A Note on the Asymptotic Stability of Wave-Type Equations with Switching Time-Delay

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SERGE NICAISE AND CRISTINA PIGNOTTI

Abstract

We consider second-order evolution equations with intermittently delayed/not-delayed damping. We give sufficient conditions for asymptotic and exponential stability, completing our previous results from Nicaise and Pignotti [*Adv. Diff. Eq.*, 17: 879–902, 2012; *J. Dyn. Diff. Eq.*, 26: 781–803, 2014]. Moreover, some concrete models are described.

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6.1 Introduction

Let H be a real Hilbert space and let $A:\mathcal{D}(A)\to H$ be a positive self-adjoint operator with a compact inverse in H. Denote by $V:=\mathcal{D}(A^{\frac{1}{2}})$ the domain of $A^{\frac{1}{2}}$. Let B_1, B_2 be time-dependent linear operators, $B_1\in\mathcal{L}(H,H)$ and $B_2(t):U\to H$, where U is a real Hilbert space with norm and inner product denoted respectively by $\|\cdot\|_U$ and $\langle\cdot,\cdot\rangle_U$. We assume that B_1 and B_2 act alternately, that is

$$B_1^*(t)B_2^*(t)=0, \quad \forall t>0.$$

Let us consider the problem

$$u_{tt}(t)+Au(t)+B_1(t)B_1^*(t)u_t(t)+B_2(t)B_2^*(t)u_t(t-\tau)=0 \quad t>0 \tag{6.1}$$

$$u(0)=u_0 \quad \text{and} \quad u_t(0)=u_1 \tag{6.2}$$

where the constant $\tau>0$ is the time delay.

Time delay effects appear in many applications and practical problems and it is by now well known that even an arbitrarily small delay in the feedback may destabilize a system which is uniformly exponentially stable in absence of delay. For some examples of this destabilizing effect of time delays we refer to [2, 3, 9, 14].

In [9] we considered the wave equation with both dampings acting simultaneously, that is $B_1(t)=\mu_1$ and $B_2(t)=\mu_2$, with $\mu_1,\mu_2\in\mathbb{R}^+$, and we proved that if $\mu_1>\mu_2$ then the system is uniformly exponentially stable. Otherwise, if $\mu_2\geq\mu_1$, that is the delay term prevails on the not-delayed one, then there are instability phenomena, namely, there are unstable solutions for arbitrarily small (large) delays.

The stabilization problem for second-order evolution equations with switching time delay is, in some sense, related to the one for second-order evolution equations damped by positive/negative feedbacks (see [6]). See also [13] for the relation between wave equation with time delay in the damping and wave equation with indefinite damping, i.e., damping which changes sign in different subsets of the domain.

We firstly studied this subject in [10], in a more general setting with respect to the problem considered here. Assuming that an observability inequality holds for the conservative model associated with (6.1) and (6.2) and, through the definition of a suitable energy, we obtained sufficient conditions ensuring asymptotic stability. Under more restrictive assumptions exponential stability estimates were also obtained.

An analogous problem has been considered in [1] for 1D models for the wave equation but with a different approach. Indeed, in [1] we obtain stability results for particular values of the time delays, related to the length of the domain, by using the D'Alembert formula.

The results of [10] are improved in [11] by removing a quite restrictive assumption on the size of the "bad" terms, i.e., the terms with time delay (cf. assumption (3.3) of [10]). Indeed, expected from [6] and from the relation between delay problems and problems with antidamping, the delay feedback operator B_2 may be also large but then has to act on small time intervals. Moreover, in [11] we consider also the case when B_1 is unbounded. In this short note, we complete the results from [10, 11] by improving the sufficient conditions that guarantee stability results but by staying in the case of bounded operators B_1.

This chapter is organized as follows. In Section 6.2, we recall the well-posedness result proved in [10] and the observability estimate in short-time proved by Haraux, Martinez, and Vancostenoble. In Section 6.3, we obtain asymptotic and exponential stability results for the abstract model under suitable conditions, completing the results of [11]. Finally, in Section 6.4, we give some concrete applications of our results.

6.2 Well-Posedness

In this section we recall a well-posedness result, proved by the authors in [10], for problem (6.1) and (6.2).

