SPECIAL LAGRANGIAN 4-FOLDS WITH SO(2) S3-SYMMETRY IN COMPLEX SPACE FORMS

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SPECIAL LAGRANGIAN 4-FOLDS WITH $SO(2) \rtimes S_3$-SYMMETRY IN COMPLEX SPACE FORMS

FRANKI DILLEN, CHRISTINE SCHARLACH, KRISTOF SCHOELS, AND LUC VRANCKEN

Abstract. In this article we obtain a classification of special Lagrangian submanifolds in complex space forms subject to an $SO(2) \rtimes S_3$-symmetry on the second fundamental form. The algebraic structure of this form has been obtained by Marianty Ionel in [7]. However, the classification of special Lagrangian submanifolds in $\mathbb{C}^4$ having this $SO(2) \rtimes S_3$ symmetry in [7] is incomplete. In this paper we give a complete classification of such submanifolds, and extend the classification to special Lagrangian submanifolds of arbitrary complex space forms with $SO(2) \rtimes S_3$-symmetry.

1. Introduction

A space (N, J, g) is called a Hermitian manifold with complex structure J and Riemannian metric g, if $g(JX, JY) = g(X, Y)$ for all X and Y. The $(0, 2)$-tensor $\omega(X, Y) = g(X, JY)$ is its symplectic form. If ω is closed, then (N, J, g) is said to be a Kähler manifold. In this case the Levi-Civita connection D of g satisfies $D\omega = 0$ as well, see [11]. A complex space form is a Kähler manifold for which the curvature tensor is given by

$$R(X, Y)Z = \epsilon\,(X \wedge Y + JX \wedge JY + 2g(X, JY)J)\,Z, \tag{1}$$

where ϵ is a real constant and $X \wedge Y$ is defined as

$$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y.$$

Every complete, simply connected complex space form of dimension n with constant holomorphic sectional curvature 4ϵ is isometric to one of the following manifolds:

(1) the standard complex space $\mathbb{C}^n$ when $\epsilon = 0$,
(2) the complex projective space $\mathbb{C}P^n(4\epsilon)$ when $\epsilon > 0$,
(3) the complex hyperbolic space $\mathbb{C}H^n(4\epsilon)$ when $\epsilon < 0$.

Because we consider submanifolds of a complex space form locally, we can restrict ourselves to those ambient spaces. By rescaling, we can even assume that $\epsilon = 0, 1, -1$.

A Lagrangian submanifold M of a Kähler manifold (N, J, g) is a submanifold such that ω vanishes identically on M and the (real) dimension of M is half the (complex) dimension of N, see [1]. This implies that J induces an orthogonal isomorphism between the tangent and the normal bundle on the submanifold. The Gauss formula is given by

$$D_X Y = \nabla_X Y + h(X, Y) = \nabla_X Y + JA(X, Y),$$

where $A = -Jh$ defines a symmetric $(1, 2)$-tensor on the submanifold, and the Weingarten formula is given by

$$D_X(JY) = J(\nabla_X Y) - A(X, Y).$$

It is easy to see that the cubic form C, defined by

$$C(X, Y, Z) = g(A(X, Y), Z)$$

is totally symmetric. For Lagrangian submanifolds of complex space forms, the equations of Gauss and Codazzi simplify to

$$R(X,Y)Z = \epsilon\,(X \wedge Y)\,Z + [A_X, A_Y]\,Z, \tag{2}$$

$$\nabla A \text{ is symmetric.} \tag{3}$$

The following theorem holds, see [3] and [5].

Theorem 1.1. *Suppose (M^n, g) is a Riemannian manifold equipped with a symmetric and g-symmetric $(1,2)$-tensor A such that (2) and (3) are satisfied for some constant ϵ. Then for every point $p \in M$ there exists a neighborhood U and a Lagrangian isometric immersion $\phi : U \to N^{2n}(4\epsilon)$ into the complex space form $N^{2n}(4\epsilon)$ such that g and JA are induced as first and second fundamental form. Such an immersion is unique up to isometries of the ambient space.*

We focus on a particular form of A assuming that there is a pointwise G-symmetry of A (or equivalently of the cubic form C), where G is a subgroup of the special orthogonal group $SO(n)$. We say that A has pointwise G-symmetry at p if for all tangent vectors X, Y in p, and all $g \in G$ the relation $A(gX, gY) = gA(X,Y)$ holds (or equivalently $C(gX, gY, gZ) = C(X,Y,Z)$ for all X, Y, Z). Furthermore, we impose a minimality condition on A at p, so for every X at p, we assume that $\mathrm{Tr}(A_X) = 0$. These manifolds are interesting, since in $\mathbb{C}^n$ the minimal Lagrangian submanifolds are precisely the special Lagrangian submanifolds of $\mathbb{C}^n$ as introduced by Harvey and Lawson [6]. If a special Lagrangian submanifold of $\mathbb{C}^n$ has G-symmetry at every point, for the same group G, then a classification result for the dimension equal to 3 has been obtained by Bryant [2]. An explicit classification for special (we also use the word "special" for "minimal" in case $c \neq 0$) Lagrangian submanifolds of complex space forms with pointwise symmetric cubic form is not yet done, but can be easily obtained from a similar classification for affine spheres in [13].

In the present paper we consider the 4-dimensional case. In particular we consider special Lagrangian 4-folds in complex space forms with pointwise symmetry. The shape of the $(1,2)$-tensor A, invariant under subgroups of $SO(4)$, has been described by M. Ionel in [7]. In the same article, the author classifies special Lagrangian 4-folds of $\mathbb{C}^4$ according to their symmetry groups. However, the classification in case the symmetry group is given by $SO(2) \rtimes S_3$ in that article is incomplete; several possible subcases including the most general one is omitted. In the present article, we give a complete classification of all special Lagrangian 4-folds in any complex space form having this particular symmetry. This settles the problem for $SO(2) \rtimes S_3$-symmetry for all ϵ. The classification for other symmetry groups remains open if $\epsilon \neq 0$.

The $SO(2) \rtimes S_3$-symmetry implies that A can be expressed as

$$\begin{array}{llll} A(X_1,X_1) = rX_1, & A(X_1,X_2) = -rX_2, & A(X_1,X_3) = 0, & A(X_1,X_4) = 0, \\ A(X_2,X_1) = -rX_2, & A(X_2,X_2) = -rX_1, & A(X_2,X_3) = 0, & A(X_2,X_4) = 0, \\ A(X_3,X_1) = 0, & A(X_3,X_2) = 0, & A(X_3,X_3) = 0, & A(X_3,X_4) = 0, \\ A(X_4,X_1) = 0, & A(X_4,X_2) = 0, & A(X_4,X_3) = 0, & A(X_4,X_4) = 0, \end{array} \tag{4}$$

in a well-chosen local orthonormal frame $\{X_1, X_2, X_3, X_4\}$. In this expression r is a strictly positive function. The $SO(2)$-symmetry is given by the free rotation in the $\{X_3, X_4\}$ plane and the S_3-symmetry is essentially obtained by rotations over an angle $2\pi/3$ in the $\{X_1, X_2\}$ plane and reflections in the $\{X_1, X_4\}$ plane. We can remark that the form of A

is exactly that of Lagrangian submanifolds attaining equality in Chen's inequality, see [4] and [5].

In order to list the different possible subcases, we introduce distributions

$$\mathcal{N}_1 = \operatorname{span}\{X_1, X_2\}, \quad \mathcal{N}_+ = \operatorname{span}\{X_1, X_2, [X_1, X_2]\}, \quad \mathcal{N}_2 = \operatorname{span}\{X_3, X_4\}.$$

We will see that $\mathcal{N}_2$ is always integrable. We obtain:

(1) If $\mathcal{N}_1 = \mathcal{N}_+$, then the submanifold is a double warped product $\mathbb{R} \times_f \mathbb{R} \times_g N^2$ where N^2 is a minimal Lagrangian submanifold in an appropriate space form.
(2) If $\mathcal{N}_1 \subsetneq \mathcal{N}_+$ and $\mathcal{N}_+$ is integrable, then the submanifold is a single warped product $\mathbb{R} \times_f N^3$ where N^3 is a special Lagrangian 3-fold with S_3-symmetry in an appropriate space form.
(3) If the smallest integrable distribution containing $\mathcal{N}_1$ is TM, then for this final case, we do not obtain an explicit expression for the immersion, but we will rewrite the equations (7) to a system of partial differential equations in 2 coordinates out of 4 coordinates defined on the submanifold. Here, techniques will be used similar to those in [8].

When we consider the different cases, we will assume the defining conditions hold on an open neighborhood of the considered point.

2. Preliminaries

2.1. **Complex space forms.** We briefly recall the basic properties of $\mathbb{C}^n$ and show how Lagrangian submanifolds of $\mathbb{C}P^n$ and $\mathbb{C}H^n$ can be lifted to subsets of $\mathbb{C}^{n+1}$.

Consider the complex vector space $\mathbb{C}^n$. Its elements can be written as n-tuples of complex numbers, so they are given as

$$\vec{z} = (z_1, \cdots, z_n), \quad z_j = x_j + iy_j, \quad x_j, y_j \in \mathbb{R}.$$

Through the map

$$\phi : \mathbb{C}^n \to \mathbb{R}^{2n} : (z_1, \cdots, z_n) \to (x_1, y_1, \cdots, x_n, y_n)$$

the space $\mathbb{C}^n$ is a real $2n$-dimensional manifold. The multiplication with the imaginary unit i translates to a linear map on $\mathbb{R}^{2n}$ given as

$$i\,(x_1, y_1, \cdots, x_n, y_n) = (-y_1, x_1, \cdots, -y_n, x_n)\,.$$

and its derivative J is given as

$$\begin{aligned} J\partial_{x_k} &= \partial_{y_k}, \\ J\partial_{y_k} &= -\partial_{x_k}. \end{aligned}$$

This squares to $-I$ and thus defines a complex structure on $\mathbb{C}^n$. On $\mathbb{C}^n$ there is also a Hermitian form given by

$$s(\vec{z}, \vec{w}) = \sum_{j=1}^{n} z_j \bar{w}_j = \sum_{j=1}^{n} (x_j u_j + y_j v_j) - i \sum_{j=1}^{n} (x_j v_j - y_j u_j).$$

The real part, which can be denoted as $\langle \vec{z}, \vec{w} \rangle$ defines the Euclidean scalar product on $\mathbb{R}^{2n}$ and induces a natural Riemannian metric on $\mathbb{C}^n$. We can see that J is an isometry and the induced Kähler form, which also coincides with the imaginary part of the Hermitian form, is closed. These structures make $\mathbb{C}^n$ into a flat Kähler manifold.

The manifold $\mathbb{C}P^n$ can be modeled as the quotient S^{2n+1}/S^1, where

$$S^{2n+1} = \{(z_0, \cdots, z_n) \in \mathbb{C}^{n+1} | \sum_{i=0}^{n} |z_i|^2 = 1\}.$$

The equivalence is given by

$$\vec{z} \sim \vec{w} \Leftrightarrow \exists \phi \in \mathbb{R}\ \forall j \in \{0, \cdots, n\} : z_j = e^{i\phi} w_j.$$

So the unit sphere S^{2n+1} is the preimage of the Hopf fibration

$$\pi : S^{2n+1} \to \mathbb{C}P^n : \vec{z} \to [\vec{z}]\,.$$

On $S^{2n+1} \subset \mathbb{C}^{n+1}$ the complex structure J induces a contact structure and the standard metric on $\mathbb{C}^{n+1}$ induces a Riemannian metric. The metric on $\mathbb{C}P^n$ that makes π a Riemannian submersion has constant holomorphic sectional curvature 4. An immersion $\phi : M \to S^{2n+1}$ is then said to be C-totally real or horizontal if $i\phi$ is orthogonal to the submanifold. It can be shown that every minimal C-totally real submanifold of S^{2n+1} can be projected onto a special Lagrangian submanifold of $\mathbb{C}P^n$ through π and conversely that a special Lagrangian submanifold in $\mathbb{C}P^n$ has a 1-parameter family of mutually isometric horizontal lifts as a minimal C-totally real submanifold in S^{2n+1}. So in order to classify special Lagrangian submanifolds in $\mathbb{C}P^n$, we can consider minimal C-totally real submanifolds in $S^{2n+1} \subset \mathbb{C}^{n+1}$, see [12]. For those submanifolds, the Gauss identity is given as

$$D_X Y = \nabla_X Y + JA(X, Y) - \langle X, Y\rangle \phi, \tag{5}$$

where D is the Levi Civita connection of $\mathbb{C}^{n+1}$.

