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## To cite this version:

Franki Dillen, Christine Scharlach, Kristof Schoels, Luc Vrancken. SPECIAL LAGRANGIAN 4FOLDS WITH SO(2) S3-SYMMETRY IN COMPLEX SPACE FORMS. Taiwanese Journal of Mathematics, TJM, 2015, 19 (3), pp.759-792. 10.11650/tjm.19.2015.4951 . hal-03148413

HAL Id: hal-03148413
https://uphf.hal.science/hal-03148413
Submitted on 5 Jul 2022

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# SPECIAL LAGRANGIAN 4-FOLDS WITH $S O(2) \rtimes S_{3}$-SYMMETRY IN COMPLEX SPACE FORMS 

FRANKI DILLEN, CHRISTINE SCHARLACH, KRISTOF SCHOELS, AND LUC VRANCKEN


#### Abstract

In this article we obtain a classification of special Lagrangian submanifolds in complex space forms subject to an $S O(2) \rtimes S_{3}$-symmetry on the second fundamental form. The algebraic structure of this form has been obtained by Marianty Ionel in [7]. However, the classification of special Lagrangian submanifolds in $\mathbb{C}^{4}$ having this $S O(2) \rtimes S_{3}$ symmetry in [7] is incomplete. In this paper we give a complete classification of such submanifolds, and extend the classification to special Lagrangian submanifolds of arbitrary complex space forms with $S O(2) \rtimes S_{3}$-symmetry.


## 1. Introduction

A space $(N, J, g)$ is called a Hermitian manifold with complex structure $J$ and Riemannian metric $g$, if $g(J X, J Y)=g(X, Y)$ for all $X$ and $Y$. The $(0,2)$-tensor $\omega(X, Y)=$ $g(X, J Y)$ is its symplectic form. If $\omega$ is closed, then $(N, J, g)$ is said to be a Kähler manifold. In this case the Levi-Civita connection $D$ of $g$ satisfies $D \omega=0$ as well, see [11]. A complex space form is a Kähler manifold for which the curvature tensor is given by

$$
\begin{equation*}
R(X, Y) Z=\epsilon(X \wedge Y+J X \wedge J Y+2 g(X, J Y) J) Z \tag{1}
\end{equation*}
$$

where $\epsilon$ is a real constant and $X \wedge Y$ is defined as

$$
(X \wedge Y) Z=g(Y, Z) X-g(X, Z) Y
$$

Every complete, simply connected complex space form of dimension $n$ with constant holomorphic sectional curvature $4 \epsilon$ is isometric to one of the following manifolds:
(1) the standard complex space $\mathbb{C}^{n}$ when $\epsilon=0$,
(2) the complex projective space $\mathbb{C} P^{n}(4 \epsilon)$ when $\epsilon>0$,
(3) the complex hyperbolic space $\mathbb{C} H^{n}(4 \epsilon)$ when $\epsilon<0$.

Because we consider submanifolds of a complex space form locally, we can restrict ourselves to those ambient spaces. By rescaling, we can even assume that $\epsilon=0,1,-1$.

A Lagrangian submanifold $M$ of a Kähler manifold $(N, J, g)$ is a submanifold such that $\omega$ vanishes identically on $M$ and the (real) dimension of $M$ is half the (complex) dimension of $N$, see [1]. This implies that $J$ induces an orthogonal isomorphism between the tangent and the normal bundle on the submanifold. The Gauss formula is given by

$$
D_{X} Y=\nabla_{X} Y+h(X, Y)=\nabla_{X} Y+J A(X, Y)
$$

where $A=-J h$ defines a symmetric (1,2)-tensor on the submanifold, and the Weingarten formula is given by

$$
D_{X}(J Y)=J\left(\nabla_{X} Y\right)-A(X, Y)
$$

It is easy to see that the cubic form $C$, defined by

$$
C(X, Y, Z)=\underset{1}{g}(A(X, Y), Z)
$$

is totally symmetric. For Lagrangian submanifolds of complex space forms, the equations of Gauss and Codazzi simplify to

$$
\begin{gather*}
R(X, Y) Z=\epsilon(X \wedge Y) Z+\left[A_{X}, A_{Y}\right] Z  \tag{2}\\
\nabla A \text { is symmetric. } \tag{3}
\end{gather*}
$$

The following theorem holds, see [3] and [5].
Theorem 1.1. Suppose $\left(M^{n}, g\right)$ is a Riemannian manifold equipped with a symmetric and $g$-symmetric $(1,2)$-tensor $A$ such that (2) and (3) are satisfied for some constant $\epsilon$. Then for every point $p \in M$ there exists a neighborhood $U$ and a Lagrangian isometric immersion $\phi: U \rightarrow N^{2 n}(4 \epsilon)$ into the complex space form $N^{2 n}(4 \epsilon)$ such that $g$ and $J A$ are induced as first and second fundamental form. Such an immersion is unique up to isometries of the ambient space.

We focus on a particular form of $A$ assuming that there is a pointwise $G$-symmetry of $A$ (or equivalently of the cubic form $C$ ), where $G$ is a subgroup of the special orthogonal group $S O(n)$. We say that $A$ has pointwise $G$-symmetry at $p$ if for all tangent vectors $X, Y$ in $p$, and all $g \in G$ the relation $A(g X, g Y)=g A(X, Y)$ holds (or equivalently $C(g X, g Y, g Z)=C(X, Y, Z)$ for all $X, Y, Z)$. Furthermore, we impose a minimality condition on $A$ at $p$, so for every $X$ at $p$, we assume that $\operatorname{Tr}\left(A_{X}\right)=0$. These manifolds are interesting, since in $\mathbb{C}^{n}$ the minimal Lagrangian submanifolds are precisely the special Lagrangian submanifolds of $\mathbb{C}^{n}$ as introduced by Harvey and Lawson 6]. If a special Lagrangian submanifold of $\mathbb{C}^{n}$ has $G$-symmetry at every point, for the same group $G$, then a classification result for the dimension equal to 3 has been obtained by Bryant [2]. An explicit classification for special (we also use the word "special" for "minimal" in case $c \neq 0$ ) Lagrangian submanifolds of complex space forms with pointwise symmetric cubic form is not yet done, but can be easily obtained from a similar classification for affine spheres in [13].

In the present paper we consider the 4-dimensional case. In particular we consider special Lagrangian 4 -folds in complex space forms with pointwise symmetry. The shape of the (1,2)-tensor $A$, invariant under subgroups of $S O(4)$, has been described by M. Ionel in [7]. In the same article, the author classifies special Lagrangian 4-folds of $\mathbb{C}^{4}$ according to their symmetry groups. However, the classification in case the symmetry group is given by $S O(2) \rtimes S_{3}$ in that article is incomplete; several possible subcases including the most general one is omitted. In the present article, we give a complete classification of all special Lagrangian 4 -folds in any complex space form having this particular symmetry. This settles the problem for $S O(2) \rtimes S_{3}$-symmetry for all $\epsilon$. The classification for other symmetry groups remains open if $\epsilon \neq 0$.