We assume that for all $n \in \mathbb{N}$, there exists $t_n > 0$ with $t_n < t_{n+1}$ and such that

$$\begin{aligned} B_2(t) = 0 \quad \forall t \in I_{2n} = [t_{2n}, t_{2n+1}), \\ B_1(t) = 0 \quad \forall t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}), \end{aligned}$$

with $B_1 \in C^1([t_{2n}, t_{2n+1}]; \mathcal{L}(H, H))$ and $B_2 \in C([t_{2n+1}, t_{2n+2}]; \mathcal{L}(U, H))$.

Moreover, we assume that $\tau \leq T_{2n}$ for all $n \in \mathbb{N}$, where T_n denotes the length of the interval I_n, i.e.,

$$T_n = t_{n+1} - t_n, \quad n \in \mathbb{N}. \tag{6.3}$$

Under these assumptions, the following result holds (see Theorem 2.1 in [10]).

Theorem 6.1 *Under the above assumptions, for any $u_0 \in V$ and $u_1 \in H$, the system (6.1) and (6.2) has a unique solution $u \in C([0, \infty); V) \cap C^1([0, \infty); H)$.*

To obtain stability results for problem (6.1) and (6.2), we assume now that for all $n \in \mathbb{N}$, there exist three positive constants m_{2n}, M_{2n}, and M_{2n+1} with $m_{2n} \le M_{2n}$ and such that for all $u \in H$ we have

(i) $m_{2n}\|u\|_H^2 \le \|B_1^*(t)u\|_H^2 \le M_{2n}\|u\|_H^2$ for $t \in I_{2n} = [t_{2n}, t_{2n+1})$, $\forall n \in \mathbb{N}$;
(ii) $\|B_2^*(t)u\|_U^2 \le M_{2n+1}\|u\|_H^2$ for $t \in I_{2n+1} = [t_{2n+1}, t_{2n+2})$, $\forall n \in \mathbb{N}$.

Denote by $E_S(\cdot)$ the standard energy for wave–type equations, i.e.,

$$E_S(t) = E_S(u;t) := \frac{1}{2}\left(\|u(t)\|_V^2 + \|u_t(t)\|_H^2\right).$$

Observe that, on the time intervals I_{2n}, $n \in \mathbb{N}$, only the standard dissipative damping acts. So, the following observability estimate holds (see theorem 3.1 of [6]).

Proposition 6.2 Assume (i). There exists a constant c, independent of the length T_{2n} of the interval I_{2n}, such that for any solution of (6.1) and (6.2) it holds

$$E_S(t_{2n+1}) \le \frac{1}{1 + c\frac{m_{2n}}{T_{2n}^{-3} + T_{2n}^{-1} + M_{2n}m_{2n}T_{2n}^{-1}}} E_S(t_{2n}), \quad n \in \mathbb{N}. \tag{6.4}$$

6.3 Stability Results

Now, as in [11], we assume

$$\inf_{n \in \mathbb{N}} \frac{m_{2n}}{M_{2n+1}} > 0. \tag{6.5}$$

Note that assumption (3.3) in [10] is instead equivalent to

$$\inf_{n \in \mathbb{N}} \frac{m_{2n}}{M_{2n+1}} > \frac{1}{2}.$$

Let us introduce the energy of the system

$$E(t) = E(u;t) := \frac{1}{2}\left(\|u(t)\|_V^2 + \|u_t(t)\|_H^2 + \frac{\xi}{2}\int_{t-\tau}^{t} \|B_2^*(s+\tau)u_t(s)\|_U^2\, ds\right), \tag{6.6}$$

where ξ is a positive number satisfying

$$\xi < \inf_{n \in \mathbb{N}} \frac{m_{2n}}{M_{2n+1}}. \tag{6.7}$$

The following estimates are proved as in proposition 3.1 of [11].