Similarly, the space $\mathbb{C}H^n$ can be modeled as H^{2n+1}/S^1, where

$$H^{2n+1} = \{(z_0, \cdots, z_n) \in \mathbb{C}_1^{n+1} | |z_0|^2 - \sum_{i=1}^{n} |z_i|^2 = 1\}.$$

The equivalence relationship determined by S^1 is the same as the one used in the projective space. The ambient space $\mathbb{C}_1^{n+1}$ is essentially the space $\mathbb{C}^{n+1}$, but equipped with the scalar product

$$\langle \vec{z}, \vec{w}\rangle_1 = \Re\left(\sum_{j=1}^{n} z_j \bar{w}_j - z_0 \bar{w}_0\right).$$

The complex structure is still obtained through multiplication with the imaginary unit i and induces a Kähler structure on $\mathbb{C}_1^{n+1}$. This metric induces a Lorentzian metric on H^{2n+1} and a metric of constant holomorphic sectional curvature -4 on $\mathbb{C}H^n$. Similar to the projective case C-totally real submanifolds $\phi : M \to H^{2n+1}$ can be defined having $i\phi$ as a normal. Each minimal C-totally real submanifold corresponds to the horizontal lift of a special Lagrangian submanifold of $\mathbb{C}H^n$. The Gauss identity is given as

$$D_X Y = \nabla_X Y + JA(X, Y) + \langle X, Y\rangle \phi, \tag{6}$$

where D is the Levi Civita connection of $\mathbb{C}_1^{n+1}$.

2.2. **Structure equations.** We can return briefly to the equations (2) and (3). We can choose an orthogonal frame $\{X_1, X_2, X_3, X_4\}$ corresponding to (4) and define the components Γ_{ij}^k and A_{ij}^k as

$$\nabla_{X_i} X_j = \sum_{k=1}^{4} \Gamma_{ij}^k X_k,$$

$$A(X_i, X_j) = \sum_{k=1}^{4} A_{ij}^k X_k.$$

Then the equations (2) and (3) can be rewritten as

$$\begin{aligned} X_i\left(\Gamma_{jk}^l\right) - X_j\left(\Gamma_{ik}^l\right) =& \epsilon\left(\delta_{jk}\delta_i^l - \delta_{ik}\delta_j^l\right) + A_{jk}^r A_{ir}^l - A_{ik}^r A_{jr}^l \\ & + \Gamma_{ik}^r \Gamma_{jr}^l - \Gamma_{jk}^r \Gamma_{ir}^l + \Gamma_{rk}^l\left(\Gamma_{ij}^r - \Gamma_{ji}^r\right), \end{aligned} \tag{7}$$

$$X_i\left(A_{jk}^l\right) - X_j\left(A_{ik}^l\right) = \left(\Gamma_{ij}^r - \Gamma_{ji}^r\right) A_{rk}^l + \Gamma_{ik}^r A_{jr}^l - \Gamma_{jk}^r A_{ir}^l - \Gamma_{ir}^l A_{jk}^r + \Gamma_{jr}^l A_{ik}^r, \tag{8}$$

where we have used the Einstein convention. We split the connection ∇ into its components and write

$$\begin{array}{r|l}
\nabla_{X_1} X_1 = a_1 X_2 + a_2 X_3 + a_3 X_4, & \nabla_{X_1} X_2 = -a_1 X_1 + a_4 X_3 + a_5 X_4, \\
\nabla_{X_2} X_1 = b_1 X_2 + b_2 X_3 + b_3 X_4, & \nabla_{X_2} X_2 = -b_1 X_1 + b_4 X_3 + b_5 X_4, \\
\nabla_{X_3} X_1 = c_1 X_2 + c_2 X_3 + c_3 X_4, & \nabla_{X_3} X_2 = -c_1 X_1 + c_4 X_3 + c_5 X_4, \\
\nabla_{X_4} X_1 = d_1 X_2 + d_2 X_3 + d_3 X_4, & \nabla_{X_4} X_2 = -d_1 X_1 + d_4 X_3 + d_5 X_4,
\end{array}$$

$$\begin{array}{r|l}
\nabla_{X_1} X_3 = -a_2 X_1 - a_4 X_2 + a_6 X_4, & \nabla_{X_1} X_4 = -a_3 X_1 - a_5 X_2 - a_6 X_3, \\
\nabla_{X_2} X_3 = -b_2 X_1 - b_4 X_2 + b_6 X_4, & \nabla_{X_2} X_4 = -b_3 X_1 - b_5 X_2 - b_6 X_3, \\
\nabla_{X_3} X_3 = -c_2 X_1 - c_4 X_2 + c_6 X_4, & \nabla_{X_3} X_4 = -c_3 X_1 - c_5 X_2 - c_6 X_3, \\
\nabla_{X_4} X_3 = -d_2 X_1 - d_4 X_2 + d_6 X_4, & \nabla_{X_4} X_4 = -d_3 X_1 - d_5 X_2 - d_6 X_3.
\end{array}$$

Equation (8) induces linear relations between the components, independent of the ambient space. The Gauss equations give further information about ∇ but use differential equations and depend on the ambient space form.

Lemma 2.1. *On a special Lagrangian submanifold M having a local $SO(2) \rtimes S_3$-symmetry there exists a frame corresponding to* (4) *such that:*

$$\begin{array}{c|c}
\nabla_{X_1} X_1 = a_1 X_2 + a_2 X_3 + a_3 X_4, & \nabla_{X_1} X_2 = -a_1 X_1 - b_2 X_3, \\
\nabla_{X_2} X_1 = b_1 X_2 + b_2 X_3, & \nabla_{X_2} X_2 = -b_1 X_1 + a_2 X_3 + a_3 X_4, \\
\nabla_{X_3} X_1 = \frac{b_2}{3} X_2, & \nabla_{X_3} X_2 = -\frac{b_2}{3} X_1, \\
\nabla_{X_4} X_1 = 0, & \nabla_{X_4} X_2 = 0, \\
\nabla_{X_1} X_3 = -a_2 X_1 + b_2 X_2 + a_6 X_4, & \nabla_{X_1} X_4 = -a_3 X_1 - a_6 X_3, \\
\nabla_{X_2} X_3 = -b_2 X_1 - a_2 X_2 + b_6 X_4, & \nabla_{X_2} X_4 = -a_3 X_2 - b_6 X_3, \\
\nabla_{X_3} X_3 = c_6 X_4, & \nabla_{X_3} X_4 = -c_6 X_3, \\
\nabla_{X_4} X_3 = d_6 X_4, & \nabla_{X_4} X_4 = -d_6 X_3.
\end{array} \tag{9}$$

Furthermore, the derivatives of r are given by

$$(X_1 + iX_2)(r) = 3ir(a_1 + ib_1), \tag{10}$$

$$X_3(r) = ra_2, \tag{11}$$

$$X_4(r) = ra_3. \tag{12}$$

Proof. This is just a straightforward application of equation (8). For instance

$$\begin{aligned}(\nabla_{X_2}A)(X_1,X_1) &= X_2(r)X_1 + 3rb_1X_2 + rb_2X_3 + rb_3X_4,\\ (\nabla_{X_1}A)(X_2,X_1) &= 3ra_1X_1 - X_1(r)X_2 - ra_4X_3 - ra_5X_4.\end{aligned}$$

Then the corresponding coordinates of both derivatives are the same. Finally we can set $b_3 = 0$, by rotating the distribution $\mathcal{N}_2$ such that X_3 lies in the direction of $\nabla_{X_1}X_2$, projected on $\mathcal{N}_2$. □

It is interesting to note that $\mathcal{N}_2$ is an integrable distribution. The distribution $\mathcal{N}_1$ however is integrable if and only if $b_2 = 0$. Applying (7), we obtain the following result.

Lemma 2.2. *The equations* (7) *on our frame of choice induce a system of differential equations given by:*

$$(X_1 + iX_2)(a_2 - ib_2) = a_3(a_6 + ib_6), \tag{13}$$

$$X_3(a_2 + ib_2) = \epsilon + a_3^2 + (a_2 + ib_2)^2, \tag{14}$$

$$X_4(a_2 + ib_2) = a_3(a_2 + ib_2), \tag{15}$$

$$(X_1 + iX_2)(a_3) = -(a_2 - ib_2)(a_6 + ib_6), \tag{16}$$

$$X_3(a_3) = 0, \tag{17}$$

$$X_4(a_3) = a_3^2 + \epsilon, \tag{18}$$

$$X_1(b_6) - X_2(a_6) = -(a_1a_6 + b_1b_6), \tag{19}$$

$$X_3(a_6 + ib_6) = \frac{5}{3}ib_2(a_6 + ib_6), \tag{20}$$

$$X_4(a_6 + ib_6) = 2a_3(a_6 + ib_6), \tag{21}$$

$$X_1(b_1) - X_2(a_1) = 2r^2 - (\epsilon + a_3^2) - \frac{5}{3}b_2^2 - a_2^2 - a_1^2 - b_1^2, \tag{22}$$

$$X_2(b_1) + X_1(a_1) = -\frac{2}{3}a_2b_2, \tag{23}$$

$$3X_3(a_1) - X_1(b_2) = 3a_1a_2 - 2b_1b_2, \tag{24}$$

$$3X_3(b_1) - X_2(b_2) = 2b_2a_1 + 3b_1a_2, \tag{25}$$

$$X_4(a_1 + ib_1) = a_3(a_1 + ib_1) + \frac{b_2}{3}(a_6 + ib_6). \tag{26}$$

Proof. This is also a straightforward application of (7). For example:

$$\begin{aligned}X_1\left(\Gamma_{23}^1\right) - X_2\left(\Gamma_{13}^1\right) &= \Gamma_{13}^r\Gamma_{2r}^1 - \Gamma_{23}^r\Gamma_{1r}^1 + \Gamma_{r3}^1\Gamma_{12}^r - \Gamma_{r3}^1\Gamma_{21}^r\\ &= a_3b_6,\\ X_1\left(\Gamma_{23}^2\right) - X_2\left(\Gamma_{13}^2\right) &= \Gamma_{13}^r\Gamma_{2r}^2 - \Gamma_{23}^r\Gamma_{1r}^2 + \Gamma_{r3}^2\Gamma_{12}^r - \Gamma_{r3}^2\Gamma_{21}^r\\ &= -a_3a_6.\end{aligned}$$

Combining both equations using the usual complex notations leads to (13). The other equations are obtained in a similar way. □

2.3. **Warped Products.** In the analysis that follows, we will often encounter warped products of manifolds. When we consider a warped product of Riemannian manifolds (M_1, g_1) and (M_2, g_2) with warping function e^f, where $f : M_1 \to \mathbb{R}$, we get a Riemannian manifold $(M_1 \times M_2, g_f)$ where $M_1 \times M_2$ as a differentiable manifolds is the product of M_1 and M_2 and the metric g_f is given as

$$g_f(X,Y) = g_1(X_1,Y_1) + e^{2f} g_2(X_2,Y_2),$$

where a vector field X is uniquely decomposed into a part X_1 tangent to M_1 and X_2 tangent to M_2. We denote this warped product as $M_1 \times_{e^f} M_2$. The following result can be obtained, see [9].