The $S O(2) \rtimes S_{3}$-symmetry implies that $A$ can be expressed as

$$
\begin{array}{ccll}
A\left(X_{1}, X_{1}\right)=r X_{1}, & A\left(X_{1}, X_{2}\right)=-r X_{2}, & A\left(X_{1}, X_{3}\right)=0, & A\left(X_{1}, X_{4}\right)=0 \\
A\left(X_{2}, X_{1}\right)=-r X_{2}, & A\left(X_{2}, X_{2}\right)=-r X_{1}, & A\left(X_{2}, X_{3}\right)=0, & A\left(X_{2}, X_{4}\right)=0 \\
A\left(X_{3}, X_{1}\right)=0, & A\left(X_{3}, X_{2}\right)=0, & A\left(X_{3}, X_{3}\right)=0, & A\left(X_{3}, X_{4}\right)=0  \tag{4}\\
A\left(X_{4}, X_{1}\right)=0, & A\left(X_{4}, X_{2}\right)=0, & A\left(X_{4}, X_{3}\right)=0, & A\left(X_{4}, X_{4}\right)=0,
\end{array}
$$

in a well-chosen local orthonormal frame $\left\{X_{1}, X_{2}, X_{3}, X_{4}\right\}$. In this expression $r$ is a strictly positive function. The $S O(2)$-symmetry is given by the free rotation in the $\left\{X_{3}, X_{4}\right\}$ plane and the $S_{3}$-symmetry is essentially obtained by rotations over an angle $2 \pi / 3$ in the $\left\{X_{1}, X_{2}\right\}$ plane and reflections in the $\left\{X_{1}, X_{4}\right\}$ plane. We can remark that the form of $A$
is exactly that of Lagrangian submanifolds attaining equality in Chen's inequality, see [4] and [5].

In order to list the different possible subcases, we introduce distributions

$$
\mathcal{N}_{1}=\operatorname{span}\left\{X_{1}, X_{2}\right\}, \quad \mathcal{N}_{+}=\operatorname{span}\left\{X_{1}, X_{2},\left[X_{1}, X_{2}\right]\right\}, \quad \mathcal{N}_{2}=\operatorname{span}\left\{X_{3}, X_{4}\right\}
$$

We will see that $\mathcal{N}_{2}$ is always integrable. We obtain:
(1) If $\mathcal{N}_{1}=\mathcal{N}_{+}$, then the submanifold is a double warped product $\mathbb{R} \times{ }_{f} \mathbb{R} \times_{g} N^{2}$ where $N^{2}$ is a minimal Lagrangian submanifold in an appropriate space form.
(2) If $\mathcal{N}_{1} \subsetneq \mathcal{N}_{+}$and $\mathcal{N}_{+}$is integrable, then the submanifold is a single warped product $\mathbb{R} \times_{f} N^{3}$ where $N^{3}$ is a special Lagrangian 3 -fold with $S_{3}$-symmetry in an appropriate space form.
(3) If the smallest integrable distribution containing $\mathcal{N}_{1}$ is $T M$, then for this final case, we do not obtain an explicit expression for the immersion, but we will rewrite the equations (7) to a system of partial differential equations in 2 coordinates out of 4 coordinates defined on the submanifold. Here, techniques will be used similar to those in [8].
When we consider the different cases, we will assume the defining conditions hold on an open neighborhood of the considered point.

## 2. Preliminaries

2.1. Complex space forms. We briefly recall the basic properties of $\mathbb{C}^{n}$ and show how Lagrangian submanifolds of $\mathbb{C} P^{n}$ and $\mathbb{C} H^{n}$ can be lifted to subsets of $\mathbb{C}^{n+1}$.

Consider the complex vector space $\mathbb{C}^{n}$. Its elements can be written as n-tuples of complex numbers, so they are given as

$$
\vec{z}=\left(z_{1}, \cdots, z_{n}\right), \quad z_{j}=x_{j}+i y_{j}, \quad x_{j}, y_{j} \in \mathbb{R}
$$

Through the map

$$
\phi: \mathbb{C}^{n} \rightarrow \mathbb{R}^{2 n}:\left(z_{1}, \cdots, z_{n}\right) \rightarrow\left(x_{1}, y_{1}, \cdots, x_{n}, y_{n}\right)
$$

the space $\mathbb{C}^{n}$ is a real $2 n$-dimensional manifold. The multiplication with the imaginary unit $i$ translates to a linear map on $\mathbb{R}^{2 n}$ given as

$$
i\left(x_{1}, y_{1}, \cdots, x_{n}, y_{n}\right)=\left(-y_{1}, x_{1}, \cdots,-y_{n}, x_{n}\right)
$$

and its derivative $J$ is given as

$$
\begin{aligned}
& J \partial_{x_{k}}=\partial_{y_{k}}, \\
& J \partial_{y_{k}}=-\partial_{x_{k}} .
\end{aligned}
$$

This squares to $-I$ and thus defines a complex structure on $\mathbb{C}^{n}$. On $\mathbb{C}^{n}$ there is also a Hermitian form given by

$$
s(\vec{z}, \vec{w})=\sum_{j=1}^{n} z_{j} \bar{w}_{j}=\sum_{j=1}^{n}\left(x_{j} u_{j}+y_{j} v_{j}\right)-i \sum_{j=1}^{n}\left(x_{j} v_{j}-y_{j} u_{j}\right) .
$$

The real part, which can be denoted as $\langle\vec{z}, \vec{w}\rangle$ defines the Euclidean scalar product on $\mathbb{R}^{2 n}$ and induces a natural Riemannian metric on $\mathbb{C}^{n}$. We can see that $J$ is an isometry and the induced Kähler form, which also coincides with the imaginary part of the Hermitian form, is closed. These structures make $\mathbb{C}^{n}$ into a flat Kähler manifold.

The manifold $\mathbb{C} P^{n}$ can be modeled as the quotient $S^{2 n+1} / S^{1}$, where

$$
S^{2 n+1}=\left\{\left.\left(z_{0}, \cdots, z_{n}\right) \in \mathbb{C}^{n+1}\left|\sum_{i=0}^{n}\right| z_{i}\right|^{2}=1\right\}
$$

The equivalence is given by

$$
\vec{z} \sim \vec{w} \Leftrightarrow \exists \phi \in \mathbb{R} \forall j \in\{0, \cdots, n\}: z_{j}=e^{i \phi} w_{j} .
$$

So the unit sphere $S^{2 n+1}$ is the preimage of the Hopf fibration

$$
\pi: S^{2 n+1} \rightarrow \mathbb{C} P^{n}: \vec{z} \rightarrow[\vec{z}] .
$$

On $S^{2 n+1} \subset \mathbb{C}^{n+1}$ the complex structure $J$ induces a contact structure and the standard metric on $\mathbb{C}^{n+1}$ induces a Riemannian metric. The metric on $\mathbb{C} P^{n}$ that makes $\pi$ a Riemannian submersion has constant holomorphic sectional curvature 4. An immersion $\phi: M \rightarrow S^{2 n+1}$ is then said to be C-totally real or horizontal if $i \phi$ is orthogonal to the submanifold. It can be shown that every minimal C-totally real submanifold of $S^{2 n+1}$ can be projected onto a special Lagrangian submanifold of $\mathbb{C} P^{n}$ through $\pi$ and conversely that a special Lagrangian submanifold in $\mathbb{C} P^{n}$ has a 1-parameter family of mutually isometric horizontal lifts as a minimal C-totally real submanifold in $S^{2 n+1}$. So in order to classify special Lagrangian submanifolds in $\mathbb{C} P^{n}$, we can consider minimal C-totally real submanifolds in $S^{2 n+1} \subset \mathbb{C}^{n+1}$, see [12]. For those submanifolds, the Gauss identity is given as

$$
\begin{equation*}
D_{X} Y=\nabla_{X} Y+J A(X, Y)-\langle X, Y\rangle \phi \tag{5}
\end{equation*}
$$

where $D$ is the Levi Civita connection of $\mathbb{C}^{n+1}$.
Similarly, the space $\mathbb{C} H^{n}$ can be modeled as $H^{2 n+1} / S^{1}$, where

$$
H^{2 n+1}=\left\{\left.\left(z_{0}, \cdots, z_{n}\right) \in \mathbb{C}_{1}^{n+1}| | z_{0}\right|^{2}-\sum_{i=1}^{n}\left|z_{i}\right|^{2}=1\right\} .
$$