Proposition 6.3 Assume (i), (ii), and (6.5). For any regular solution of problem (6.1) and (6.2) the energy is decreasing on the intervals I_{2n}, $n \in \mathbb{N}$, and

$$E'(t) \leq -\frac{m_{2n}}{2}\|u_t\|_H^2. \tag{6.8}$$

Moreover, on the intervals I_{2n+1}, $n \in \mathbb{N}$,

$$E'(t) \leq \frac{M_{2n+1}}{2}\left(\xi + \frac{1}{\xi}\right)\|u_t\|_H^2. \tag{6.9}$$

The following theorem, due to the more general assumption (6.5) and due to the more general setting, improves and generalizes theorem 3.3 of [10].

Theorem 6.4 *Assume* (i), (ii)*, and* (6.5)*. If*

$$\sum_{n=0}^{\infty} M_{2n+1}T_{2n+1} < +\infty, \tag{6.10}$$

and

$$\sum_{n=0}^{\infty} \ln\left(\frac{1}{1 + c\frac{m_{2n}}{T_{2n}^{-3}+T_{2n}^{-1}+M_{2n}m_{2n}T_{2n}^{-1}}} + \tau\xi M_{2n+1}\right) = -\infty, \tag{6.11}$$

then system (6.1) and (6.2) is asymptotically stable, that is any solution u of (6.1) and (6.2) satisfies $E_S(u,t) \to 0$ *for* $t \to +\infty$*.*

Proof Note that (6.9) implies

$$E'(t) \leq M_{2n+1}\left(\xi + \frac{1}{\xi}\right)E(t), \quad t \in I_{2n+1} = [t_{2n+1}, t_{2n+2}), \;\; n \in \mathbb{N}.$$

Then we have

$$E(t_{2n+2}) \leq e^{(\xi+\frac{1}{\xi})M_{2n+1}T_{2n+1}}E(t_{2n+1}), \quad \forall n \in \mathbb{N}. \tag{6.12}$$

Now, note that

$$E(t_{2n+1}) = E_S(t_{2n+1}) + \frac{\xi}{2}\int_{t_{2n+1}-\tau}^{t_{2n+1}} \|B_2^*(s+\tau)u_t(s)\|_U^2\, ds.$$

Now observe that, since $T_{2n} \geq \tau, n \in \mathbb{N}$, the variable $s+\tau$ in the above integral belongs to $I_{2n+1} \cup I_{2n+2}$. If $s+\tau \in I_{2n+1}$, then $\|B_2^*(s+\tau)u_t(s)\|_U^2 \leq M_{2n+1}\|u_t(s)\|_H^2$. Otherwise, if $s+\tau \in I_{2n+2}$, then $B_2^*(s+\tau)u_t(s) = 0$. Then,

$$E(t_{2n+1}) \leq E_S(t_{2n+1}) + \frac{\xi}{2}M_{2n+1}\int_{t_{2n+1}-\tau}^{t_{2n+1}} \|u_t(s)\|_H^2\, ds. \tag{6.13}$$

Now, since $(t_{2n+1}-\tau, t_{2n+1})\subset I_{2n}$ and in I_{2n} the system is dissipative, from (6.13) we deduce

$$\begin{aligned} E(t_{2n+1}) &\leq E_S(t_{2n+1}) + \tau M_{2n+1}\xi E_S(t_{2n+1}-\tau) \\ &\leq E_S(t_{2n+1}) + \tau M_{2n+1}\xi E_S(t_{2n}). \end{aligned} \tag{6.14}$$

Combining Proposition 6.2 and (6.14) we then obtain

$$E(t_{2n+1}) \leq \left(\frac{1}{1 + c\frac{m_{2n}}{T_{2n}^{-3}+T_{2n}^{-1}+M_{2n}m_{2n}T_{2n}^{-1}}} + \tau\xi M_{2n+1} \right) E_S(t_{2n}), \tag{6.15}$$

and therefore

$$\begin{aligned} E_S(t_{2n+2}) &\leq E(t_{2n+2}) \\ &\leq e^{(\xi+\frac{1}{\xi})M_{2n+1}T_{2n+1}} \left(\frac{1}{1 + c\frac{m_{2n}}{T_{2n}^{-3}+T_{2n}^{-1}+M_{2n}m_{2n}T_{2n}^{-1}}} + \tau\xi M_{2n+1} \right) E_S(t_{2n}). \end{aligned} \tag{6.16}$$