Theorem 2.1. *Consider a Riemannian manifold (M,g) with Levi-Civita connection ∇ and suppose that on a neighborhood of $p \in M$ there are orthogonal distributions $\mathcal{N}_1$ and $\mathcal{N}_2$ such that*

$$\begin{aligned} &\forall X,Y \in \mathcal{N}_1 (\textit{i.e. } X \textit{ and } Y \textit{ are sections of } \mathcal{N}_1) : \nabla_X Y \in \mathcal{N}_1, \\ &\forall X,Y \in \mathcal{N}_2 : \nabla_X Y = \tilde{\nabla}_X Y + g(X,Y)H, \end{aligned}$$

where $\tilde{\nabla}$ is the projection of ∇ on $\mathcal{N}_2$ and $H \in \mathcal{N}_1$. Then there exists a function $f : M \to \mathbb{R}$ such that on a neighborhood of p, M can be written as $M_1 \times_{e^f} M_2$, where M_i is an integral manifold of $\mathcal{N}_i$.
If furthermore $H = \lambda H_0$, where $\|H_0\| = 1$, and $X(\lambda) = 0$ for every $X \in \mathcal{N}_2$, then $f : M_1 \to \mathbb{R}$ and $H = -\operatorname{grad} e^f$.

The first part of the theorem constructs a twisted product, the second part reduces this to a warped product. This will be useful in choosing coordinates, since the product structures allows for coordinates to be chosen on each factor separately. In particular, if $\dim(\mathcal{N}_1) = 1$, then any non vanishing vector field in $\mathcal{N}_1$ can be fixed as a useful coordinate vector field on M.

3. Submanifolds in $\mathbb{C}^4$.

3.1. **The case where $b_2 = 0$.** The assumption that X_3 lies along $\nabla_{X_1} X_2$ becomes redundant since the latter has no $\mathcal{N}_2$ component. Instead, we can choose X_3 in the direction of $\nabla_{X_1} X_1$, projected on $\mathcal{N}_2$. Hence without loss of generality we can assume that $a_3 = 0$. The equations (7) show that in this case either $a_2 = 0$ or $a_6 = b_6 = c_6 = d_6 = 0$. First we will assume that $a_2 \neq 0$.

Theorem 3.1. *Consider M a special Lagrangian submanifold in $\mathbb{C}^4$ having $SO(2) \rtimes S_3$-symmetry and an orthogonal frame corresponding to (4). Suppose that $\mathcal{N}_1$ is an integrable distribution and $\nabla_{X_1} X_1$ is nowhere contained within this distribution. Then M is locally congruent to*

$$F(t,s,u,v) = (t, s\phi(u,v)) \tag{27}$$

where ϕ is the horizontal lift of a special Lagrangian submanifold of $\mathbb{C}P^2$ to the unit sphere in $\mathbb{C}^3$.

Proof. Taking into account every component that vanishes in (9), we find

$$\begin{array}{c|c}
\nabla_{X_1}X_1 = a_1X_2 + a_2X_3 & \nabla_{X_1}X_2 = -a_1X_1, \\
\nabla_{X_2}X_1 = b_1X_2 & \nabla_{X_2}X_2 = -b_1X_1 + a_2X_3, \\
\nabla_{X_3}X_1 = 0 & \nabla_{X_3}X_2 = 0, \\
\nabla_{X_4}X_1 = 0 & \nabla_{X_4}X_2 = 0, \\
\nabla_{X_1}X_3 = -a_2X_1 & \nabla_{X_1}X_4 = 0, \\
\nabla_{X_2}X_3 = -a_2X_2 & \nabla_{X_2}X_4 = 0, \\
\nabla_{X_3}X_3 = 0 & \nabla_{X_3}X_4 = 0, \\
\nabla_{X_4}X_3 = 0 & \nabla_{X_4}X_4 = 0.
\end{array}$$

We find that the distributions span$\{X_4\}$ and span$\{X_1, X_2, X_3\}$ satisfy the conditions for a warped product $\mathbb{R} \times_{e^f} N^3$. But $X_4(f) = 0$, hence f is a constant. M is a standard Riemannian product $\mathbb{R} \times N^3$ and its immersion can be written, up to an isometry as

$$F(t, x) = (t, \psi(x)), \; \psi : N^3 \to \mathbb{C}^4.$$

The immersion ψ is contained in the subspace orthogonal to both X_4 and JX_4, since they both are constant unit normals along N^3. Now it is also obvious that span$\{X_3\}$ and $\mathcal{N}_1$ satisfy the conditions for a warped product. So N^3 can be decomposed as $\mathbb{R} \times_{e^g} N^2$ and $X_3(g) = -a_2$. Then X_3 can be associated with a coordinate s on the manifold and it follows that

$$D_{X_3}X_3 = \frac{\partial^2 F}{\partial s^2} = 0 \Rightarrow F = As + B.$$

Both A and B are independent of (s, t). Calculating (7), one has

$$X_3(a_2) = \frac{\partial a_2}{\partial s} = a_2^2.$$

The solution of this equation, after a translation of the s-coordinate is given as $a_2 = -\frac{1}{s}$. The derivatives of X_3 to X_1 and X_2 are

$$\begin{aligned}
D_{X_i}X_3 &= \frac{\partial F_* X_i}{\partial s} = A_* X_i \\
&= \frac{1}{s}X_i = A_*X_i + \frac{B_*X_i}{s} \\
&\Rightarrow B_* = 0.
\end{aligned}$$

So B is a constant vector along the submanifold and vanishes when applying a translation. It is easy to see that $X_3 = A$ and is orthogonal to $X_i = sA_*(X_i)$, for $i \in \{1, 2\}$. Hence everywhere along A, the position vector is orthogonal to the tangent space. Thus A has constant length. Calculating the other covariant derivatives yields for example for $i, j \in \{1, 2\}$ that

$$\begin{aligned}
A_*X_i &= \frac{F_*X_i}{s}, \\
D_{X_i}(A_*X_j) &= \frac{D_{X_i}(F_*X_j)}{s} = A_*\left(\tilde{\nabla}_{X_i}X_j\right) + JA_*(K(X_i, X_j)) - \frac{1}{s^2}\delta_{ij}\phi.
\end{aligned} \tag{28}$$

Here $\tilde{\nabla}$ is the connection restricted to N^2. Combining this with the other equations in (7), it follows that A is a C-totally real immersion in $S^5 \subset \mathbb{C}^3$. Furthermore, the components a_1 and b_1 have no other restrictions on them except satisfying the Gauss equations for a minimal C-totally real submanifold of S^5. This proves the theorem. □

The case $a_2 = 0$ was the only case that was studied in [7]. We can quote the following result from [7].

Theorem 3.2. *Consider M a special Lagrangian submanifold in $\mathbb{C}^4$ having a local $SO(2) \rtimes S_3$-symmetry group and an orthogonal frame corresponding to (4). Suppose that $\mathcal{N}_1$ is an integrable distribution and $\nabla_{X_1} X_1$ is contained within this distribution. Then M is locally congruent to*

$$F(t, s, u, v) = (t, s, \phi(u, v)) \tag{29}$$

where $\phi : \mathbb{C} \to \mathbb{C}^2$ is a special Lagrangian surface.

Remark 3.1. *As proved in [7], a special Lagrangian surface in $\mathbb{C}^2$, with complex coordinates $x_1 + iy_1$ and $x_2 + iy_2$ is a holomorphic curve in $\mathbb{C}^2$ with complex coordinates $x_1 - ix_2$ and $y_1 + iy_2$, and conversely.*

3.2. **The case where $b_2 \neq 0$.** Now the distribution $\mathcal{N}_1$ is no longer integrable. The simplest case one can hope for is that there is a 3-dimensional integrable distribution containing $\mathcal{N}_1$. Such a distribution should contain at least X_3 since

$$[X_1, X_2] \quad \mathrm{mod}\ \mathcal{N}_1 \parallel X_3.$$

Using the fact that $b_2 \neq 0$, the equations (7) reduce (9) to

$$\begin{array}{c|c} \nabla_{X_1} X_1 = a_1 X_2 + a_2 X_3 + a_3 X_4 & \nabla_{X_1} X_2 = -a_1 X_1 - b_2 X_3, \\ \nabla_{X_2} X_1 = b_1 X_2 + b_2 X_3 & \nabla_{X_2} X_2 = -b_1 X_1 + a_2 X_3 + a_3 X_4, \\ \nabla_{X_3} X_1 = \frac{b_2}{3} X_2 & \nabla_{X_3} X_2 = -\frac{b_2}{3} X_1, \\ \nabla_{X_4} X_1 = 0 & \nabla_{X_4} X_2 = 0, \\ \nabla_{X_1} X_3 = -a_2 X_1 + b_2 X_2 + a_6 X_4 & \nabla_{X_1} X_4 = -a_3 X_1 - a_6 X_3, \\ \nabla_{X_2} X_3 = -b_2 X_1 - a_2 X_2 + b_6 X_4 & \nabla_{X_2} X_4 = -a_3 X_2 - b_6 X_3, \\ \nabla_{X_3} X_3 = a_3 X_4 & \nabla_{X_3} X_4 = -a_3 X_3, \\ \nabla_{X_4} X_3 = 0 & \nabla_{X_4} X_4 = 0. \end{array} \tag{30}$$

It is apparent that the condition that $\mathcal{N}_+$ is integrable is given by $a_6 + ib_6 = 0$. We consider this case first.

Theorem 3.3. *Suppose M is a special Lagrangian submanifold in $\mathbb{C}^4$ with $SO(2) \rtimes S_3$-symmetry, such that $\mathcal{N}_1$ is not an integrable distribution, but $\mathcal{N}_+$ is. Then the submanifold, up to isometry, can be given locally by either*

$$F(t, s, u, v) = (t, \phi(s, u, v)), \tag{31}$$

where ϕ is a special Lagrangian submanifold with S_3-symmetry in $\mathbb{C}^3$, or

$$F(t, s, u, v) = t\phi(s, u, v) \tag{32}$$

where ϕ is the horizontal lift of a special Lagrangian submanifold with local S_3-symmetry in $\mathbb{C}P^3$ to the unit sphere in $\mathbb{C}^4$.

Proof. We find according to (30) and (7) that span$\{X_4\}$ and $\mathcal{N}_+$ satisfy the conditions for a warped product. So M can be decomposed as $\mathbb{R} \times_{e^f} N^3$, where $X_4(f) = -a_3$. We can solve

$$X_4(a_3) = \frac{\partial a_3}{\partial t} = a_3^2.$$

This equation has 2 possible solutions.
First, we assume $a_3 = 0$. In this case M is simply the manifold $\mathbb{R} \times N^3$. Hence the immersion, up to isometry, can be given as

$$F(t, s, u, v) = (t, \phi(s, u, v)),$$

where ϕ is a 3-fold immersed in the subspace $\mathbb{C}^3$ orthogonal to X_4 and JX_4. Similar calculations as in (28) show that this can be any special Lagrangian submanifold in $\mathbb{C}^3$, given the presence of an S_3-symmetry in the second fundamental form.
The second solution, after a translation of t, is given by $a_3 = -\frac{1}{t}$. The calculations are similar to the case where $b_2 = 0$ and $a_2 \neq 0$. This gives the required result. □

The last case in $\mathbb{C}^4$ is the one where there is no integrable distribution containing $\mathcal{N}_1$ other than the whole tangent bundle. In this case, we can no longer rely on an obvious warped product structure. We can attempt to introduce a set of independent coordinates and reduce (7) to a system of PDE's on $\mathbb{C}^4$ using as little functions as possible. We now use (13) to (26) to construct a coordinate frame from $\{X_1, X_2, X_3, X_4\}$. Since $\mathcal{N}_2$ is integrable, it is a good idea to choose $X_4 = T$ and $\mu X_3 = S$. Requiring that $[S, T] = 0$ implies that

$$[\mu X_3, X_4] = \mu\,[X_3, X_4] - X_4(\mu) X_3 = -\left(\mu a_3 + X_4(\mu)\right) X_3 = 0.$$

We can find such a μ by taking $\mu = \frac{1}{\sqrt{|\epsilon + a_3^2|}}$. The equation $a_3^2 + \epsilon = 0$ implies that a_3 is a constant and hence $(a_2 - ib_2)(a_6 + ib_6) = 0$. This will correspond to the integrability of either $\mathcal{N}_1$ or $\mathcal{N}_+$. Therefore μ is well defined.