The equivalence relationship determined by $S^{1}$ is the same as the one used in the projective space. The ambient space $\mathbb{C}_{1}^{n+1}$ is essentially the space $\mathbb{C}^{n+1}$, but equipped with the scalar product

$$
\langle\vec{z}, \vec{w}\rangle_{1}=\Re\left(\sum_{j=1}^{n} z_{j} \bar{w}_{j}-z_{0} \bar{w}_{0}\right) .
$$

The complex structure is still obtained through multiplication with the imaginary unit $i$ and induces a Kähler structure on $\mathbb{C}_{1}^{n+1}$. This metric induces a Lorentzian metric on $H^{2 n+1}$ and a metric of constant holomorphic sectional curvature -4 on $\mathbb{C} H^{n}$. Similar to the projective case C-totally real submanifolds $\phi: M \rightarrow H^{2 n+1}$ can be defined having $i \phi$ as a normal. Each minimal C-totally real submanifold corresponds to the horizontal lift of a special Lagrangian submanifold of $\mathbb{C} H^{n}$. The Gauss identity is given as

$$
\begin{equation*}
D_{X} Y=\nabla_{X} Y+J A(X, Y)+\langle X, Y\rangle \phi \tag{6}
\end{equation*}
$$

where $D$ is the Levi Civita connection of $\mathbb{C}_{1}^{n+1}$.
2.2. Structure equations. We can return briefly to the equations (21) and (3). We can choose an orthogonal frame $\left\{X_{1}, X_{2}, X_{3}, X_{4}\right\}$ corresponding to (4) and define the components $\Gamma_{i j}^{k}$ and $A_{i j}^{k}$ as

$$
\begin{aligned}
\nabla_{X_{i}} X_{j} & =\sum_{k=1}^{4} \Gamma_{i j}^{k} X_{k} \\
A\left(X_{i}, X_{j}\right) & =\sum_{k=1}^{4} A_{i j}^{k} X_{k} .
\end{aligned}
$$

Then the equations (2) and (3) can be rewritten as

$$
\begin{align*}
& X_{i}\left(\Gamma_{j k}^{l}\right)-X_{j}\left(\Gamma_{i k}^{l}\right)=\epsilon\left(\delta_{j k} \delta_{i}^{l}-\delta_{i k} \delta_{j}^{l}\right)+A_{j k}^{r} A_{i r}^{l}-A_{i k}^{r} A_{j r}^{l} \\
&+\Gamma_{i k}^{r} \Gamma_{j r}^{l}-\Gamma_{j k}^{r} \Gamma_{i r}^{l}+\Gamma_{r k}^{l}\left(\Gamma_{i j}^{r}-\Gamma_{j i}^{r}\right)  \tag{7}\\
& X_{i}\left(A_{j k}^{l}\right)-X_{j}\left(A_{i k}^{l}\right)=\left(\Gamma_{i j}^{r}-\Gamma_{j i}^{r}\right) A_{r k}^{l}+\Gamma_{i k}^{r} A_{j r}^{l}-\Gamma_{j k}^{r} A_{i r}^{l}-\Gamma_{i r}^{l} A_{j k}^{r}+\Gamma_{j r}^{l} A_{i k}^{r}, \tag{8}
\end{align*}
$$

where we have used the Einstein convention. We split the connection $\nabla$ into its components and write

$$
\begin{aligned}
& \begin{array}{ll}
\nabla_{X_{1}} X_{1}=a_{1} X_{2}+a_{2} X_{3}+a_{3} X_{4}, \quad \nabla_{X_{1}} X_{2}=-a_{1} X_{1}+a_{4} X_{3}+a_{5} X_{4}, ~
\end{array} \\
& \nabla_{X_{2}} X_{1}=b_{1} X_{2}+b_{2} X_{3}+b_{3} X_{4}, \quad \nabla_{X_{2}} X_{2}=-b_{1} X_{1}+b_{4} X_{3}+b_{5} X_{4}, \\
& \nabla_{X_{3}} X_{1}=c_{1} X_{2}+c_{2} X_{3}+c_{3} X_{4}, \quad \nabla_{X_{3}} X_{2}=-c_{1} X_{1}+c_{4} X_{3}+c_{5} X_{4}, \\
& \nabla_{X_{4}} X_{1}=d_{1} X_{2}+d_{2} X_{3}+d_{3} X_{4}, \quad \nabla_{X_{4}} X_{2}=-d_{1} X_{1}+d_{4} X_{3}+d_{5} X_{4}, \\
& \begin{array}{l|l}
\nabla_{X_{1}} X_{3}=-a_{2} X_{1}-a_{4} X_{2}+a_{6} X_{4}, & \nabla_{X_{1}} X_{4}=-a_{3} X_{1}-a_{5} X_{2}-a_{6} X_{3},
\end{array} \\
& \nabla_{X_{2}} X_{3}=-b_{2} X_{1}-b_{4} X_{2}+b_{6} X_{4}, \quad \nabla_{X_{2}} X_{4}=-b_{3} X_{1}-b_{5} X_{2}-b_{6} X_{3}, \\
& \nabla_{X_{3}} X_{3}=-c_{2} X_{1}-c_{4} X_{2}+c_{6} X_{4}, \quad \nabla_{X_{3}} X_{4}=-c_{3} X_{1}-c_{5} X_{2}-c_{6} X_{3}, \\
& \nabla_{X_{4}} X_{3}=-d_{2} X_{1}-d_{4} X_{2}+d_{6} X_{4}, \quad \nabla_{X_{4}} X_{4}=-d_{3} X_{1}-d_{5} X_{2}-d_{6} X_{3} .
\end{aligned}
$$

Equation (8) induces linear relations between the components, independent of the ambient space. The Gauss equations give further information about $\nabla$ but use differential equations and depend on the ambient space form.