Since (6.16) holds for any $n\in\mathbb{N}$ we can deduce

$$E_S(t_{2n+2}) \leq \prod_{p=0}^{n} e^{(\xi+\frac{1}{\xi})M_{2p+1}T_{2p+1}} \left(\frac{1}{1 + c\frac{m_{2p}}{T_{2p}^{-3}+T_{2p}^{-1}+M_{2p}m_{2p}T_{2p}^{-1}}} + \tau\xi M_{2p+1} \right) E_S(0). \tag{6.17}$$

Now observe that the standard energy $E_S(\cdot)$ is not decreasing. However, for $t\in[t_{2n}, t_{2n+1})$, only the standard dissipative damping acts and so

$$E_S(t) \leq E_S(t_{2n}), \quad \forall t\in[t_{2n}, t_{2n+1}). \tag{6.18}$$

Moreover, for $t\in[t_{2n+1}, t_{2n+2})$, it results

$$E_S(t) \leq E(t) \leq e^{(\xi+\frac{1}{\xi})M_{2n+1}T_{2n+1}} E(t_{2n+1}), \tag{6.19}$$

where in the second inequality we have used (6.12).

Then, by (6.15) and (6.17)–(6.19), asymptotic stability occurs if

$$\sum_{n=0}^{\infty} \left[\left(\xi + \frac{1}{\xi} \right) M_{2n+1}T_{2n+1} + \ln\tilde{c}_n \right] = -\infty, \tag{6.20}$$

where

$$\tilde{c}_n = \frac{1}{1 + c\frac{m_{2n}}{T_{2n}^{-3}+T_{2n}^{-1}+M_{2n}m_{2n}T_{2n}^{-1}}} + \tau\xi M_{2n+1}. \tag{6.21}$$

Thus, if

$$\sum_{n=0}^{\infty} M_{2n+1} T_{2n+1} < +\infty, \qquad \sum_{n=0}^{\infty} \ln \tilde{c}_n = -\infty,$$

system (6.1) and (6.2) is asymptotically stable. □

We now show that under additional assumptions on the coefficients T_n, m_n, M_n an exponential stability result holds.

Theorem 6.5 *Assume* $(i), (ii)$*, and* (6.5). *Assume also that*

$$T_{2n} = T^* \quad \forall n \in \mathbb{N}, \tag{6.22}$$

with $T^* \geq \tau$, *and*

$$T_{2n+1} = \tilde{T} \quad \forall n \in \mathbb{N}. \tag{6.23}$$

Moreover, assume that

$$\sup_{n \in \mathbb{N}} e^{(\xi + \frac{1}{\xi}) M_{2n+1} \tilde{T}} \tilde{c}_n = d < 1, \tag{6.24}$$

where $\tilde{c}_n$ *is as in* (6.21). *Then, there exist two positive constants* γ, μ *such that*

$$E_S(t) \leq \gamma e^{-\mu t} E_S(0), \quad t > 0, \tag{6.25}$$

for every solution of problem $(6.1) - (6.2)$.

Proof From (6.16) and (6.24) we obtain

$$E_S(T^* + \tilde{T}) \leq d E_S(0),$$

and also

$$E_S(n(T^* + \tilde{T})) \leq d^n E_S(0), \quad \forall n \in \mathbb{N}.$$

Then, the standard energy $E_S(\cdot)$ satisfies an exponential estimate like (6.25) (see lemma 1 of [5]). □

Remark 6.6 In the assumptions of Theorem 6.5, from (6.17) we can see that exponential stability also holds if instead of (6.24) we assume

$$\exists n \in \mathbb{N} \quad \text{such that} \quad \prod_{p=k(n+1)}^{k(n+1)+n} e^{(\xi + \frac{1}{\xi}) M_{2p+1} \tilde{T}} c_p \leq d < 1, \;\; \forall k = 0, 1, 2, \ldots$$

6.4 Examples

Here, we illustrate some concrete examples falling in our previous abstract setting, namely the wave equation, the elasticity system, the Midlin–Timoshenko model, the Petrowsky system.

6.4.1 The Wave Equation

As a concrete application let us consider the wave equation with internal damping. More precisely, let $\Omega \subset \mathbb{R}^d$ (with d a positive natural number) be an open bounded domain with a boundary $\partial\Omega$ of class C^2. We denote by ω a subset of Ω.