Vector fields U and V can be sought such that every couple out of $\{S, T, U, V\}$ commutes. Such an attempt can be made, writing

$$U + iV = (\rho_1 - i\rho_2)\Big((X_1 + iX_2) + (\alpha_1 + i\beta_1)S + (\alpha_2 + i\beta_2)T\Big) \tag{33}$$

We rename the following expressions:

$$\begin{aligned} \rho &= \rho_1 - i\rho_2, \\ \gamma_j &= \alpha_j + i\beta_j \qquad j \in \{1, 2\}. \end{aligned}$$

After calculating the Lie brackets of these four vector fields, the following conditions on the introduced functions make the vector fields commute:

$$X_4(\rho) = -a_3\rho, \tag{34}$$

$$X_3(\rho) = -\left(a_2 + \frac{2}{3}ib_2\right)\rho, \tag{35}$$

$$(X_1 - iX_2)(\rho) = (b_1 + ia_1)\,\rho, \tag{36}$$

$$X_4(\gamma_1) = -\frac{1}{\mu}(a_6 + ib_6) + a_3\gamma_1, \tag{37}$$

$$X_3(\gamma_1) = \frac{1}{\mu^2}(X_1 + iX_2)(\mu) + \gamma_1\left(a_2 + \frac{2}{3}ib_2\right), \tag{38}$$

$$X_2(\alpha_1) - X_1(\beta_1) = a_1\alpha_1 + b_1\beta_1 - \frac{2}{\mu}b_2, \tag{39}$$

$$X_4(\gamma_2) = a_3\gamma_2, \tag{40}$$

$$X_3(\gamma_2) = (a_6 + ib_6) + \left(a_2 + \frac{2}{3}ib_2\right)\gamma_2, \tag{41}$$

$$X_2(\alpha_2) - X_1(\beta_2) = a_1\alpha_2 + b_1\beta_2. \tag{42}$$

The following result can be obtained.

Lemma 3.1. *Suppose f and g are real valued functions on the manifold satisfying*

$$\begin{aligned} S(f) &= 0, \quad T(f) = -1,\\ S(g) &= -1, \quad T(g) = 0, \end{aligned}$$

and defining

$$\begin{aligned} X_1(f) &= \alpha_2, \quad X_2(f) = \beta_2,\\ X_1(g) &= \alpha_1, \quad X_2(g) = \beta_1, \end{aligned}$$

then the functions α_i and β_i obtained this way satisfy the conditions (37) *to* (42).

It is interesting to see that this way the vector fields

$$\tilde{U} = X_1 + \alpha_1 S + \alpha_2 T,$$
$$\tilde{V} = X_2 + \beta_1 S + \beta_2 T,$$

satisfy $\tilde{U}(f) = \tilde{U}(g) = \tilde{V}(f) = \tilde{V}(g) = 0$. Furthermore $\tilde{U}$ and $\tilde{V}$ are independent of one-another and they span the distribution which is the intersection of the kernel of $\mathrm{d}\,f$ and $\mathrm{d}\,g$. Note that this distribution is indeed 2-dimensional since both forms have a hyperplane as a kernel and these kernels do not coincide, since the 1-forms are linearly independent. Using the dimension theorem, they have a 2-dimensional intersection. Construction (33) is just a complex rotation of these two vector fields in that distribution. This way, it is clear that f and g serve as coordinates s and t conjugate to S and T.

Proof. Apply the relation

$$[X_i, X_j]\,(f) = X_i X_j(f) - X_j X_i(f) = \nabla_{X_i} X_j(f) - \nabla_{X_j} X_i(f)$$

on both functions, using (30). ∎

A suitable function for f is easily found, since $S(a_3) = 0$. Let f be a function of a_3, then

$$X_4(f) = f'(\epsilon + a_3^2) = -1 \Leftrightarrow f' = -\frac{1}{\epsilon + a_3^2}.$$

Hence f can be given by

$$f = -\int \frac{1}{\epsilon + a_3^2}\,\mathrm{d}\,a_3.$$

This also determines γ_2 completely, since using (13) yields

$$\begin{aligned}\gamma_2 &= (X_1 + iX_2)(f) = f'(X_1 + iX_2)(a_3)\\ &= \frac{a_2a_6 + b_2b_6}{\epsilon + a_3^2} + i\frac{a_2b_6 - b_2a_6}{\epsilon + a_3^2}.\end{aligned}$$

As for the function g, the complex valued function $z = \mu(a_2 + ib_2)$ can be considered and calculations show

$$\begin{aligned}X_4(z) &= -\mu a_3(a_2 + ib_2) + \mu a_3(a_2 + ib_2) = 0,\\ S(z) &= \mu^2\left(\epsilon + a_3^2 + (a_2 + ib_2)^2\right) = \mathrm{sg}\left(\epsilon + a_3^2\right) + z^2.\end{aligned}$$

Rewriting $\tilde\epsilon = \mathrm{sg}(\epsilon + a_3^2)$, we find that z is useful as long as $z^2 + \tilde\epsilon \neq 0$. When $\tilde\epsilon = +1$, this occurs when $a_2 = 0$ and $|b_2| = \sqrt{\epsilon + a_3^2}$. When $\tilde\epsilon = -1$, this occurs when $|a_2| = \sqrt{|\epsilon + a_3^2|}$ and $b_2 = 0$, resulting in $\mathcal{N}_1$ being integrable.

First we assume that $z^2 \neq -\tilde\epsilon$. Then the function g can be calculated as the real part of a function G of z given by

$$S(G) = (\tilde\epsilon + z^2)G' = -1 \Leftrightarrow G' = -\frac{1}{\tilde\epsilon + z^2}.$$

A function ρ still has to be constructed satisfying (34) to (36). Define a function H as

$$H = \rho^3 r(z^2 + \tilde\epsilon)\,|\epsilon + a_3^2|.$$

This function is a constant on the submanifold and can be taken to be equal to 1. This defines a function ρ satisfying the necessary conditions.

Using the Frobenius theorem in [10], a coordinate frame on the submanifold is given by

$$\begin{aligned}X_4 &= T,\\ X_3 &= \frac{1}{\mu}S,\\ X_1 + iX_2 &= \frac{U + iV}{\rho} - \gamma_1 S - \gamma_2 T.\end{aligned}$$

We can describe the dependence of $a_6 + ib_6$ on (s,t) by writing

$$a_6 + ib_6 = \frac{k_3 + ik_4}{\rho}\sqrt{|a_3^2 + \epsilon|}\left(\bar z^2 + \tilde\epsilon\right)^{\frac{-1}{2}}.$$

The functions k_3 and k_4 depend solely on (u,v). This expression is obtained from (20) and (21). The rest of the equations in (7) can be rewritten and solved. Applying our method for $\epsilon = 0$, we find after a translation of the coordinates that

$$\begin{aligned}
&a_3 = -\frac{1}{t},\\
x = \frac{\sin(2s)}{\cos(2s)+\cosh(2k_1)} &\Rightarrow a_2 = -\frac{\sin(2s)}{t(\cos(2s)+\cosh(2k_1))},\\
y = \frac{\sinh(2k_1)}{\cos(2s)+\cosh(2k_1)} &\Rightarrow b_2 = -\frac{\sinh(2k_1)}{t(\cos(2s)+\cosh(2k_1))},\\
&r = \frac{e^{k_2}}{t\sqrt{\cos(2s)+\cosh(2k_1)}}.
\end{aligned}$$

Here the functions k_1 and k_2 depend solely on (u, v). Then we can use (13) to find an expression for γ_1 in terms of the coordinates. Equation (10) can be used to find an expression for a_1 and b_1 in terms of the coordinates. We obtain

$$\begin{aligned}
\gamma_1 &= \frac{(k_3+ik_4)\cos(s-ik_1)+t\left(\frac{\partial k_1}{\partial v}-i\frac{\partial k_1}{\partial u}\right)}{t\rho},\\
a_1 &= \frac{2^{\frac{2}{3}}e^{\frac{2}{3}k_2}}{3t^3\left(\cos(2s)+\cosh(2k_1)\right)^2}\Big(t\big((\cos(2s)+\cosh(2k_1))(\rho_1\frac{\partial k_2}{\partial v}+\rho_2\frac{\partial k_2}{\partial u})\\
&\quad+\sin(2s)(\rho_1\frac{\partial k_1}{\partial u}-\rho_2\frac{\partial k_1}{\partial v})-\sinh(2k_1)(\rho_1\frac{\partial k_1}{\partial v}+\rho_2\frac{\partial k_1}{\partial u})\big)\\
&\quad+\sinh(2k_1)\left(\cos(s)\cosh(k_1)(k_4\rho_2-k_3\rho_1)+\sin(s)\sinh(k_1)(k_4\rho_1+k_3\rho_2)\right)\Big),\\
b_1 &= \frac{2^{\frac{2}{3}}e^{\frac{2}{3}k_2}}{3t^3\left(\cos(2s)+\cosh(2k_1)\right)^2}\Big(t\big((\cos(2s)+\cosh(2k_1))(\rho_2\frac{\partial k_2}{\partial v}-\rho_1\frac{\partial k_2}{\partial u})\\
&\quad+\sin(2s)(\rho_2\frac{\partial k_1}{\partial u}+\rho_1\frac{\partial k_1}{\partial v})+\sinh(2k_1)(\rho_1\frac{\partial k_1}{\partial u}-\rho_2\frac{\partial k_1}{\partial v})\big)\\
&\quad+\sinh(2k_1)\left(\sin(s)\sinh(k_1)(k_4\rho_2-k_3\rho_1)-\cos(s)\cosh(k_1)(k_4\rho_1+k_3\rho_2)\right)\Big).
\end{aligned}$$

Now every function on the submanifold is expressed in terms of (s, t, u, v), possibly indirectly through $\{k_1, k_2, k_3, k_4\}$. Demanding that the other Gauss equations are satisfied gives partial differential equations for k_i, given by

$$\begin{aligned}
\frac{\partial k_4}{\partial u}-\frac{\partial k_3}{\partial v} &= 2\tanh(k_1)\left(k_3\frac{\partial k_1}{\partial v}-k_4\frac{\partial k_1}{\partial u}\right),\\
\frac{\partial k_4}{\partial v}+\frac{\partial k_3}{\partial u} &= -2\coth(k_1)\left(k_3\frac{\partial k_1}{\partial u}+k_4\frac{\partial k_1}{\partial v}\right),\\
\Delta k_2 &= 3*2^{\frac{1}{3}}e^{-\frac{2k_2}{3}}\left(-e^{2k_2}+\cosh(2k_1)\right),\\
\Delta k_1 &= -2^{\frac{1}{3}}e^{-\frac{2k_2}{3}}\sinh(2k_1).
\end{aligned} \tag{43}$$