Lemma 2.1. On a special Lagrangian submanifold $M$ having a local $S O(2) \rtimes S_{3}$-symmetry there exists a frame corresponding to (4) such that:

$$
\begin{align*}
& \nabla_{X_{1}} X_{1}=a_{1} X_{2}+a_{2} X_{3}+a_{3} X_{4}, \\
& \nabla_{X_{2}} X_{1}=b_{1} X_{2}+b_{2} X_{3}, \\
& \nabla_{X_{3}} X_{1}=\frac{b_{2}}{3} X_{2}, \\
& \nabla_{X_{4}} X_{1}=0, \\
& \nabla_{X_{1}} X_{3}=-a_{2} X_{1}+b_{2} X_{2}+a_{6} X_{4},  \tag{9}\\
& \nabla_{X_{2}} X_{3}=-b_{2} X_{1}-a_{2} X_{2}+b_{6} X_{4}, \\
& \nabla_{X_{3}} X_{3}=c_{6} X_{4}, \\
& \nabla_{X_{4}} X_{3}=d_{6} X_{4}, \\
& \nabla_{X_{1}} X_{2}=-a_{1} X_{1}-b_{2} X_{3}, \\
& \nabla_{X_{2}} X_{2}=-b_{1} X_{1}+a_{2} X_{3}+a_{3} X_{4}, \\
& \nabla_{X_{3}} X_{2}=-\frac{b_{2}}{3} X_{1}, \\
& \nabla_{X_{4}} X_{2}=0, \\
& \nabla_{X_{1}} X_{4}=-a_{3} X_{1}-a_{6} X_{3}, \\
& \nabla_{X_{2}} X_{4}=-a_{3} X_{2}-b_{6} X_{3}, \\
& \nabla_{X_{3}} X_{4}=-c_{6} X_{3}, \\
& \nabla_{X_{4}} X_{4}=-d_{6} X_{3} .
\end{align*}
$$

Furthermore, the derivatives of $r$ are given by

$$
\begin{align*}
\left(X_{1}+i X_{2}\right)(r) & =3 i r\left(a_{1}+i b_{1}\right),  \tag{10}\\
X_{3}(r) & =r a_{2}  \tag{11}\\
X_{4}(r) & =r a_{3} . \tag{12}
\end{align*}
$$

Proof. This is just a straightforward application of equation (8). For instance

$$
\begin{aligned}
& \left(\nabla_{X_{2}} A\right)\left(X_{1}, X_{1}\right)=X_{2}(r) X_{1}+3 r b_{1} X_{2}+r b_{2} X_{3}+r b_{3} X_{4} \\
& \left(\nabla_{X_{1}} A\right)\left(X_{2}, X_{1}\right)=3 r a_{1} X_{1}-X_{1}(r) X_{2}-r a_{4} X_{3}-r a_{5} X_{4} .
\end{aligned}
$$

Then the corresponding coordinates of both derivatives are the same. Finally we can set $b_{3}=0$, by rotating the distribution $\mathcal{N}_{2}$ such that $X_{3}$ lies in the direction of $\nabla_{X_{1}} X_{2}$, projected on $\mathcal{N}_{2}$.

It is interesting to note that $\mathcal{N}_{2}$ is an integrable distribution. The distribution $\mathcal{N}_{1}$ however is integrable if and only if $b_{2}=0$. Applying (7), we obtain the following result.

Lemma 2.2. The equations (17) on our frame of choice induce a system of differential equations given by:

$$
\begin{align*}
\left(X_{1}+i X_{2}\right)\left(a_{2}-i b_{2}\right) & =a_{3}\left(a_{6}+i b_{6}\right),  \tag{13}\\
X_{3}\left(a_{2}+i b_{2}\right) & =\epsilon+a_{3}^{2}+\left(a_{2}+i b_{2}\right)^{2},  \tag{14}\\
X_{4}\left(a_{2}+i b_{2}\right) & =a_{3}\left(a_{2}+i b_{2}\right),  \tag{15}\\
\left(X_{1}+i X_{2}\right)\left(a_{3}\right) & =-\left(a_{2}-i b_{2}\right)\left(a_{6}+i b_{6}\right),  \tag{16}\\
X_{3}\left(a_{3}\right) & =0,  \tag{17}\\
X_{4}\left(a_{3}\right) & =a_{3}^{2}+\epsilon,  \tag{18}\\
X_{1}\left(b_{6}\right)-X_{2}\left(a_{6}\right) & =-\left(a_{1} a_{6}+b_{1} b_{6}\right),  \tag{19}\\
X_{3}\left(a_{6}+i b_{6}\right) & =\frac{5}{3} i b_{2}\left(a_{6}+i b_{6}\right),  \tag{20}\\
X_{4}\left(a_{6}+i b_{6}\right) & =2 a_{3}\left(a_{6}+i b_{6}\right),  \tag{21}\\
X_{1}\left(b_{1}\right)-X_{2}\left(a_{1}\right) & =2 r^{2}-\left(\epsilon+a_{3}^{2}\right)-\frac{5}{3} b_{2}^{2}-a_{2}^{2}-a_{1}^{2}-b_{1}^{2},  \tag{22}\\
X_{2}\left(b_{1}\right)+X_{1}\left(a_{1}\right) & =-\frac{2}{3} a_{2} b_{2},  \tag{23}\\
3 X_{3}\left(a_{1}\right)-X_{1}\left(b_{2}\right) & =3 a_{1} a_{2}-2 b_{1} b_{2},  \tag{24}\\
3 X_{3}\left(b_{1}\right)-X_{2}\left(b_{2}\right) & =2 b_{2} a_{1}+3 b_{1} a_{2},  \tag{25}\\
X_{4}\left(a_{1}+i b_{1}\right) & =a_{3}\left(a_{1}+i b_{1}\right)+\frac{b_{2}}{3}\left(a_{6}+i b_{6}\right) . \tag{26}
\end{align*}
$$

Proof. This is also a straightforward application of (7). For example:

$$
\begin{aligned}
X_{1}\left(\Gamma_{23}^{1}\right)-X_{2}\left(\Gamma_{13}^{1}\right) & =\Gamma_{13}^{r} \Gamma_{2 r}^{1}-\Gamma_{23}^{r} \Gamma_{1 r}^{1}+\Gamma_{r 3}^{1} \Gamma_{12}^{r}-\Gamma_{r 3}^{1} \Gamma_{21}^{r} \\
& =a_{3} b_{6}, \\
X_{1}\left(\Gamma_{23}^{2}\right)-X_{2}\left(\Gamma_{13}^{2}\right) & =\Gamma_{13}^{r} \Gamma_{2 r}^{2}-\Gamma_{23}^{r} \Gamma_{1 r}^{2}+\Gamma_{r 3}^{2} \Gamma_{12}^{r}-\Gamma_{r 3}^{2} \Gamma_{21}^{r} \\
& =-a_{3} a_{6} .
\end{aligned}
$$

Combining both equations using the usual complex notations leads to (13). The other equations are obtained in a similar way.
2.3. Warped Products. In the analysis that follows, we will often encounter warped products of manifolds. When we consider a warped product of Riemannian manifolds $\left(M_{1}, g_{1}\right)$ and $\left(M_{2}, g_{2}\right)$ with warping function $e^{f}$, where $f: M_{1} \rightarrow \mathbb{R}$, we get a Riemannian
manifold $\left(M_{1} \times M_{2}, g_{f}\right)$ where $M_{1} \times M_{2}$ as a differentiable manifolds is the product of $M_{1}$ and $M_{2}$ and the metric $g_{f}$ is given as

$$
g_{f}(X, Y)=g_{1}\left(X_{1}, Y_{1}\right)+e^{2 f} g_{2}\left(X_{2}, Y_{2}\right)
$$

where a vector field $X$ is uniquely decomposed into a part $X_{1}$ tangent to $M_{1}$ and $X_{2}$ tangent to $M_{2}$. We denote this warped product as $M_{1} \times{ }_{e} M_{2}$. The following result can be obtained, see 9].