Let us consider the initial boundary value problem

$$u_{tt}(x,t) - \Delta u(x,t) + b_1(t)u_t(x,t) + b_2(t)\chi_\omega u_t(x,t-\tau) = 0 \quad \text{in } \Omega \times (0,+\infty) \tag{6.26}$$

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,+\infty) \tag{6.27}$$

$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega \tag{6.28}$$

with initial data $(u_0,u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and b_1, b_2 in $L^\infty(0,+\infty)$ such that

$$b_1(t)b_2(t) = 0, \quad \forall t > 0.$$

Moreover, we assume

(i_w) $0 < m_{2n} \le b_1(t) \le M_{2n}$, $b_2(t) = 0$, for all $t \in I_{2n} = [t_{2n}, t_{2n+1})$, and $b_1 \in C^1(\bar{I}_{2n})$, for all $n \in \mathbb{N}$;
(ii_w) $|b_2(t)| \le M_{2n+1}$, $b_1(t) = 0$, for all $t \in I_{2n+1} = [t_{2n+1}, t_{2n+2})$, and $b_2 \in C(\bar{I}_{2n+1})$, for all $n \in \mathbb{N}$.

This problem enters into our previous framework, if we take $H = L^2(\Omega)$ and the operator A defined by

$$A : \mathcal{D}(A) \to H : u \to -\Delta u,$$

where $\mathcal{D}(A) = H_0^1(\Omega) \cap H^2(\Omega)$.

The operator A is a self-adjoint and positive operator with a compact inverse in H and is such that $V = \mathcal{D}(A^{1/2}) = H_0^1(\Omega)$. We then define the operator B_1 as

$$B_1 : H \to H : v \to \sqrt{b_1}v, \tag{6.29}$$

and, denote $U := L^2(\Omega)$, the operator B_2 as

$$B_2 : U \to H : v \to \sqrt{b_2}\tilde{v}, \tag{6.30}$$

where $\tilde{v} \in L^2(\Omega)$ is the extension of v by zero outside ω.

It is easy to verify that $B_1 B_1^*(\varphi) = b_1\varphi$ and $B_2 B_2^*(\varphi) = b_2\varphi\chi_\omega$, for $\varphi \in H$.

This shows that problem (6.26) and (6.28) enters in the abstract framework (6.1) and (6.2). Moreover, (i_w) and (ii_w) easily imply (i) and (ii) of Section 6.3. Therefore we can restate Proposition 6.3. Now, the energy functional is

$$E(t) = \frac{1}{2}\int_\Omega \{u_t^2(x,t) + |\nabla u(x,t)|^2\}dx + \frac{\xi}{2}\int_{t-\tau}^t |b_2(s+\tau)| \int_\omega u_t^2(x,s)dx\,ds. \tag{6.31}$$

Proposition 6.7 Assume (i_w), (ii_w), and (6.5). Then, for every regular solution of problem (6.26)–(6.28) the energy is decreasing on the intervals I_{2n}, $n \in \mathbb{N}$, and

$$E'(t) \leq -\frac{m_{2n}}{2}\int_\Omega u_t^2(x,t)dx. \tag{6.32}$$

Moreover, on the intervals I_{2n+1}, $n \in \mathbb{N}$,

$$E'(t) \leq \frac{M_{2n+1}}{2}\left(\xi + \frac{1}{\xi}\right)\int_\Omega u_t^2(x,t)dx. \tag{6.33}$$

Thus, the stability results of Theorems 6.4 and 6.5 apply to this model.