Now we return to the case where $-1 = z^2$ and $\tilde{\epsilon} = 1$. We assume first that ϵ isn't specified. In this case $a_2 = 0$, $b_2 = \pm\sqrt{\epsilon + a_3^2}$ and $S(z) = 0$, so z is insufficient to construct the function g. Equations (7) are reduced to

$$(X_1 - iX_2)(a_6 + ib_6) = -i(b_1 + ia_1)(a_6 + ib_6), \tag{44}$$

$$X_3(a_6 + ib_6) = \pm i\frac{5}{3}\sqrt{\epsilon + a_3^2}(a_6 + ib_6), \tag{45}$$

$$X_4(a_6 + ib_6) = 2a_3(a_6 + ib_6), \tag{46}$$

$$X_1(b_1) - X_2(a_1) = 2r^2 - \frac{8}{3}(\epsilon + a_3^2) - a_1^2 - b_1^2, \tag{47}$$

$$X_1(a_1) + X_2(b_1) = 0, \tag{48}$$

$$X_4(a_1 + ib_1) = a_3(a_1 + ib_1) \pm \sqrt{\epsilon + a_3^2}\frac{(a_6 + ib_6)}{3}, \tag{49}$$

$$X_3(a_1 + ib_1) = \frac{i}{3}\left(a_3(a_6 + ib_6) \pm 2\sqrt{\epsilon + a_3^2}(a_1 + ib_1)\right). \tag{50}$$

The first equation is obtained from applying integrability on a_3. Now we define

$$w = \frac{a_6 + ib_6}{\epsilon + a_3^2},$$

which after derivation gives

$$X_4(w) = -2a_3\frac{(a_6 + ib_6)}{\epsilon + a_3^2} + 2a_3\frac{a_6 + ib_6}{\epsilon + a_3^2} = 0,$$
$$S(w) = \pm i\frac{5}{3}w.$$

The resulting differential equation for a function G of w will be

$$S(G) = \pm G' i\frac{5}{3}w = -1 \Leftrightarrow G' = \pm i\frac{3}{5w}.$$

The solution is that G is a logarithm of w. We find that H defined by

$$H = w^2(\epsilon + a_3^2)^2\rho^5 r$$

is a constant and hence can be used to express ρ. We can thus solve w as

$$w = e^{k_1 \pm i\frac{5}{3}s} = e^{k_1}\left(\cos(\frac{5}{3}s) \pm i\sin(\frac{5}{3}s)\right).$$

Applying (18),(46),(45),(12) and (11) when $\epsilon = 0$ yields

$$a_3 = -\frac{1}{t},$$
$$a_6 = e^{k_1}\frac{\cos(\frac{5}{3}s)}{t^2},$$
$$b_6 = \pm e^{k_1}\frac{\sin(\frac{5}{3}s)}{t^2},$$
$$r = \frac{e^{k_2}}{t}.$$

The equation (10) now gives $a_1 + ib_1$ immediately without going through γ_1 because of (11). The final unknown, γ_1 can then be determined using (44). When we pick $b_2 = a_3$, we obtain

$$\gamma_{1+} = \frac{-5e^{k_1+\frac{5si}{3}} + e^{\frac{2k_1+k_2}{5}+\frac{2si}{3}} t\left((\frac{\partial k_2}{\partial v} - 3\frac{\partial k_1}{\partial v}) - i(\frac{\partial k_2}{\partial u} - 3\frac{\partial k_1}{\partial u})\right)}{5t^2},$$

$$a_{1+} = \frac{e^{k_1}\cos(\frac{5s}{3}) + e^{\frac{2k_1+k_2}{5}} t\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial v} + \sin(\frac{2s}{3})\frac{\partial k_2}{\partial u}\right)}{3t^2},$$

$$b_{1+} = \frac{e^{k_1}\sin(\frac{5s}{3}) - e^{\frac{2k_1+k_2}{5}} t\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial u} - \sin(\frac{2s}{3})\frac{\partial k_2}{\partial v}\right)}{3t^2},$$

and for $b_2 = -a_3$ we obtain

$$\gamma_{1-} = \frac{-5e^{k_1-\frac{5si}{3}} + e^{\frac{2k_1+k_2}{5}-\frac{2si}{3}} t\left((3\frac{\partial k_1}{\partial v} - \frac{\partial k_2}{\partial v}) - i(3\frac{\partial k_1}{\partial u} - \frac{\partial k_2}{\partial u})\right)}{5t^2},$$

$$a_{1-} = \frac{-e^{k_1}\cos(\frac{5s}{3}) + e^{\frac{2k_1+k_2}{5}} t\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial v} - \sin(\frac{2s}{3})\frac{\partial k_2}{\partial u}\right)}{3t^2},$$

$$b_{1-} = \frac{e^{k_1}\sin(\frac{5s}{3}) - e^{\frac{2k_1+k_2}{5}} t\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial u} + \sin(\frac{2s}{3})\frac{\partial k_2}{\partial v}\right)}{3t^2}.$$

Equations (47) and (39) result in restrictions on the functions k_1 and k_2 of (u, v) given by

$$\begin{aligned} \Delta k_2 &= e^{-\frac{2}{5}(2k_1+k_2)}(8 - 6e^{2k_2}), \\ \Delta k_1 &= e^{-\frac{2}{5}(2k_1+k_2)}(6 - 2e^{2k_2}). \end{aligned} \tag{51}$$

These equations are valid for both $b_2 = \pm a_3$. Using the constructed functions, the rest of the Gauss equations don't impose further conditions. We can summarize this result in the following theorem.

Theorem 3.4. *Each special Lagrangian submanifold of $\mathbb{C}^4$ with $SO(2) \rtimes S_3$-symmetry where the only integral distribution containing $\mathcal{N}_1$ is the tangent bundle, can be constructed in the way above using either functions $\{k_1, k_2, k_3, k_4\}$ subject to (43) or functions $\{k_1, k_2\}$ subject to (51). Conversely, each such a construction results in such a submanifold, unique up to local isometry.*

In the upcoming sections we will consider the construction for $\epsilon = \pm 1$.

4. Submanifolds in $\mathbb{C}P^4$.

4.1. **The case where $b_2 = 0$.** This means that both $\mathcal{N}_1$ and $\mathcal{N}_2$ are integrable distributions. We can assume $a_3 = 0$. However, the Gauss equation

$$X_3(a_2) = 1 + a_2^2 \tag{52}$$

no longer allows for a_2 being a constant. The following result is obtained:

Theorem 4.1. *Suppose M is a special Lagrangian submanifold in $\mathbb{C}P^4$ having $SO(2) \rtimes S_3$-symmetry. Suppose $\mathcal{N}_1$ is integrable. Then M can be lifted horizontally to a submanifold in the unit sphere of $\mathbb{C}^5$ through F and this lift is congruent to*

$$F(t, s, u, v) = (\phi(u, v)\cos(s), \sin(s)\cos(t), \sin(s)\sin(t)), \tag{53}$$

where ϕ is the horizontal lift of a special Lagrangian submanifold in $\mathbb{C}P^2$ to the unit sphere in $\mathbb{C}^3$.

Proof. Equations (7) reduce ∇ to

$$\begin{array}{c|c}
\nabla_{X_1}X_1 = a_1X_2 + a_2X_3 & \nabla_{X_1}X_2 = -a_1X_1, \\
\nabla_{X_2}X_1 = b_1X_2 & \nabla_{X_2}X_2 = -b_1X_1 + a_2X_3, \\
\nabla_{X_3}X_1 = 0 & \nabla_{X_3}X_2 = 0, \\
\nabla_{X_4}X_1 = 0 & \nabla_{X_4}X_2 = 0, \\
\nabla_{X_1}X_3 = -a_2X_1 & \nabla_{X_1}X_4 = 0, \\
\nabla_{X_2}X_3 = -a_2X_2 & \nabla_{X_2}X_4 = 0, \\
\nabla_{X_3}X_3 = 0 & \nabla_{X_3}X_4 = 0, \\
\nabla_{X_4}X_3 = \frac{X_4}{a_2} & \nabla_{X_4}X_4 = -\frac{X_3}{a_2}.
\end{array}$$

The distributions $\mathcal{N}_1$ and $\mathcal{N}_2$ satisfy the conditions for a warped product $N_2 \times_{e^f} N_1$. Furthermore, the distributions span$\{X_3\}$ and span$\{X_4\}$ satisfy those of a warped product and we can write $M = \mathbb{R} \times_{e^g} \mathbb{R} \times_{e^f} N_1$. The functions f and g depend solely on the parameter corresponding to X_3 and are given by $X_3(f) = -a_2$ and $X_3(g) = \frac{1}{a_2}$. We can assume $X_3 = \frac{\partial}{\partial s}$ on the submanifold. We can also find a function $\mu(s)$ such that $\mu X_4 = \frac{\partial}{\partial t}$. To find a suitable μ, we solve

$$[X_3, \mu X_4] = \left(X_3(\mu) - \frac{\mu}{a_2}\right) X_4 = \left(\mu'(1 + a_2^2) - \frac{\mu}{a_2}\right) X_4 = 0.$$

The function $\mu = \frac{a_2}{\sqrt{1+a_2^2}}$ satisfies this equation. We can find $a_2(s)$ by solving

$$\frac{\partial a_2}{\partial s} = 1 + a_2^2 \Rightarrow a_2 = \tan(s).$$

Hence $\mu(s) = \sin(s)$ and we calculate for $i \in \{1, 2\}$ that

$$\begin{aligned}
D_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s} &= \frac{\partial^2 F}{\partial s^2} = -F \\
&\Rightarrow F = A\cos(s) + B\sin(s), \\
D_{\frac{\partial}{\partial t}}\frac{\partial}{\partial s} &= \frac{\partial^2 F}{\partial t \partial s} = -\frac{\partial A}{\partial t}\sin(s) + \frac{\partial B}{\partial t}\cos(s) \\
&= \cot(s)\frac{\partial F}{\partial t} = \frac{\cos(s)^2}{\sin(s)}\frac{\partial A}{\partial t} + \cos(s)\frac{\partial B}{\partial t} \\
&\Rightarrow \frac{\partial A}{\partial t} = 0, \\
D_{X_i}\frac{\partial}{\partial s} &= \frac{\partial F_* X_i}{\partial s} = -A_*X_i\sin(s) + B_*X_i\cos(s) \\
&= -\tan(s)X_i = -A_*X_i\sin(s) - \frac{\sin(s)^2}{\cos(s)}B_*X_i \\
&\Rightarrow B_*X_i = 0.
\end{aligned}$$

So A is the immersion of N_1 and B is a curve tangent to X_4. Because F lies in the unit sphere, one has

$$\langle F, F\rangle = \cos(s)^2\langle A, A\rangle + \sin(s)^2\langle B, B\rangle + \sin(2s)\langle A, B\rangle$$

which implies that A and B have both unit length and are orthogonal. We can also calculate

$$\begin{aligned} D_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} &= -\cos(s)\sin(s)\frac{\partial F}{\partial s} - \sin(s)^2 F = -\sin(s) B \\ &= \frac{\partial^2 F}{\partial t^2} = \sin(s)\frac{\partial^2 B}{\partial t^2} \\ &\Rightarrow B = B_1 \cos(t) + B_2 \sin(t). \end{aligned}$$

Vector fields B_1 and B_2 are constant, normalized and orthogonal. This follows from the fact that $\langle B, B\rangle = 1$. Finally similar to (28), A can be shown to be any special Lagrangian submanifold in $\mathbb{C}P^2$ lifted to the unit sphere in $\mathbb{C}^3$ orthogonal to B_1 and B_2 directions. Fixing B_1 and B_2 by an isometry leads to (53). □

4.2. **The case where $b_2 \neq 0$.** When $\mathcal{N}_+$ is integrable, so when $a_6 = b_6 = 0$, the equations for ∇ are given by (30). We have:

Theorem 4.2. *Suppose M is a special Lagrangian submanifold in $\mathbb{C}P^4$ having a local $SO(2) \rtimes S_3$ symmetry. Suppose $\mathcal{N}_+$ is integrable. Then M can be lifted horizontally to a submanifold in the unit sphere of $\mathbb{C}^5$ through F and is locally isometric to*

$$F(t, s, u, v) = (\phi(s, u, v)\cos(t), \sin(t)), \tag{54}$$

where ϕ is the horizontal lift of a special Lagrangian submanifold in $\mathbb{C}P^3$ with S_3-symmetry to the unit sphere in $\mathbb{C}^4$.