Theorem 2.1. Consider a Riemannian manifold ( $M, g$ ) with Levi-Civita connection $\nabla$ and suppose that on a neighborhood of $p \in M$ there are orthogonal distributions $\mathcal{N}_{1}$ and $\mathcal{N}_{2}$ such that

$$
\begin{aligned}
& \forall X, Y \in \mathcal{N}_{1}\left(\text { i.e. } X \text { and } Y \text { are sections of } \mathcal{N}_{1}\right): \nabla_{X} Y \in \mathcal{N}_{1}, \\
& \forall X, Y \in \mathcal{N}_{2}: \nabla_{X} Y=\tilde{\nabla}_{X} Y+g(X, Y) H,
\end{aligned}
$$

where $\tilde{\nabla}$ is the projection of $\nabla$ on $\mathcal{N}_{2}$ and $H \in \mathcal{N}_{1}$. Then there exists a function $f: M \rightarrow$ $\mathbb{R}$ such that on a neighborhood of $p, M$ can be written as $M_{1} \times_{e f} M_{2}$, where $M_{i}$ is an integral manifold of $\mathcal{N}_{i}$.
If furthermore $H=\lambda H_{0}$, where $\left\|H_{0}\right\|=1$, and $X(\lambda)=0$ for every $X \in \mathcal{N}_{2}$, then $f: M_{1} \rightarrow \mathbb{R}$ and $H=-\operatorname{grad} e^{f}$.

The first part of the theorem constructs a twisted product, the second part reduces this to a warped product. This will be useful in choosing coordinates, since the product structures allows for coordinates to be chosen on each factor separately. In particular, if $\operatorname{dim}\left(\mathcal{N}_{1}\right)=1$, then any non vanishing vector field in $\mathcal{N}_{1}$ can be fixed as a useful coordinate vector field on $M$.

## 3. SUBMANIFOLDS IN $\mathbb{C}^{4}$.

3.1. The case where $b_{2}=0$. The assumption that $X_{3}$ lies along $\nabla_{X_{1}} X_{2}$ becomes redundant since the latter has no $\mathcal{N}_{2}$ component. Instead, we can choose $X_{3}$ in the direction of $\nabla_{X_{1}} X_{1}$, projected on $\mathcal{N}_{2}$. Hence without loss of generality we can assume that $a_{3}=0$. The equations (7) show that in this case either $a_{2}=0$ or $a_{6}=b_{6}=c_{6}=d_{6}=0$. First we will assume that $a_{2} \neq 0$.

Theorem 3.1. Consider $M$ a special Lagrangian submanifold in $\mathbb{C}^{4}$ having $S O(2) \rtimes S_{3^{-}}$ symmetry and an orthogonal frame corresponding to (4). Suppose that $\mathcal{N}_{1}$ is an integrable distribution and $\nabla_{X_{1}} X_{1}$ is nowhere contained within this distribution. Then $M$ is locally congruent to

$$
\begin{equation*}
F(t, s, u, v)=(t, s \phi(u, v)) \tag{27}
\end{equation*}
$$

where $\phi$ is the horizontal lift of a special Lagrangian submanifold of $\mathbb{C} P^{2}$ to the unit sphere in $\mathbb{C}^{3}$.

Proof. Taking into account every component that vanishes in (9), we find

$$
\begin{array}{c|c}
\nabla_{X_{1}} X_{1}=a_{1} X_{2}+a_{2} X_{3} & \nabla_{X_{1}} X_{2}=-a_{1} X_{1}, \\
\nabla_{X_{2}} X_{1}=b_{1} X_{2} & \nabla_{X_{2}} X_{2}=-b_{1} X_{1}+a_{2} X_{3}, \\
\nabla_{X_{3}} X_{1}=0 & \nabla_{X_{3}} X_{2}=0 \\
\nabla_{X_{4} X_{1}=0}=0 & \nabla_{X_{4} X_{2}=0}^{\nabla_{X_{1}} X_{3}=-a_{2} X_{1}} \\
\nabla_{X_{2}} X_{3}=-a_{2} X_{2} & \nabla_{X_{1}} X_{4}=0 \\
\nabla_{X_{3}} X_{3}=0 & \nabla_{X_{2} X_{4}}=0 \\
\nabla_{X_{4}} X_{3}=0 & \nabla_{X_{3}} X_{4}=0 \\
& \nabla_{X_{4}} X_{4}=0
\end{array}
$$

We find that the distributions $\operatorname{span}\left\{X_{4}\right\}$ and $\operatorname{span}\left\{X_{1}, X_{2}, X_{3}\right\}$ satisfy the conditions for a warped product $\mathbb{R} \times_{e^{f}} N^{3}$. But $X_{4}(f)=0$, hence $f$ is a constant. $M$ is a standard Riemannian product $\mathbb{R} \times N^{3}$ and its immersion can be written, up to an isometry as

$$
F(t, x)=(t, \psi(x)), \psi: N^{3} \rightarrow \mathbb{C}^{4}
$$

The immersion $\psi$ is contained in the subspace orthogonal to both $X_{4}$ and $J X_{4}$, since they both are constant unit normals along $N^{3}$. Now it is also obvious that $\operatorname{span}\left\{X_{3}\right\}$ and $\mathcal{N}_{1}$ satisfy the conditions for a warped product. So $N^{3}$ can be decomposed as $\mathbb{R} \times{ }_{e}{ }^{g} N^{2}$ and $X_{3}(g)=-a_{2}$. Then $X_{3}$ can be associated with a coordinate $s$ on the manifold and it follows that

$$
D_{X_{3}} X_{3}=\frac{\partial^{2} F}{\partial s^{2}}=0 \Rightarrow F=A s+B
$$

Both $A$ and $B$ are independent of $(s, t)$. Calculating (7), one has

$$
X_{3}\left(a_{2}\right)=\frac{\partial a_{2}}{\partial s}=a_{2}^{2}
$$

The solution of this equation, after a translation of the $s$-coordinate is given as $a_{2}=-\frac{1}{s}$. The derivatives of $X_{3}$ to $X_{1}$ and $X_{2}$ are

$$
\begin{aligned}
D_{X_{i}} X_{3} & =\frac{\partial F_{*} X_{i}}{\partial s}=A_{*} X_{i} \\
& =\frac{1}{s} X_{i}=A_{*} X_{i}+\frac{B_{*} X_{i}}{s} \\
& \Rightarrow B_{*}=0 .
\end{aligned}
$$

So $B$ is a constant vector along the submanifold and vanishes when applying a translation. It is easy to see that $X_{3}=A$ and is orthogonal to $X_{i}=s A_{*}\left(X_{i}\right)$, for $i \in\{1,2\}$. Hence everywhere along $A$, the position vector is orthogonal to the tangent space. Thus $A$ has constant length. Calculating the other covariant derivatives yields for example for $i, j \in\{1,2\}$ that

$$
\begin{align*}
A_{*} X_{i} & =\frac{F_{*} X_{i}}{s}  \tag{28}\\
D_{X_{i}}\left(A_{*} X_{j}\right) & =\frac{D_{X_{i}}\left(F_{*} X_{j}\right)}{s}=A_{*}\left(\tilde{\nabla}_{X_{i}} X_{j}\right)+J A_{*}\left(K\left(X_{i}, X_{j}\right)\right)-\frac{1}{s^{2}} \delta_{i j} \phi
\end{align*}
$$

Here $\tilde{\nabla}$ is the connection restricted to $N^{2}$. Combining this with the other equations in (7), it follows that $A$ is a C-totally real immersion in $S^{5} \subset \mathbb{C}^{3}$. Furthermore, the components $a_{1}$ and $b_{1}$ have no other restrictions on them except satisfying the Gauss equations for a minimal C-totally real submanifold of $S^{5}$. This proves the theorem.