6.4.2 The Elasticity System

In the same setting than in the previous section, we consider the following elastodynamic system

$$u_{tt}(x,t) - \mu\Delta u(x,t) - (\lambda+\mu)\nabla \operatorname{div} u(x,t) + b_1(t)u_t(x,t)$$
$$+ b_2(t)\chi_\omega u_t(x,t-\tau) = 0 \quad \text{in } \Omega \times (0,+\infty) \tag{6.34}$$
$$u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,+\infty) \tag{6.35}$$
$$u(x,0) = u_0(x) \quad \text{and} \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega \tag{6.36}$$

with initial data $(u_0, u_1) \in H_0^1(\Omega)^d \times L^2(\Omega)^d$ and b_1, b_2 satisfying the same assumptions as in Section 6.4.1. Here the state variable u is vector-valued and λ, μ are the Lamé coefficients that are positive real numbers. Finally for a (smooth enough) vector-valued function $v : \Omega \to \mathbb{R}^d$, div v is its standard divergence, namely

$$\operatorname{div} v = \sum_{j=1}^d \partial_j v_j.$$

As before this problem enters into our abstract setting, once we take $H = L^2(\Omega)^d$, and A defined by

$$A : \mathcal{D}(A) \to H : u \to -\mu\Delta u(x,t) - (\lambda+\mu)\nabla \operatorname{div} u,$$

where $\mathcal{D}(A) = H_0^1(\Omega)^d \cap H^2(\Omega)^d$.

The operator A is a self-adjoint and positive operator with a compact inverse in H and is such that $V = \mathcal{D}(A^{1/2}) = H_0^1(\Omega)^d$ equipped with the inner product

$$(u,v)_V = \int_\Omega \Big(\mu \sum_{i,j=1}^d \partial_i u_j \partial_i v_j + (\lambda+\mu) \operatorname{div} u \operatorname{div} v\Big) dx, \quad \forall u, v \in H_0^1(\Omega)^d.$$

We further define $U = L^2(\omega)^d$ and the operators B_i, $i = 1, 2$, as follows:

$$B_1 : U \to H : v \to \sqrt{b_1} v,$$
$$B_2 : U \to H : v \to \sqrt{b_2} \tilde{v},$$

$\tilde{v}$ being again the extension of v by zero outside ω. As before $B_1 B_1^*(\varphi) = b_1\varphi$, and $B_2 B_2^*(\varphi) = b_2\varphi\chi_\omega$, for any $\varphi \in H$. So, problem (6.34)–(6.36) enters in the abstract framework (6.1) and (6.2). Moreover, (i_w) and (ii_w) easily imply (i) and (ii) of Section 6.2. Therefore, the results of Section 6.3 apply also to the system (6.34)–(6.36) with the energy defined by

$$E(t) = \frac{1}{2}\int_\Omega \Big\{|u_t|^2(x,t) + \mu \sum_{i,j=1}^d (\partial_i u_j(x,t))^2 + (\lambda+\mu)(\operatorname{div} u(x,t))^2\Big\} dx$$
$$+ \frac{\xi}{2}\int_{t-\tau}^t |b_2(s+\tau)| \int_\omega |u_t(x,s)|^2 dx\, ds.$$

6.4.3 The Mindlin–Timoshenko Model

In the same setting as in Section 6.4.1, we consider the internal stabilization of the following Mindlin–Timoshenko (beam/plate) model (for similar models, see chapter 5 of [8], chapters 2 and 4 of [7], [4, 12]).

$$w_{tt}(x,t) = \operatorname{div}(K(\nabla w + u))(x,t) - a_1(t) w_t(x,t)$$
$$-a_2(t)\chi_\omega w_t(x, t-\tau) \quad \text{in } \Omega \times (0, +\infty) \tag{6.37}$$

$$u_{tt}(x,t) = \operatorname{div} C\epsilon(u)(x,t) - K(\nabla w + u)(x,t) - b_1(t) u_t(x,t)$$
$$-b_2(t)\chi_\omega u_t(x, t-\tau) \quad \text{in } \Omega \times (0, +\infty) \tag{6.38}$$

$$u(x,t)=0, \quad w(x,t)=0 \quad \text{on } \partial\Omega\times(0,+\infty) \tag{6.39}$$

$$u(x,0)=u^0(x),\ u_t(x,0)=u^1(x),\ \ w(x,0)=w^0(x),\ w_t(x,0)=w^1(x) \ \text{ in } \Omega. \tag{6.40}$$

If $d=1$ (resp. $d=2$) the scalar variable w represents the displacement of the beam (resp. plate) in the vertical direction, while the vectorial variable $u=(u_i)_{i=1}^d$ is the angles of rotation of a filament of the beam (resp. plate).