Proof. The manifold is a warped product $\mathbb{R} \times_{e^f} N^3$. Solving the Gauss equation

$$X_4(a_3) = \frac{\partial a_3}{\partial t} = 1 + a_3^2$$

yields $a_3 = \tan(t)$. For $i \in \{1, 2, 3\}$ this implies

$$\begin{aligned} D_{X_4} X_4 &= \frac{\partial^2 F}{\partial t^2} = -F \\ &\Rightarrow F = A\cos(t) + B\sin(t), \\ D_{X_i} X_4 &= -A_* X_i \sin(t) + B_* X_i \cos(t) \\ &= -\tan(t) X_i = -A_* X_i \sin(t) - B_* X_i \frac{\sin(t)^2}{\cos(t)} \\ &\Rightarrow B_* = 0. \end{aligned}$$

Thus B is a constant vector field along the submanifold and A is an immersion of a 3-fold N^3. Using the fact that F is of unit length, A and B are orthogonal and of unit length. Using calculations similar to (28) A is a C-totally real submanifold in S^7 having local S_3-symmetry, where S^7 lies in the subspace orthogonal to B and JB. Applying a suitable isometry results in (54). □

The method to solve the case where the only integrable distribution containing $\mathcal{N}_1$ is the tangent bundle, has been analyzed earlier for a non-specific complex space form. We can now fill in $\epsilon = 1$ and we find for $z^2 \neq -1$ that

$$\begin{aligned}
a_3 &= \tan(t), \\
a_2 &= \frac{\sin(2s)}{\cos(t)(\cos(2s)+\cosh(2k_1))}, \\
b_2 &= \frac{\sinh(2k_1)}{\cos(t)(\cos(2s)+\cosh(2k_1))}, \\
r &= \frac{e^{k_2}}{\cos(t)\sqrt{\cos(2s)+\cosh(2k_1)}}, \\
a_6 + ib_6 &= \frac{k_3+ik_4}{\rho}\sqrt{1+a_3^2}\left(1+\bar{z}^2\right)^{-\frac{1}{2}},
\end{aligned}$$

where the functions k_i depend only on (u,v). Solving for (7), we obtain furthermore that

$$\begin{aligned}
\gamma_1 &= \frac{-\tan(t)(k_3+ik_4)\cos(s-ik_1)+\left(\frac{\partial k_1}{\partial v} - i\frac{\partial k_1}{\partial u}\right)}{\rho}, \\
a_1 &= \frac{2^{\frac{2}{3}}e^{\frac{2}{3}k_2}}{3\cos(t)^2\left(\cos(2s)+\cosh(2k_1)\right)^2}\Big((\cos(2s)+\cosh(2k_1))(\rho_1\frac{\partial k_2}{\partial v}+\rho_2\frac{\partial k_2}{\partial u}) \\
&\quad + \sin(2s)(\rho_1\frac{\partial k_1}{\partial u}-\rho_2\frac{\partial k_1}{\partial v}) - \sinh(2k_1)(\rho_1\frac{\partial k_1}{\partial v}+\rho_2\frac{\partial k_1}{\partial u}) \\
&\quad - \tan(t)\sinh(2k_1)\left(\cos(s)\cosh(k_1)(k_4\rho_2-k_3\rho_1)+\sin(s)\sinh(k_1)(k_4\rho_1+k_3\rho_2)\right)\Big), \\
b_1 &= \frac{2^{\frac{2}{3}}e^{\frac{2}{3}k_2}}{3\cos(t)^2\left(\cos(2s)+\cosh(2k_1)\right)^2}\Big((\cos(2s)+\cosh(2k_1))(\rho_2\frac{\partial k_2}{\partial v}-\rho_1\frac{\partial k_2}{\partial u}) \\
&\quad + \sin(2s)(\rho_2\frac{\partial k_1}{\partial u}+\rho_1\frac{\partial k_1}{\partial v}) + \sinh(2k_1)(\rho_1\frac{\partial k_1}{\partial u}-\rho_2\frac{\partial k_1}{\partial v}) \\
&\quad - \sinh(2k_1)\tan(t)\left(\sin(s)\sinh(k_1)(k_4\rho_2-k_3\rho_1)-\cos(s)\cosh(k_1)(k_4\rho_1+k_3\rho_2)\right)\Big).
\end{aligned}$$

The other equations in (7) impose restrictions on $\{k_1,k_2,k_3,k_4\}$ given by

$$\begin{aligned}
\frac{\partial k_4}{\partial u} - \frac{\partial k_3}{\partial v} &= 2\tanh(k_1)\left(k_3\frac{\partial k_1}{\partial v} - k_4\frac{\partial k_1}{\partial u}\right), \\
\frac{\partial k_4}{\partial v} + \frac{\partial k_3}{\partial u} &= -2\coth(k_1)\left(k_3\frac{\partial k_1}{\partial u} + k_4\frac{\partial k_1}{\partial v}\right), \\
\Delta k_1 &= e^{-\frac{2k_2}{3}}\frac{\sinh(2k_1)}{2}\left(-2^{\frac{4}{3}} + e^{\frac{2k_2}{3}}(k_3^2+k_4^2)\right), \\
\Delta k_2 &= 3 * 2^{\frac{1}{3}}e^{-\frac{2k_2}{3}}\left(\cosh(2k_1) - e^{2k_2}\right).
\end{aligned} \tag{55}$$

When $a_2 = 0$ and $b_2 = \pm\sqrt{1+a_3^2}$, we find

$$a_6 = \frac{e^{k_1}\cos(\frac{5}{3}s)}{\cos(t)^2};$$
$$b_6 = \pm\frac{e^{k_1}\sin(\frac{5}{3}s)}{\cos(t)^2};$$
$$r = \frac{e^{k_2}}{\cos(t)}.$$

Furthermore, we obtain for $b_2 = \sqrt{1+a_3^2}$ that

$$\gamma_{1+} = \frac{-5e^{k_1+\frac{5si}{3}}\tan(t) + e^{\frac{2k_1+k_2}{5}+\frac{2si}{3}}\left((\frac{\partial k_2}{\partial v} - 3\frac{\partial k_1}{\partial v}) - i(\frac{\partial k_2}{\partial u} - 3\frac{\partial k_1}{\partial u})\right)}{5\cos(t)},$$
$$a_{1+} = \frac{e^{k_1}\cos(\frac{5s}{3})\tan(t) + e^{\frac{2k_1+k_2}{5}}\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial v} + \sin(\frac{2s}{3})\frac{\partial k_2}{\partial u}\right)}{3\cos(t)},$$
$$b_{1+} = \frac{e^{k_1}\sin(\frac{5s}{3})\tan(t) - e^{\frac{2k_1+k_2}{5}}\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial u} - \sin(\frac{2s}{3})\frac{\partial k_2}{\partial v}\right)}{3\cos(t)},$$

and for $b_2 = -\sqrt{1+a_3^2}$ we obtain

$$\gamma_{1-} = \frac{-5e^{k_1-\frac{5si}{3}}\tan(t) + e^{\frac{2k_1+k_2}{5}-\frac{2si}{3}}\left((3\frac{\partial k_1}{\partial v} - \frac{\partial k_2}{\partial v}) - i(3\frac{\partial k_1}{\partial u} - \frac{\partial k_2}{\partial u})\right)}{5\cos(t)},$$
$$a_{1-} = \frac{-e^{k_1}\cos(\frac{5s}{3})\tan(t) + e^{\frac{2k_1+k_2}{5}}\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial v} - \sin(\frac{2s}{3})\frac{\partial k_2}{\partial u}\right)}{3\cos(t)},$$
$$b_{1-} = \frac{e^{k_1}\sin(\frac{5s}{3})\tan(t) - e^{\frac{2k_1+k_2}{5}}\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial u} + \sin(\frac{2s}{3})\frac{\partial k_2}{\partial v}\right)}{3\cos(t)}.$$

Solving the last equations in (7) implies restrictions on the functions $k_1(u,v)$ and $k_2(u,v)$ given by

$$\begin{aligned} \Delta k_1 &= 2e^{-\frac{2(2k_1+k_2)}{5}}\left(3 - e^{2k_1} - e^{2k_2}\right) \\ \Delta k_2 &= e^{-\frac{2(2k_1+k_2)}{5}}\left(8 - e^{2k_1} - 6e^{2k_2}\right). \end{aligned} \tag{56}$$

These equations are valid for both $b_2 = \pm\sqrt{1+a_3^2}$. We summarize this in the following theorem.

Theorem 4.3. *Each special Lagrangian submanifold of $\mathbb{C}P^4$ with $SO(2) \rtimes S_3$-symmetry where the only integral distribution containing $\mathcal{N}_1$ is the tangent bundle, can be constructed in the way above using either functions $\{k_1, k_2, k_3, k_4\}$ subject to (55) or functions $\{k_1, k_2\}$ subject to (56). Conversely, each such a construction results in such a submanifold, unique up to local isometry.*

5. Submanifolds in $\mathbb{C}H^4$.

5.1. **The case where $b_2 = 0$.** This is the case where $\mathcal{N}_1$ is an integrable distribution. We assume that $a_3 = 0$. Similar to the case in $\mathbb{C}P^4$ we have that M is a double warped product $\mathbb{R} \times_{e^g} \mathbb{R} \times_{e^f} N^2$. the function a_2 depends only on the coordinate s and is given by

$$\frac{\partial a_2}{\partial s} = a_2^2 - 1.$$

This equation has 3 possible solutions, depending on the initial conditions. For $a_2(0) = 1$, it is a constant. For $a_2(0) > 1$ it is given as $a_2 = -\coth(s)$. Finally for $a_2(0) < 1$, it is given as $a_2(s) = -\tanh(s)$. The connection ∇ is given by

$$\begin{array}{c|c}
\nabla_{X_1} X_1 = a_1 X_2 + a_2 X_3 & \nabla_{X_1} X_2 = -a_1 X_1, \\
\nabla_{X_2} X_1 = b_1 X_2 & \nabla_{X_2} X_2 = -b_1 X_1 + a_2 X_3, \\
\nabla_{X_3} X_1 = 0 & \nabla_{X_3} X_2 = 0, \\
\nabla_{X_4} X_1 = 0 & \nabla_{X_4} X_2 = 0, \\
\nabla_{X_1} X_3 = -a_2 X_1 & \nabla_{X_1} X_4 = 0, \\
\nabla_{X_2} X_3 = -a_2 X_2 & \nabla_{X_2} X_4 = 0, \\
\nabla_{X_3} X_3 = 0 & \nabla_{X_3} X_4 = 0, \\
\nabla_{X_4} X_3 = -\frac{X_4}{a_2} & \nabla_{X_4} X_4 = \frac{X_3}{a_2}.
\end{array}$$

We have the following result.