The case $a_{2}=0$ was the only case that was studied in [7]. We can quote the following result from [7].

Theorem 3.2. Consider $M$ a special Lagrangian submanifold in $\mathbb{C}^{4}$ having a local $S O(2) \rtimes$ $S_{3}$-symmetry group and an orthogonal frame corresponding to (4). Suppose that $\mathcal{N}_{1}$ is an integrable distribution and $\nabla_{X_{1}} X_{1}$ is contained within this distribution. Then $M$ is locally congruent to

$$
\begin{equation*}
F(t, s, u, v)=(t, s, \phi(u, v)) \tag{29}
\end{equation*}
$$

where $\phi: \mathbb{C} \rightarrow \mathbb{C}^{2}$ is a special Lagrangian surface.
Remark 3.1. As proved in [7], a special Lagrangian surface in $\mathbb{C}^{2}$, with complex coordinates $x_{1}+i y_{1}$ and $x_{2}+i y_{2}$ is a holomorphic curve in $\mathbb{C}^{2}$ with complex coordinates $x_{1}-i x_{2}$ and $y_{1}+i y_{2}$, and conversely.
3.2. The case where $b_{2} \neq 0$. Now the distribution $\mathcal{N}_{1}$ is no longer integrable. The simplest case one can hope for is that there is a 3-dimensional integrable distribution containing $\mathcal{N}_{1}$. Such a distribution should contain at least $X_{3}$ since

$$
\left[X_{1}, X_{2}\right] \quad \bmod \mathcal{N}_{1} \| X_{3} .
$$

Using the fact that $b_{2} \neq 0$, the equations (7) reduce (9) to

$$
\begin{array}{c|c}
\nabla_{X_{1}} X_{1}=a_{1} X_{2}+a_{2} X_{3}+a_{3} X_{4} & \nabla_{X_{1}} X_{2}=-a_{1} X_{1}-b_{2} X_{3}, \\
\nabla_{X_{2}} X_{1}=b_{1} X_{2}+b_{2} X_{3} & \nabla_{X_{2}} X_{2}=-b_{1} X_{1}+a_{2} X_{3}+a_{3} X_{4}, \\
\nabla_{X_{3}} X_{1}=\frac{b_{2} X_{2}}{3} & \nabla_{X_{3}} X_{2}=-\frac{b_{2}}{3} X_{1}, \\
\nabla_{X_{4} X_{1}}=0 & \nabla_{X_{4}} X_{2}=0,  \tag{30}\\
\nabla_{X_{1}} X_{3}=-a_{2} X_{1}+b_{2} X_{2}+a_{6} X_{4} & \nabla_{X_{1} X_{4}=-a_{3} X_{1}-a_{6} X_{3}}^{\nabla_{X_{2}} X_{3}=-b_{2} X_{1}-a_{2} X_{2}+b_{6} X_{4}} \quad \nabla_{X_{2} X_{4}=-a_{3} X_{2}-b_{6} X_{3}} \quad \nabla_{X_{3} X_{4}=-a_{3} X_{3}} \\
\nabla_{X_{3}} X_{3}=a_{3} X_{4} & \nabla_{X_{4}} X_{4}=0 .
\end{array}
$$

It is apparent that the condition that $\mathcal{N}_{+}$is integrable is given by $a_{6}+i b_{6}=0$. We consider this case first.

Theorem 3.3. Suppose $M$ is a special Lagrangian submanifold in $\mathbb{C}^{4}$ with $S O(2) \rtimes S_{3}$ symmetry, such that $\mathcal{N}_{1}$ is not an integrable distribution, but $\mathcal{N}_{+}$is. Then the submanifold, up to isometry, can be given locally by either

$$
\begin{equation*}
F(t, s, u, v)=(t, \phi(s, u, v)), \tag{31}
\end{equation*}
$$

where $\phi$ is a special Lagrangian submanifold with $S_{3}$-symmetry in $\mathbb{C}^{3}$, or

$$
\begin{equation*}
F(t, s, u, v)=t \phi(s, u, v) \tag{32}
\end{equation*}
$$

where $\phi$ is the horizontal lift of a special Lagrangian submanifold with local $S_{3}$-symmetry in $\mathbb{C} P^{3}$ to the unit sphere in $\mathbb{C}^{4}$.

Proof. We find according to (30) and (7) that $\operatorname{span}\left\{X_{4}\right\}$ and $\mathcal{N}_{+}$satisfy the conditions for a warped product. So $M$ can be decomposed as $\mathbb{R} \times_{e f} N^{3}$, where $X_{4}(f)=-a_{3}$. We can solve

$$
X_{4}\left(a_{3}\right)=\frac{\partial a_{3}}{\partial t}=a_{3}^{2} .
$$

This equation has 2 possible solutions.
First, we assume $a_{3}=0$. In this case $M$ is simply the manifold $\mathbb{R} \times N^{3}$. Hence the immersion, up to isometry, can be given as

$$
F(t, s, u, v)=(t, \phi(s, u, v))
$$

where $\phi$ is a 3 -fold immersed in the subspace $\mathbb{C}^{3}$ orthogonal to $X_{4}$ and $J X_{4}$. Similar calculations as in (28) show that this can be any special Lagrangian submanifold in $\mathbb{C}^{3}$, given the presence of an $S_{3}$-symmetry in the second fundamental form.
The second solution, after a translation of $t$, is given by $a_{3}=-\frac{1}{t}$. The calculations are similar to the case where $b_{2}=0$ and $a_{2} \neq 0$. This gives the required result.

The last case in $\mathbb{C}^{4}$ is the one where there is no integrable distribution containing $\mathcal{N}_{1}$ other than the whole tangent bundle. In this case, we can no longer rely on an obvious warped product structure. We can attempt to introduce a set of independent coordinates and reduce (7) to a system of PDE's on $\mathbb{C}^{4}$ using as little functions as possible. We now use (13) to (26) to construct a coordinate frame from $\left\{X_{1}, X_{2}, X_{3}, X_{4}\right\}$. Since $\mathcal{N}_{2}$ is integrable, it is a good idea to choose $X_{4}=T$ and $\mu X_{3}=S$. Requiring that $[S, T]=0$ implies that

$$
\left[\mu X_{3}, X_{4}\right]=\mu\left[X_{3}, X_{4}\right]-X_{4}(\mu) X_{3}=-\left(\mu a_{3}+X_{4}(\mu)\right) X_{3}=0
$$

We can find such a $\mu$ by taking $\mu=\frac{1}{\sqrt{\left|\epsilon+a_{3}^{2}\right|}}$. The equation $a_{3}^{2}+\epsilon=0$ implies that $a_{3}$ is a constant and hence $\left(a_{2}-i b_{2}\right)\left(a_{6}+i b_{6}\right)=0$. This will correspond to the integrability of either $\mathcal{N}_{1}$ or $\mathcal{N}_{+}$. Therefore $\mu$ is well defined.