Here K belongs to $L^\infty(\Omega)^{d\times d}$, is symmetric and positive definite, i.e., there exists a positive constant k_0 such that

$$X^\top K(x)X\geq k_0, \quad \forall X\in \mathbb{R}^d, \text{ for a.e. } x\in\Omega,$$

while $C=(c_{ijk\ell})$ is a tensor such that

$$c_{ijk\ell}=c_{ji\ell k}=c_{k\ell ij}\in L^\infty(\Omega), \tag{6.41}$$

all indices running over the integers $1,\dots,d$. These quantities are related to the constitutive materials of the beam/plate.

As usual for $u=(u_i)_{i=1}^d$, $\epsilon(u)$ is the linear strain tensor defined by

$$\epsilon(u)=(\epsilon_{ij}(u))_{i,j=1}^d \quad \text{with } \epsilon_{ij}(u)=\frac{1}{2}(\partial_i u_j+\partial_j u_i).$$

For a $d\times d$ matrix $\epsilon=(\epsilon_{ij})_{i,j=1}^d$ the product $C\epsilon=((C\epsilon)_{ij})_{i,j=1}^d$ is the $d\times d$ matrix given by

$$(C\epsilon)_{ij}=\sum_{k,\ell=1}^d c_{ijk\ell}\epsilon_{k\ell}.$$

As usual we assume that C is positive definite in the sense that there exists $\mu_0>0$ such that

$$C(x)\epsilon:\epsilon\geq\mu_0|\epsilon|^2, \quad \forall\epsilon\in\mathbb{R}^{d\times d}, \text{ for a.e. } x\in\Omega. \tag{6.42}$$

We further recall that for a (smooth enough) matrix-valued function $w=(w_{ij}):\Omega\to\mathbb{R}^{d\times d}$, div w is its divergence line by line, i.e.,

$$\operatorname{div} w=\Big(\sum_{j=1}^d \partial_j w_{ij}\Big)_{i=1}^d.$$

Finally we require that the functions b_1, b_2 satisfy the assumptions (i_w) and (ii_w) from Section 6.4.1, and similarly for a_1 and a_2, we suppose that

(i_{mt}) $0<m_{2n}\leq a_1(t)\leq M_{2n}$, $a_2(t)=0$, for all $t\in I_{2n}=[t_{2n},t_{2n+1})$, and $a_1\in C^1(\bar{I}_{2n})$, for all $n\in\mathbb{N}$;

(ii_{mt}) $|a_2(t)| \leq M_{2n+1},\ a_1(t) = 0$, for all $t \in I_{2n+1} = [t_{2n+1}, t_{2n+2})$, and $a_2 \in C(\bar{I}_{2n+1})$, for all $n \in \mathbb{N}$.

System (6.37)–(6.39) can be viewed as a coupling between the wave equation in w with the dynamical elastic system in u with internal feedbacks with delays.

Again this problem enters into our abstract setting by using Friedrichs extension theorem. Namely we take $H = L^2(\Omega)^d \times L^2(\Omega)$ with its natural inner product

$$((u,w),(u^*,w^*))_H = \int_\Omega \Big(u \cdot \bar{u}^* + w\bar{w}^* \Big) dx, \quad \forall (u,w),(u^*,w^*) \in H,$$

$V = H_0^1(\Omega)^d \times H_0^1(\Omega)$ that is clearly compactly embedded into H and the sesquilinear and symmetric form

$$a(U,U^*) = \int_\Omega \Big(C\varepsilon(u) : \varepsilon(\bar{u}^*) + K(\nabla w + u) \cdot (\nabla \bar{w}^* + \bar{u}^*) \Big) dx$$

with $U = (u,w),\ U^* = (u^*,w^*) \in V$. Indeed using Korn's and Poincaré's inequalities, it is not difficult to check that this sesquilinear form is coercive in V, namely there exists $\alpha > 0$ such that

$$a(U,U) \geq \alpha \|U\|^2_{H^1(\Omega)^d \times H^1(\Omega)}, \quad \forall U \in V.$$

