varepsilon,\frac{b}{2}+\varepsilon\right] \;\middle|\; \big|W_1(p)\big| > 0\right\}\,,$$

under additional hypotheses on the potential and the initial datum.
To prove the following corollary, we apply the same method as the one employed to prove the optimality of the time-decay rate $t^{-\frac{1}{2}}$ for the term $S_1(t)u_0$; see Corollary 5.5.

6.11 Corollary. *Suppose that the hypotheses of Theorem 6.9 are satisfied. Then the time-decay rate $t^{-\frac{1}{2}}$ for the term $S_2(t)u_0$ is attained along any space-time direction $\mathfrak{D}(\overline{p})$ such that $\overline{p} \in \mathbb{R}\backslash\big[\frac{a}{2}-\varepsilon,\frac{b}{2}+\varepsilon\big]$ satisfies $\big|W_1(\overline{p})\big| > 0$. For such a real number $\overline{p}$, we have especially $\mathfrak{D}(\overline{p}) \subset \mathfrak{C}(p_1+a,p_2+b)$.*

Proof. Suppose that there exists $\overline{p} \in \mathbb{R}\backslash\big[\frac{a}{2}-\varepsilon,\frac{b}{2}+\varepsilon\big]$ such that $\big|W_1(\overline{p})\big| > 0$. Since the support of W_1 is contained in $[p_1+a,p_2+b]$ according to Proposition 6.4, the real number $\overline{p}$ belongs to this interval and so $\mathfrak{D}(\overline{p}) \subset \mathfrak{C}(p_1+a,p_2+b)$. Then for all $(t,x) \in \mathfrak{D}(\overline{p})$, the modulus of the coefficient of the expansion given in Theorem 6.9 is equal to

$$\left|\frac{1}{2\sqrt{\pi}}\,e^{-i\frac{\pi}{4}}\,e^{it\overline{p}^2}\,W_1(\overline{p})\right| = \frac{1}{2\sqrt{\pi}}\big|W_1(\overline{p})\big| > 0\,,$$

proving that $t^{-\frac{1}{2}}$ is attained along $\mathfrak{D}(\overline{p})$. □

6.12 Proposition. *Suppose that the hypotheses of Theorem 6.9 are satisfied. Assume in addition that $p_1 > \frac{b}{2}-a+\varepsilon$ or $p_2 < \frac{a}{2}-b-\varepsilon$ and*

$$\forall\, p \in (a,b)\quad \begin{cases} \big|\Re\,\widehat{V}(p)\big| > 0 \\ \big|\Im\,\widehat{V}(p)\big| > 0 \end{cases} \qquad \text{and} \qquad \forall\, p \in (p_1,p_2)\quad \begin{cases} \big|\Re\,\widehat{u_0}(p)\big| > 0 \\ \big|\Im\,\widehat{u_0}(p)\big| > 0 \end{cases}\;;$$

here $\Re\, z$ and $\Im\, z$ are respectively the real part and the imaginary part of $z \in \mathbb{C}$. Then *i) If* $b>0$ *then*

$$(p_2+\eta_1, p_2+b) \subseteq \left\{ p \in \mathbb{R} \setminus \left[\frac{a}{2}-\varepsilon, \frac{b}{2}+\varepsilon\right] \,\middle|\, \big|W_1(p)\big| > 0 \right\} ,$$

where $\eta_1 := \max\{0, a\}$.

ii) If $a<0$ *then*

$$(p_1+a, p_1+\eta_2) \subseteq \left\{ p \in \mathbb{R} \setminus \left[\frac{a}{2}-\varepsilon, \frac{b}{2}+\varepsilon\right] \,\middle|\, \big|W_1(p)\big| > 0 \right\} ,$$

where $\eta_2 := \min\{0, b\}$.

iii) If $0 \in (a,b)$ *then*

$$(p_1+a, p_1) \cup (p_2, p_2+b) \subseteq \left\{ p \in \mathbb{R} \setminus \left[\frac{a}{2}-\varepsilon, \frac{b}{2}+\varepsilon\right] \,\middle|\, \big|W_1(p)\big| > 0 \right\} .$$

Proof. First of all, we note that the hypothesis $p_1 > \frac{b}{2} - a + \varepsilon$ or $p_2 < \frac{a}{2} - b - \varepsilon$ implies

$$\big[p_1+a, p_2+b\big] \subset \mathbb{R} \setminus \left[\frac{a}{2}-\varepsilon, \frac{b}{2}+\varepsilon\right] , \tag{44}$$

according to Remark 6.8. Now we define the function $g_{\overline{p}} : \mathbb{R}\backslash\{2\overline{p}\} \longrightarrow \mathbb{C}$ by

$$g_{\overline{p}}(y) := \frac{\widetilde{V}(y)\, \widehat{u_0}(\overline{p}-y)}{y-2\overline{p}} = \frac{\widehat{V}(y)\, \widehat{u_0}(\overline{p}-y)}{y\,(y-2\overline{p})} ,$$

where $\overline{p} \in \mathbb{R}\backslash\big[\frac{a}{2}-\varepsilon, \frac{b}{2}+\varepsilon\big]$; we note that

$$\forall\, \overline{p} \in \mathbb{R} \setminus \left[\frac{a}{2}-\varepsilon, \frac{b}{2}+\varepsilon\right] \qquad W_1(\overline{p}) = \int_a^b g_{\overline{p}}(y)\, dy ,$$

and, by following the argument of Remark 6.2, that the support of $g_{\overline{p}}$ is actually contained in the interval $I_{\overline{p}} = [a,b] \cap [\overline{p}-p_2, \overline{p}-p_1]$. We distinguish the three cases:

i) If $b>0$ then we choose $\overline{p} \in (p_2+\eta_1, p_2+b)$, where $\eta_1 := \max\{0,a\}$. This implies especially

$$I_{\overline{p}} = [\overline{p}-p_2, b] \subset (0, +\infty) ,$$

and since $(p_2+\eta_1, p_2+b) \subset [p_1+a, p_2+b]$, the real number $\overline{p}$ belongs to $\mathbb{R}\backslash\big[\frac{a}{2}-\varepsilon, \frac{b}{2}+\varepsilon\big]$ according to (44). Thanks to the hypotheses on $\widehat{V}$ and $\widehat{u_0}$, we observe that either the real part or the imaginary part of the function $g_{\overline{p}}$ is positive or negative in the interior of $I_{\overline{p}}$, namely $(\overline{p}-p_2, b)$. It follows that the modulus of $W_1(\overline{p})$ is positive for any $\overline{p} \in (p_2+\eta_1, p_2+b)$.

ii) If $a<0$ then we choose $\overline{p} \in (p_1+a, p_1+\eta_2) \subset [p_1+a, p_2+b]$, where $\eta_2 := \min\{0,b\}$, implying especially

$$I_{\overline{p}} = [a, \overline{p}-p_1] \subset (-\infty, 0) ;$$

as above, $\overline{p} \in \mathbb{R}\backslash\big[\frac{a}{2}-\varepsilon, \frac{b}{2}+\varepsilon\big]$. By following the same argument, we deduce that $\big|W_1(\overline{p})\big|$ is positive for any $\overline{p} \in (p_1+a, p_1+\eta_2)$.

iii) If $0 \in (a,b)$, we follow the same argument as above to show that $\big|W_1(\overline{p})\big| > 0$ for any $\overline{p} \in (p_1+a, p_1) \cup (p_2, p_2+b)$. Note that we have in the present case: $\eta_1 = \eta_2 = 0$.

□

7 Some results from functional analysis

In this last section, we start by recalling some results from semigroup theory that we use in Section 2. They permit to define the notion of classical solution for the Schrödinger equation (1) and to prove existence and uniqueness of this solution. Then a result concerning the possibility of interchanging integration with the application of bounded operators is recalled.

Let us remark that the results of this section are not proved but quoted from the literature containing their proofs. Furthermore, in the present section, the operators A, B, C and the semigroups $\big(T(t)\big)_{t\geqslant 0}$, $\big(S(t)\big)_{t\geqslant 0}$ are general and do not refer to the particular objects which are considered in the preceding sections of this paper.

We start by recalling the notion of a classical solution for an abstract evolution equation (see [9, Chapter II, Definition 6.1]). If an operator generates a semigroup on a Banach space, then the classical solution of the evolution equation given by this operator exists, is unique and corresponds to the orbit of the initial value under the semigroup (see [9, Chapter II, Proposition 6.2]).

7.1 Definition and Proposition. *Consider the initial value problem*

$$\begin{cases} \dot{u}(t) = A\,u(t) \\ u(0) = v \end{cases} , \tag{45}$$

for $t \geqslant 0$, where $A : D(A) \subset X \longrightarrow X$ is the generator of a semigroup $\big(T(t)\big)_{t\geqslant 0}$ on the Banach space X.
A function $u : \mathbb{R}_+ \longrightarrow X$ is called a classical solution of (45) *if u is continuously differentiable with respect to X, $u(t) \in D(A)$ for all $t \geqslant 0$, and u satisfies* (45).
If $v \in D(A)$ then the function

$$u : t \in \mathbb{R}_+ \longmapsto u(t) = T(t)\,v \ ,$$

is the unique classical solution of (45).

In the following theorem, we recall that the sum of a generator of a semigroup on X and a bounded operator on X generates a semigroup as well (see [9, Chapter III, Bounded Perturbation Theorem 1.3]).

7.2 Theorem. *Let $\big(A, D(A)\big)$ be the generator of a strongly continuous semigroup on a Banach space X. If B is a bounded operator from X into itself, i.e. $B \in \mathcal{L}(X)$, then the operator $\big(C, D(C)\big) := \big(A + B, D(A)\big)$ generates a strongly continuous semigroup on X.*

A representation of the semigroup generated by the operator $\big(A + B, D(A)\big)$ as a series is recalled in the following theorem: it is called *Dyson-Phillips series* (see [9, Chapter III, Theorem 1.10]).

7.3 Theorem. *Let $\big(A, D(A)\big)$ be the generator of a strongly continuous semigroup $\big(T(t)\big)_{t\geqslant 0}$ on a Banach space X, and let $B \in \mathcal{L}(X)$. The strongly continuous semigroup $\big(S(t)\big)_{t\geqslant 0}$*

generated by $\big(C, D(C)\big) := \big(A+B, D(A)\big)$ *can be obtained as*

$$\lim_{N\to+\infty} \left\| S(t) - \sum_{n=1}^{N} S_n(t) \right\|_{\mathcal{L}(X)} = 0 \; ,$$

where $S_1(t) := T(t)$ *and*

$$\forall\, v \in X \qquad S_{n+1}(t)v := \int_0^t S_n(t-\tau)\, B\, T(\tau) v \, d\tau \; .$$

In the final result, we recall that Bochner-type integration and the application of bounded operators can be interchanged (see [17, Chapter V, Section 5, Corollary 2]).

7.4 Proposition. *Let* A *be a bounded operator acting between two Banach spaces* X *and* Y *and let* $J \subseteq \mathbb{R}$ *be an interval. If* $F : J \longrightarrow X$ *is a Bochner-integrable function, then* $AF : J \longrightarrow Y$ *is also a Bochner-integrable function and*

$$A\left(\int_J F(s)\, ds \right) = \int_J A\, F(s)\, ds \; .$$

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