Vector fields $U$ and $V$ can be sought such that every couple out of $\{S, T, U, V\}$ commutes. Such an attempt can be made, writing

$$
\begin{equation*}
U+i V=\left(\rho_{1}-i \rho_{2}\right)\left(\left(X_{1}+i X_{2}\right)+\left(\alpha_{1}+i \beta_{1}\right) S+\left(\alpha_{2}+i \beta_{2}\right) T\right) \tag{33}
\end{equation*}
$$

We rename the following expressions:

$$
\begin{aligned}
\rho & =\rho_{1}-i \rho_{2}, \\
\gamma_{j} & =\alpha_{j}+i \beta_{j}
\end{aligned} \quad j \in\{1,2\} .
$$

After calculating the Lie brackets of these four vector fields, the following conditions on the introduced functions make the vector fields commute:

$$
\begin{align*}
X_{4}(\rho) & =-a_{3} \rho,  \tag{34}\\
X_{3}(\rho) & =-\left(a_{2}+\frac{2}{3} i b_{2}\right) \rho,  \tag{35}\\
\left(X_{1}-i X_{2}\right)(\rho) & =\left(b_{1}+i a_{1}\right) \rho,  \tag{36}\\
X_{4}\left(\gamma_{1}\right) & =-\frac{1}{\mu}\left(a_{6}+i b_{6}\right)+a_{3} \gamma_{1},  \tag{37}\\
X_{3}\left(\gamma_{1}\right) & =\frac{1}{\mu^{2}}\left(X_{1}+i X_{2}\right)(\mu)+\gamma_{1}\left(a_{2}+\frac{2}{3} i b_{2}\right),  \tag{38}\\
X_{2}\left(\alpha_{1}\right)-X_{1}\left(\beta_{1}\right) & =a_{1} \alpha_{1}+b_{1} \beta_{1}-\frac{2}{\mu} b_{2},  \tag{39}\\
X_{4}\left(\gamma_{2}\right) & =a_{3} \gamma_{2},  \tag{40}\\
X_{3}\left(\gamma_{2}\right) & =\left(a_{6}+i b_{6}\right)+\left(a_{2}+\frac{2}{3} i b_{2}\right) \gamma_{2},  \tag{41}\\
X_{2}\left(\alpha_{2}\right)-X_{1}\left(\beta_{2}\right) & =a_{1} \alpha_{2}+b_{1} \beta_{2} . \tag{42}
\end{align*}
$$

The following result can be obtained.
Lemma 3.1. Suppose $f$ and $g$ are real valued functions on the manifold satisfying

$$
\begin{gathered}
S(f)=0, \quad T(f)=-1, \\
S(g)=-1, \quad T(g)=0,
\end{gathered}
$$

and defining

$$
\begin{array}{ll}
X_{1}(f)=\alpha_{2}, & X_{2}(f)=\beta_{2}, \\
X_{1}(g)=\alpha_{1}, & X_{2}(g)=\beta_{1},
\end{array}
$$

then the functions $\alpha_{i}$ and $\beta_{i}$ obtained this way satisfy the conditions (37) to (42).
It is interesting to see that this way the vector fields

$$
\begin{aligned}
\tilde{U} & =X_{1}+\alpha_{1} S+\alpha_{2} T, \\
\tilde{V} & =X_{2}+\beta_{1} S+\beta_{2} T,
\end{aligned}
$$

satisfy $\tilde{U}(f)=\tilde{U}(g)=\tilde{V}(f)=\tilde{V}(g)=0$. Furthermore $\tilde{U}$ and $\tilde{V}$ are independent of oneanother and they span the distribution which is the intersection of the kernel of $\mathrm{d} f$ and $\mathrm{d} g$. Note that this distribution is indeed 2-dimensional since both forms have a hyperplane as a kernel and these kernels do not coincide, since the 1-forms are linearly independent. Using the dimension theorem, they have a 2-dimensional intersection. Construction (33) is just a complex rotation of these two vector fields in that distribution. This way, it is clear that $f$ and $g$ serve as coordinates $s$ and $t$ conjugate to $S$ and $T$.

Proof. Apply the relation

$$
\left[X_{i}, X_{j}\right](f)=X_{i} X_{j}(f)-X_{j} X_{i}(f)=\nabla_{X_{i}} X_{j}(f)-\nabla_{X_{j}} X_{i}(f)
$$

on both functions, using (30).

A suitable function for $f$ is easily found, since $S\left(a_{3}\right)=0$. Let $f$ be a function of $a_{3}$, then

$$
X_{4}(f)=f^{\prime}\left(\epsilon+a_{3}^{2}\right)=-1 \Leftrightarrow f^{\prime}=-\frac{1}{\epsilon+a_{3}^{2}}
$$

Hence $f$ can be given by

$$
f=-\int \frac{1}{\epsilon+a_{3}^{2}} \mathrm{~d} a_{3} .
$$

This also determines $\gamma_{2}$ completely, since using (13) yields

$$
\begin{aligned}
\gamma_{2} & =\left(X_{1}+i X_{2}\right)(f)=f^{\prime}\left(X_{1}+i X_{2}\right)\left(a_{3}\right) \\
& =\frac{a_{2} a_{6}+b_{2} b_{6}}{\epsilon+a_{3}^{2}}+i \frac{a_{2} b_{6}-b_{2} a_{6}}{\epsilon+a_{3}^{2}}
\end{aligned}
$$

As for the function $g$, the complex valued function $z=\mu\left(a_{2}+i b_{2}\right)$ can be considered and calculations show

$$
\begin{aligned}
X_{4}(z) & =-\mu a_{3}\left(a_{2}+i b_{2}\right)+\mu a_{3}\left(a_{2}+i b_{2}\right)=0 \\
S(z) & =\mu^{2}\left(\epsilon+a_{3}^{2}+\left(a_{2}+i b_{2}\right)^{2}\right)=\operatorname{sg}\left(\epsilon+a_{3}^{2}\right)+z^{2}
\end{aligned}
$$

Rewriting $\tilde{\epsilon}=\operatorname{sg}\left(\epsilon+a_{3}^{2}\right)$, we find that $z$ is useful as long as $z^{2}+\tilde{\epsilon} \neq 0$. When $\tilde{\epsilon}=+1$, this occurs when $a_{2}=0$ and $\left|b_{2}\right|=\sqrt{\epsilon+a_{3}^{2}}$. When $\tilde{\epsilon}=-1$, this occurs when $\left|a_{2}\right|=\sqrt{\left|\epsilon+a_{3}^{2}\right|}$ and $b_{2}=0$, resulting in $\mathcal{N}_{1}$ being integrable.

First we assume that $z^{2} \neq-\tilde{\epsilon}$. Then the function $g$ can be calculated as the real part of a function $G$ of $z$ given by

$$
S(G)=\left(\tilde{\epsilon}+z^{2}\right) G^{\prime}=-1 \Leftrightarrow G^{\prime}=-\frac{1}{\tilde{\epsilon}+z^{2}}
$$

A function $\rho$ still has to be constructed satisfying (34) to (36). Define a function $H$ as

$$
H=\rho^{3} r\left(z^{2}+\tilde{\epsilon}\right)\left|\epsilon+a_{3}^{2}\right| .
$$

This function is a constant on the submanifold and can be taken to be equal to 1 . This defines a function $\rho$ satisfying the necessary conditions.