Hence it is well known that the operator A associated with the triple (a, V, H) is a self-adjoint and positive operator with a compact inverse in H with $\mathcal{D}(A^{1/2}) = V$. This operator is defined by

$$\mathcal{D}(A) = \{U \in V : \exists F_U \in H \text{ such that } a(U,U^*) = (F_H, U^*)_H, \forall U^* \in V\},$$

and

$$AU = F_U, \quad \forall U \in \mathcal{D}(A).$$

By the definition of a, it is easy to see that

$$\mathcal{D}(A) = \big\{(u,w) \in V : \text{div } C\varepsilon(u) \in L^2(\Omega)^d \text{ and } \text{div}(K(\nabla w + u)) \in L^2(\Omega)\big\},$$

and then

$$A(u,w) = -(\text{div } C\varepsilon(u), \text{div}(K(\nabla w + u))), \quad \forall (u,w) \in \mathcal{D}(A).$$

We further define $U = H$ and the operators $B_i, i = 1,2$, as follows:

$$B_1 : (u,w) \to H : v \to (\sqrt{b_1} v, \sqrt{a_1} w)$$
$$B_2 : (u,w) \to H : v \to (\sqrt{b_2} \tilde{v}, \sqrt{a_2} \tilde{w}),$$

$\tilde{v}$ being again the extension of v by zero outside ω. Clearly one has $B_1B_1^*(u,w)=(b_1u,a_1w)$ and $B_2B_2^*(u,w)=(b_2u\chi_\omega,a_2w\chi_\omega)$, for any $(u,w)\in H$. So, problem (6.37)–(6.39) enters in the abstract framework (6.1) and (6.2). Moreover, (i_w), (ii_w), (i_{mt}), and (ii_{mt}) imply (i) and (ii) from Section 6.2, and consequently the results of Section 6.3 apply also to this system with the energy defined by

$$E(t)=\frac{1}{2}\int_\Omega\Big\{(C\varepsilon(u)(x,t)):\varepsilon(\bar{u}(x,t)))+K(\nabla w+u)(x,t))\cdot(\nabla\bar{w}+\bar{u})(x,t))\Big\}dx$$
$$+\frac{\xi}{2}\int_{t-\tau}^t\int_\omega(|b_2(s+\tau)||u_t(x,s)|^2+|a_2(s+\tau)||w_t(x,s)|^2)dx\,ds.$$

6.4.4 The Petrovsky System

Let $\Omega\subset\mathbb{R}^d$ be an open bounded set with a boundary $\partial\Omega$ of class C^4 and let ω be any fixed subset of Ω.

Let us consider the initial boundary value problem

$$u_{tt}(x,t)+\Delta^2u(x,t)+b_1(t)u_t(x,t)+b_2(t)\chi_\omega u_t(x,t-\tau)=0 \quad \text{in } \Omega\times(0,+\infty) \tag{6.43}$$

$$u(x,t)=\Delta u(x,t)=0 \quad \text{on } \partial\Omega\times(0,+\infty) \tag{6.44}$$

$$u(x,0)=u_0(x) \quad \text{and} \quad u_t(x,0)=u_1(x) \quad \text{in } \Omega \tag{6.45}$$

with initial data $(u_0,u_1)\in H^2(\Omega)\cap H_0^1(\Omega)\times L^2(\Omega)$ and b_1,b_2 satisfying the same assumptions as in Section 6.4.1.

Now, we take $H=L^2(\Omega)$ and let A be the operator

$$A:\mathcal{D}(A)\to H:\ u\to\Delta^2u, \tag{6.46}$$

where

$$\mathcal{D}(A)=\{v\in H_0^1(\Omega)\cap H^4(\Omega):\Delta u=0 \text{ on } \partial\Omega\}.$$

The operator A is self-adjoint and positive, has a compact inverse in H and satisfies $\mathcal{D}(A^{1/2})=H^2(\Omega)\cap H_0^1(\Omega)$. We then define $U=L^2(\omega)$ and the operators $B_i, i=1,2$, by (6.29) and (6.30). So, problem (6.43)–(6.45) enters in the abstract framework (6.1) and (6.2). Moreover, (i_w) and (ii_w) easily imply (i) and (ii) of Section 6.2.

Therefore, the results of Section 6.3 apply also to the plate model.

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