Theorem 5.1. *Suppose M is a special Lagrangian submanifold in $\mathbb{C}H^4$ having a local $SO(2) \rtimes S_3$-symmetry. Suppose $\mathcal{N}_1$ is integrable. Then M can be lifted horizontally to a submanifold in H^9 through F and is locally isometric to either*

$$F(t,s,u,v) = (\sin(t)\sinh(s), \cos(t)\sinh(s), \phi(u,v)\cosh(s)), \tag{57}$$

where ϕ is the horizontal lift of a special Lagrangian submanifold of $\mathbb{C}H^2$ to H^5 in case $a_2^2 < 1$, or

$$F(t,s,u,v) = (\phi(u,v)\sinh(s), \cos(t)\cosh(s), \sin(t)\cosh(s)), \tag{58}$$

where ϕ is the horizontal lift of a special Lagrangian submanifold of $\mathbb{C}P^2$ to S^5 in case $a_2^2 > 1$, or

$$F(t,s,u,v) = \Big((\phi(u,v),t)\, e^{-s}, -\frac{1}{2} e^{-s}, \left(\|(\phi(u,v),t)\|^2 + i f(u,v) \right) e^{-s} + e^s \Big), \tag{59}$$

where ϕ is a special Lagrangian surface in $\mathbb{C}^2$ and f is the integral of the differential form

$$2 \sum_{i=1}^{2} \left(x^i \, \mathrm{d}\, y^i - y^i \, \mathrm{d}\, x^i \right)$$

on $\mathbb{C}^2$ in case $a_2^2 = 1$.

Proof. We can check similar to the case in $\mathbb{C}P^4$ that $M = \mathbb{R} \times_{e^g} \mathbb{R} \times_{e^f} \times N^2$, where f and g are functions on the first factor, determined by $X_3(g) = -\frac{1}{a_2}$ and $X_3(f) = -a_2$. We can treat the cases separately for each solution to $a_2(s)$.

Assume $a_2 = -\tanh(s)$, then it is easy to see that $\frac{\partial}{\partial t} = -\sinh(s)X_4$ commutes with $\frac{\partial}{\partial s}$. The Gauss identity now implies for $i \in \{1, 2\}$ that

$$\begin{aligned}
D_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s} &= \frac{\partial^2 F}{\partial s^2} = F \\
&\Rightarrow F = A\sinh(s) + B\cosh(s), \\
D_{\frac{\partial}{\partial T}}\frac{\partial}{\partial s} &= \frac{\partial^2 F}{\partial t \partial s} = \frac{\partial A}{\partial t}\cosh(s) + \frac{\partial B}{\partial t}\sinh(s) \\
&= \coth(s)\frac{\partial F}{\partial t} = \frac{\partial A}{\partial t}\cosh(s) + \frac{\partial B}{\partial t}\frac{\cosh(s)^2}{\sinh(s)} \\
&\Rightarrow \frac{\partial B}{\partial t} = 0, \\
D_{X_i}\frac{\partial}{\partial s} &= \frac{\partial F_* X_i}{\partial s} = A_* X_i \cosh(s) + B_* X_i \sinh(s) \\
&= \tanh(s) X_i = A_* X_i \frac{\sinh(s)^2}{\cosh(s)} + B_* X_i \sinh(s) \\
&\Rightarrow A_* X_i = 0.
\end{aligned}$$

Using the fact that $\langle F, F\rangle_1 = -1$, we get that $\langle B, B\rangle_1 = -\langle A, A\rangle_1 = -1$ and $\langle A, B\rangle_1 = 0$. Furthermore, we find

$$\begin{aligned}
D_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} &= \frac{\partial^2 F}{\partial t^2} = \frac{\partial^2 A}{\partial t^2}\sinh(s) \\
&= -\cosh(s)\sinh(s)\frac{\partial F}{\partial s} + \sinh(s)^2 F = -A\sinh(s) \\
&\Rightarrow A = A_1\cos(t) + A_2\sin(t).
\end{aligned}$$

Because A has unit length, so do A_1 and A_2 and they are both orthogonal. Calculations similar to (28) show that B can be taken as the horizontal lift of any special Lagrangian submanifold in $\mathbb{C}H^2$ and applying a suitable isometry gives (57).
For $a_2 = -\coth(s)$ calculations similar to the previous case result in (58).
Finally we assume $a_2 = 1$. Then the vector field given by $\frac{\partial}{\partial t} = e^{-s}X_4$ commutes with $\frac{\partial}{\partial s}$. We can calculate for $i \in \{1, 2\}$ that

$$\begin{aligned}
D_{\frac{\partial}{\partial s}}\frac{\partial}{\partial s} &= \frac{\partial^2 F}{\partial s^2} = F \Rightarrow F = Ae^s + Be^{-s}, \\
D_{\frac{\partial}{\partial t}}\frac{\partial}{\partial s} &= \frac{\partial^2 F}{\partial t \partial s} = \frac{\partial A}{\partial t}e^s - \frac{\partial B}{\partial t}e^{-s} \\
&= -\frac{\partial F}{\partial t} = -\frac{\partial A}{\partial t}e^s - \frac{\partial B}{\partial t}e^{-s} \\
&\Rightarrow \frac{\partial A}{\partial t} = 0, \\
D_{X_i}\frac{\partial}{\partial s} &= \frac{\partial F_* X_i}{\partial s} = A_* X_i e^s - B_* X_i e^{-s} \\
&= -F_* X_i = -\left(A_* X_i e^s + B_* X_i e^{-s}\right) \\
&\Rightarrow A_* = 0.
\end{aligned}$$

Using the fact that $\langle F, F\rangle_1 = -1$, we obtain that A and B are vector fields with 0 length and they satisfy $\langle A, B\rangle_1 = -\frac{1}{2}$. Further calculations show

$$\begin{aligned}
D_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} &= \frac{\partial^2 F}{\partial t^2} = e^{-s}\frac{\partial^2 B}{\partial t^2} \\
&= e^{-2s}\left(\frac{\partial F}{\partial s} + F\right) = 2Ae^{-s} \\
&\Rightarrow B = At^2 + B_1 t + B_2, \\
D_{X_i}\frac{\partial}{\partial t} &= \frac{\partial F_* X_i}{\partial t} = B_{1*}X_i e^{-s} = 0 \\
&\Rightarrow B_{1*} = 0.
\end{aligned}$$

We can conclude that F has the form

$$F = \left(At^2 + B_1 t + \phi\right)e^{-s} + Ae^{s}$$

Here, ϕ is an immersion of a 2-fold in $\mathbb{C}^5_1$ tangent to $\mathcal{N}_1$. Calculating the scalar products of B and A, we get

$$\begin{aligned}
\langle A, B\rangle_1 &= t\langle A, B_1\rangle_1 + \langle A, \phi\rangle_1 = -\frac{1}{2} \\
&\Rightarrow \langle A, B_1\rangle_1 = 0 \bigwedge \langle A, \phi\rangle_1 = -\frac{1}{2}, \\
\langle B, B\rangle_1 &= t^2\left(\langle B_1, B_1\rangle_1 - 1\right) + 2t\langle B_1, \phi\rangle_1 + \langle \phi, \phi\rangle_1 = 0 \\
&\Rightarrow \langle B_1, B_1\rangle_1 = 1 \bigwedge \langle B_1, \phi\rangle_1 = 0 \bigwedge \langle \phi, \phi\rangle_1 = 0.
\end{aligned} \tag{60}$$

We can shift to a different standard basis of $\mathbb{C}^5_1$ such that

$$\langle \vec{z}, \vec{w}\rangle = \Re\left(\sum_{j=1}^{3} z_j\bar{w}_j + z_4\bar{w}_5 + z_5\bar{w}_4\right).$$

In this case the constant light-like vector A and time-like B_1, after applying a suitable isometry, can be chosen to be

$$\begin{aligned}
A &= (0,0,0,0,1), \\
B_1 &= (0,0,1,0,0).
\end{aligned}$$

We can write $\phi = (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5)$ where $\phi_j = x_j + iy_j$. Then (60) implies

$$\begin{aligned}
x_4 &= -\frac{1}{2}, \\
x_3 &= 0, \\
x_5 - 2y_4y_5 &= \sum_{j=1}^{3}|\phi_j|^2.
\end{aligned}$$

We can use the fact that both F and iF are orthogonal to the tangent space in $\mathbb{C}^5_1$ and this results in

$$\begin{aligned}\phi_4 &= -\frac{1}{2},\\ \phi_3 &= 0,\\ \mathrm{d}\, y_5 &= 2\sum_{i=1}^{2}(x_i \,\mathrm{d}\, y_i - y_i \,\mathrm{d}\, x_i).\end{aligned}$$

This last equation is integrable if and only its derivative equals 0 along the submanifold. But this derivative is nothing more than a multiple of the Kähler form on $\mathbb{C}^2$ spanned by the first 2 complex coordinates. In other words, for such a submanifold to exist, the projection of ϕ onto the first 2 coordinates should be a Lagrangian submanifold in $\mathbb{C}^2$. Calculating the Gauss identity on $D_{X_i}X_j$ we find that the metric on this immersion is given by

$$\langle \phi_* X_i, \phi_* X_j \rangle = e^{2s}\delta_{ij},$$

where $\langle a, b\rangle$ is the standard scalar product on $\mathbb{C}^2$ and ϕ here is the restriction to the first 2 complex coordinates. Because $\langle F_* X_i, F_* X_j\rangle = \delta_{ij}$ and because $\phi_{3*} = 0$ and $\phi_{4*} = 0$ this condition is included in the warped product structure. Using calculations like (28) we conclude that (ϕ_1, ϕ_2) can be any special Lagrangian 2-fold in $\mathbb{C}^2$. The result is summarized in (59). □

5.2. **The case where $b_2 \neq 0$.** First we assume that $\mathcal{N}_+$ is an integrable distribution. This is equivalent to $a_6 + ib_6 = 0$. The connection is given by (30), resulting in a warped product structure $\mathbb{R} \times_{e^f} N^3$. The equation

$$X_4(a_3) = \frac{\partial a_3}{\partial t} = a_3^2 - 1$$

has a solution given as $|a_3| = 1$, $a_3 = -\tanh(t)$ or $a_3 = -\coth(t)$, depending on the initial value of a_3. Using an analysis similar to the case of $\mathbb{C}P^4$ and the case above gives the following result.

Theorem 5.2. *Suppose M is a special Lagrangian submanifold in $\mathbb{C}H^4$ having a local $SO(2) \rtimes S_3$-symmetry. Suppose $\mathcal{N}_1$ is non-integrable, but is contained in the integrable $\mathcal{N}_+$ distribution. Then M can be lifted horizontally to a submanifold in H^9 through F and is locally isometric to either*

$$F(t, s, u, v) = (\sinh(t), \phi(s, u, v)\cosh(t)), \tag{61}$$

where ϕ is the horizontal lift of a special Lagrangian submanifold of $\mathbb{C}H^3$ with a local S_3-symmetry to H^7 in case $a_3^2 < 1$, or

$$F(t, s, u, v) = (\phi(s, u, v)\sinh(t), \cosh(t)), \tag{62}$$

where ϕ is the horizontal lift of a special Lagrangian submanifold of $\mathbb{C}P^3$ with a local S_3-symmetry to S^7 in case $a_3^2 > 1$, or

$$F(t, s, u, v) = \left(\phi(s, u, v)e^{-t}, -e^{-t}/2, \left(\|\phi(s, u, v)\|^2 + if(s, u, v)\right) e^{-t} + e^t\right), \tag{63}$$

where ϕ is a special Lagrangian submanifold with a local S_3-symmetry in $\mathbb{C}^3$ and f is the integral of the differential form

$$2\sum_{i=1}^{3}\left(x^i\,\mathrm{d}\,y^i - y^i\,\mathrm{d}\,x^i\right).$$

Finally we assume that there is no integrable distribution that contains $\mathcal{N}_1$ except for the tangent bundle. We return to the analysis as done for $\mathbb{C}^4$, but set $\epsilon = -1$. The result will depend on the initial value of a_3. First assume that $a_3^2 < 1$, then $\tilde{\epsilon} = -1$. We find functions $\{k_1, k_2, k_3, k_4\}$ of (u, v) such that

$$\begin{aligned}
a_3 &= -\tanh(t),\\
a_2 &= -\frac{\sinh(2s)}{\cosh(t)\left(\cosh(2s)+\cos(2k_1)\right)},\\
b_2 &= -\frac{\sin(2k_1)}{\cosh(t)\left(\cosh(2s)+\cos(2k_1)\right)},\\
r &= \frac{e^{k_2}}{\cosh(t)\sqrt{\cosh(2s)+\cos(2k_1)}},\\
a_6 + ib_6 &= \frac{k_3+ik_4}{\rho}\sqrt{1-a_3^2}\left(1-\bar{z}^2\right)^{-\frac{1}{2}}.
\end{aligned}$$

Using (7) as earlier, we obtain a_1, b_1, γ_1 as

$$\begin{aligned}
\gamma_1 &= \frac{-\tanh(t)(k_3+ik_4)\cosh(s-ik_1) + \left(\frac{\partial k_1}{\partial v} - i\frac{\partial k_1}{\partial u}\right)}{\rho},\\
a_1 &= \frac{2^{\frac{2}{3}}e^{\frac{2}{3}k_2}}{3\cosh(t)^2\left(\cosh(2s)+\cos(2k_1)\right)^2}\Big((\cosh(2s)+\cos(2k_1))(\rho_1\frac{\partial k_2}{\partial v}+\rho_2\frac{\partial k_2}{\partial u})\\
&\quad + \sinh(2s)(\rho_2\frac{\partial k_1}{\partial v}-\rho_1\frac{\partial k_1}{\partial u}) + \sin(2k_1)(\rho_1\frac{\partial k_1}{\partial v}+\rho_2\frac{\partial k_1}{\partial u})\\
&\quad + \sin(2k_1)\tanh(t)\left(\cosh(s)\cos(k_1)(k_4\rho_2-k_3\rho_1) - \sinh(s)\sin(k_1)(k_4\rho_1+k_3\rho_2)\right)\Big),\\
b_1 &= \frac{2^{\frac{2}{3}}e^{\frac{2}{3}k_2}}{3\cosh(t)^2\left(\cosh(2s)+\cos(2k_1)\right)^2}\Big((\cosh(2s)+\cos(2k_1))(\rho_2\frac{\partial k_2}{\partial v}-\rho_1\frac{\partial k_2}{\partial u})\\
&\quad - \sinh(2s)(\rho_2\frac{\partial k_1}{\partial u}+\rho_1\frac{\partial k_1}{\partial v}) - \sin(2k_1)(\rho_1\frac{\partial k_1}{\partial u}-\rho_2\frac{\partial k_1}{\partial v}))\\
&\quad - \sin(2k_1)\tanh(t)\left(\sinh(s)\sin(k_1)(k_4\rho_2-k_3\rho_1) + \cosh(s)\cos(k_1)(k_4\rho_1+k_3\rho_2)\right)\Big).
\end{aligned}$$

The other equations in (7) put restrictions on $\{k_1, k_2, k_3, k_4\}$ given by

$$
\begin{aligned}
\frac{\partial k_4}{\partial u} - \frac{\partial k_3}{\partial v} &= 2\tan(k_1)\left(k_4\frac{\partial k_1}{\partial u} - k_3\frac{\partial k_1}{\partial v}\right),\\
\frac{\partial k_4}{\partial v} + \frac{\partial k_3}{\partial u} &= -2\cot(k_1)\left(k_3\frac{\partial k_1}{\partial u} + k_4\frac{\partial k_1}{\partial v}\right),\\
\Delta k_1 &= \frac{\sin(2k_1)}{2}\left(2^{\frac{4}{3}}e^{-\frac{2}{3}k_2} + k_3^2 + k_4^2\right),\\
\Delta k_2 &= -3 * 2^{\frac{1}{3}}e^{-\frac{2}{3}k_2}\left(e^{2k_2} + \cos(2k_1)\right).
\end{aligned}
\tag{64}
$$

Then we set $a_3^2 > 1$ and assume $z^2 \neq -1$. We then find

$$
\begin{aligned}
a_3 &= -\coth(t),\\
a_2 &= \frac{\sin(2s)}{\sinh(t)\left(\cos(2s) + \cosh(2k_1)\right)},\\
b_2 &= \frac{\sinh(2k_1)}{\sinh(t)\left(\cos(2s) + \cosh(2k_1)\right)},\\
r &= \frac{e^{k_2}}{\sinh(t)\sqrt{\cos(2s) + \cosh(2k_1)}},\\
a_6 + ib_6 &= \frac{k_3 + ik_4}{\rho}\sqrt{a_3^2 - 1}\left(1 + \bar{z}^2\right)^{-\frac{1}{2}}.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\gamma_1 &= \frac{\coth(t)(k_3 + ik_4)\cos(s - ik_1) + \left(\frac{\partial k_1}{\partial v} - i\frac{\partial k_1}{\partial u}\right)}{\rho},\\
a_1 &= \frac{2^{\frac{2}{3}}e^{\frac{2}{3}k_2}}{3\sinh(t)^2\left(\cos(2s) + \cosh(2k_1)\right)^2}\Big((\cos(2s) + \cosh(2k_1))(\rho_1\frac{\partial k_2}{\partial v} + \rho_2\frac{\partial k_2}{\partial u})\\
&\quad + \sin(2s)(\rho_1\frac{\partial k_1}{\partial u} - \rho_2\frac{\partial k_1}{\partial v}) - \sinh(2k_1)(\rho_1\frac{\partial k_1}{\partial v} + \rho_2\frac{\partial k_1}{\partial u})\\
&\quad + \sinh(2k_1)\coth(t)\left(\cos(s)\cosh(k_1)(k_4\rho_2 - k_3\rho_1) + \sin(s)\sinh(k_1)(k_4\rho_1 + k_3\rho_2)\right)\Big),\\
b_1 &= \frac{2^{\frac{2}{3}}e^{\frac{2}{3}k_2}}{3\sinh(t)^2\left(\cosh(2s) + \cos(2k_1)\right)^2}\Big((\cosh(2s) + \cos(2k_1))(\rho_2\frac{\partial k_2}{\partial v} - \rho_1\frac{\partial k_2}{\partial u})\\
&\quad + \sin(2s)(\rho_2\frac{\partial k_1}{\partial u} + \rho_1\frac{\partial k_1}{\partial v}) + \sinh(2k_1)(\rho_1\frac{\partial k_1}{\partial u} - \rho_2\frac{\partial k_1}{\partial v}))\\
&\quad - \sinh(2k_1)\coth(t)\left(\sin(s)\sinh(k_1)(k_3\rho_1 - k_4\rho_2) + \cos(s)\cosh(k_1)(k_4\rho_1 + k_3\rho_2)\right)\Big).
\end{aligned}
$$

The functions $\{k_1, k_2, k_3, k_4\}$ have to satisfy

$$\begin{aligned}
\frac{\partial k_4}{\partial u} - \frac{\partial k_3}{\partial v} &= 2\tanh(k_1)\left(k_3\frac{\partial k_1}{\partial v} - k_4\frac{\partial k_1}{\partial u}\right),\\
\frac{\partial k_4}{\partial v} + \frac{\partial k_3}{\partial u} &= -2\coth(k_1)\left(k_3\frac{\partial k_1}{\partial u} + k_4\frac{\partial k_1}{\partial v}\right),\\
\Delta k_1 &= -\sinh(2k_1)\left(2^{\frac{1}{3}}e^{-\frac{2}{3}k_2} + \frac{k_3^2+k_4^2}{2}\right),\\
\Delta k_2 &= 3 * 2^{\frac{1}{3}}e^{-\frac{2}{3}k_2}\left(\cosh(2k_1) - e^{2k_2}\right).
\end{aligned} \tag{65}$$

Finally, assume $a_3^2 > 1$ and $z^2 = -1$. Then we find

$$\begin{aligned}
a_3 &= -\coth(t),\\
a_6 &= \frac{e^{k_1}\cos(\frac{5}{3}s)}{\sinh(t)^2},\\
b_6 &= \pm\frac{e^{k_1}\sin(\frac{5}{3}s)}{\sinh(t)^2},\\
r &= \frac{e^{k_2}}{\sinh(t)}.
\end{aligned}$$

We obtain for $b_2 = \sqrt{a_3^2 - 1}$ that

$$\begin{aligned}
\gamma_{1+} &= \frac{5e^{k_1+\frac{5si}{3}}\coth(t) + e^{\frac{2k_1+k_2}{5}+\frac{2si}{3}}\left((\frac{\partial k_2}{\partial v} - 3\frac{\partial k_1}{\partial v}) - i(\frac{\partial k_2}{\partial u} - 3\frac{\partial k_1}{\partial u})\right)}{5\sinh(t)},\\
a_{1+} &= \frac{-e^{k_1}\cos(\frac{5s}{3})\coth(t) + e^{\frac{2k_1+k_2}{5}}\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial v} + \sin(\frac{2s}{3})\frac{\partial k_2}{\partial u}\right)}{3\sinh(t)},\\
b_{1+} &= \frac{-e^{k_1}\sin(\frac{5s}{3})\coth(t) - e^{\frac{2k_1+k_2}{5}}\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial u} - \sin(\frac{2s}{3})\frac{\partial k_2}{\partial v}\right)}{3\sinh(t)},
\end{aligned}$$

and for $b_2 = -\sqrt{a_3^2 - 1}$ we obtain

$$\begin{aligned}
\gamma_{1-} &= \frac{5e^{k_1-\frac{5si}{3}}\coth(t) + e^{\frac{2k_1+k_2}{5}-\frac{2si}{3}}\left((3\frac{\partial k_1}{\partial v} - \frac{\partial k_2}{\partial v}) - i(3\frac{\partial k_1}{\partial u} - \frac{\partial k_2}{\partial u})\right)}{5\sinh(t)},\\
a_{1-} &= \frac{e^{k_1}\cos(\frac{5s}{3})\coth(t) + e^{\frac{2k_1+k_2}{5}}\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial v} - \sin(\frac{2s}{3})\frac{\partial k_2}{\partial u}\right)}{3\sinh(t)},\\
b_{1-} &= \frac{-e^{k_1}\sin(\frac{5s}{3})\coth(t) - e^{\frac{2k_1+k_2}{5}}\left(\cos(\frac{2s}{3})\frac{\partial k_2}{\partial u} + \sin(\frac{2s}{3})\frac{\partial k_2}{\partial v}\right)}{3\sinh(t)}.
\end{aligned}$$

Solving the other Gauss equations results in the relations

$$\begin{aligned}
\Delta k_1 &= e^{-\frac{2}{5}(2k_1+k_2)}\left(6 + 2e^{2k_1} - 2e^{2k_2}\right),\\
\Delta k_2 &= e^{-\frac{2}{5}(2k_1+k_2)}\left(8 + e^{2k_1} - 6e^{2k_2}\right).
\end{aligned} \tag{66}$$

These equations are valid for both $b_2 = \pm\sqrt{a_3^2 - 1}$. We can conclude with the following proposition.

Theorem 5.3. *Each special Lagrangian submanifold of $\mathbb{C}H^4$ with $SO(2) \rtimes S_3$-symmetry where the only integral distribution containing $\mathcal{N}_1$ is the whole tangent bundle, can be constructed in the way above using functions $\{k_1, k_2, k_3, k_4\}$ subject to* (64) *in case $a_3^2(0) < 1$, subject to* (65) *in case $a_3^2(0) > 1$, or functions $\{k_1, k_2\}$ subject to* (66) *when $a_3^2(0) > 1$ and $z^2 = -1$. Conversely, each such a construction results in such a submanifold, unique up to local isometry.*

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