Using the Frobenius theorem in [10], a coordinate frame on the submanifold is given by

$$
\begin{aligned}
X_{4} & =T \\
X_{3} & =\frac{1}{\mu} S \\
X_{1}+i X_{2} & =\frac{U+i V}{\rho}-\gamma_{1} S-\gamma_{2} T .
\end{aligned}
$$

We can describe the dependence of $a_{6}+i b_{6}$ on $(s, t)$ by writing

$$
a_{6}+i b_{6}=\frac{k_{3}+i k_{4}}{\rho} \sqrt{\left|a_{3}^{2}+\epsilon\right|}\left(\bar{z}^{2}+\tilde{\epsilon}\right)^{\frac{-1}{2}} .
$$

The functions $k_{3}$ and $k_{4}$ depend solely on $(u, v)$. This expression is obtained from (20) and (21). The rest of the equations in (7) can be rewritten and solved. Applying our
method for $\epsilon=0$, we find after a translation of the coordinates that

$$
\begin{aligned}
& a_{3}=-\frac{1}{t}, \\
& x=\frac{\sin (2 s)}{\cos (2 s)+\cosh \left(2 k_{1}\right)} \Rightarrow a_{2}=-\frac{\sin (2 s)}{t\left(\cos (2 s)+\cosh \left(2 k_{1}\right)\right)}, \\
& y=\frac{\sinh \left(2 k_{1}\right)}{\cos (2 s)+\cosh \left(2 k_{1}\right)} \Rightarrow b_{2}=-\frac{\sinh \left(2 k_{1}\right)}{t\left(\cos (2 s)+\cosh \left(2 k_{1}\right)\right)}, \\
& r=\frac{e^{k_{2}}}{t \sqrt{\cos (2 s)+\cosh \left(2 k_{1}\right)}} .
\end{aligned}
$$

Here the functions $k_{1}$ and $k_{2}$ depend solely on $(u, v)$. Then we can use (13) to find an expression for $\gamma_{1}$ in terms of the coordinates. Equation (10) can be used to find an expression for $a_{1}$ and $b_{1}$ in terms of the coordinates. We obtain

$$
\begin{aligned}
\gamma_{1}= & \frac{\left(k_{3}+i k_{4}\right) \cos \left(s-i k_{1}\right)+t\left(\frac{\partial k_{1}}{\partial v}-i \frac{\partial k_{1}}{\partial u}\right)}{t \rho}, \\
a_{1}= & \frac{2^{\frac{2}{3}} e^{\frac{2}{3} k_{2}}}{3 t^{3}\left(\cos (2 s)+\cosh \left(2 k_{1}\right)\right)^{2}}\left(t \left(\left(\cos (2 s)+\cosh \left(2 k_{1}\right)\right)\left(\rho_{1} \frac{\partial k_{2}}{\partial v}+\rho_{2} \frac{\partial k_{2}}{\partial u}\right)\right.\right. \\
& \left.+\sin (2 s)\left(\rho_{1} \frac{\partial k_{1}}{\partial u}-\rho_{2} \frac{\partial k_{1}}{\partial v}\right)-\sinh \left(2 k_{1}\right)\left(\rho_{1} \frac{\partial k_{1}}{\partial v}+\rho_{2} \frac{\partial k_{1}}{\partial u}\right)\right) \\
& \left.+\sinh \left(2 k_{1}\right)\left(\cos (s) \cosh \left(k_{1}\right)\left(k_{4} \rho_{2}-k_{3} \rho_{1}\right)+\sin (s) \sinh \left(k_{1}\right)\left(k_{4} \rho_{1}+k_{3} \rho_{2}\right)\right)\right), \\
b_{1}= & \frac{2^{\frac{2}{3}} e^{\frac{2}{3} k_{2}}}{3 t^{3}\left(\cos (2 s)+\cosh \left(2 k_{1}\right)\right)^{2}}\left(t \left(\left(\cos (2 s)+\cosh \left(2 k_{1}\right)\right)\left(\rho_{2} \frac{\partial k_{2}}{\partial v}-\rho_{1} \frac{\partial k_{2}}{\partial u}\right)\right.\right. \\
& \left.+\sin (2 s)\left(\rho_{2} \frac{\partial k_{1}}{\partial u}+\rho_{1} \frac{\partial k_{1}}{\partial v}\right)+\sinh \left(2 k_{1}\right)\left(\rho_{1} \frac{\partial k_{1}}{\partial u}-\rho_{2} \frac{\partial k_{1}}{\partial v}\right)\right) \\
& \left.+\sinh \left(2 k_{1}\right)\left(\sin (s) \sinh \left(k_{1}\right)\left(k_{4} \rho_{2}-k_{3} \rho_{1}\right)-\cos (s) \cosh \left(k_{1}\right)\left(k_{4} \rho_{1}+k_{3} \rho_{2}\right)\right)\right) .
\end{aligned}
$$

Now every function on the submanifold is expressed in terms of $(s, t, u, v)$, possibly indirectly through $\left\{k_{1}, k_{2}, k_{3}, k_{4}\right\}$. Demanding that the other Gauss equations are satisfied gives partial differential equations for $k_{i}$, given by

$$
\begin{align*}
\frac{\partial k_{4}}{\partial u}-\frac{\partial k_{3}}{\partial v} & =2 \tanh \left(k_{1}\right)\left(k_{3} \frac{\partial k_{1}}{\partial v}-k_{4} \frac{\partial k_{1}}{\partial u}\right) \\
\frac{\partial k_{4}}{\partial v}+\frac{\partial k_{3}}{\partial u} & =-2 \operatorname{coth}\left(k_{1}\right)\left(k_{3} \frac{\partial k_{1}}{\partial u}+k_{4} \frac{\partial k_{1}}{\partial v}\right)  \tag{43}\\
\Delta k_{2} & =3 * 2^{\frac{1}{3}} e^{-\frac{2 k_{2}}{3}}\left(-e^{2 k_{2}}+\cosh \left(2 k_{1}\right)\right) \\
\Delta k_{1} & =-2^{\frac{1}{3}} e^{-\frac{2 k_{2}}{3}} \sinh \left(2 k_{1}\right) .
\end{align*}
$$

Now we return to the case where $-1=z^{2}$ and $\tilde{\epsilon}=1$. We assume first that $\epsilon$ isn't specified. In this case $a_{2}=0, b_{2}= \pm \sqrt{\epsilon+a_{3}^{2}}$ and $S(z)=0$, so $z$ is insufficient to construct the
function $g$. Equations (77) are reduced to

$$
\begin{align*}
\left(X_{1}-i X_{2}\right)\left(a_{6}+i b_{6}\right) & =-i\left(b_{1}+i a_{1}\right)\left(a_{6}+i b_{6}\right),  \tag{44}\\
X_{3}\left(a_{6}+i b_{6}\right) & = \pm i \frac{5}{3} \sqrt{\epsilon+a_{3}^{2}}\left(a_{6}+i b_{6}\right),  \tag{45}\\
X_{4}\left(a_{6}+i b_{6}\right) & =2 a_{3}\left(a_{6}+i b_{6}\right),  \tag{46}\\
X_{1}\left(b_{1}\right)-X_{2}\left(a_{1}\right) & =2 r^{2}-\frac{8}{3}\left(\epsilon+a_{3}^{2}\right)-a_{1}^{2}-b_{1}^{2},  \tag{47}\\
X_{1}\left(a_{1}\right)+X_{2}\left(b_{1}\right) & =0,  \tag{48}\\
X_{4}\left(a_{1}+i b_{1}\right) & =a_{3}\left(a_{1}+i b_{1}\right) \pm \sqrt{\epsilon+a_{3}^{2}} \frac{\left(a_{6}+i b_{6}\right)}{3},  \tag{49}\\
X_{3}\left(a_{1}+i b_{1}\right) & =\frac{i}{3}\left(a_{3}\left(a_{6}+i b_{6}\right) \pm 2 \sqrt{\epsilon+a_{3}^{2}}\left(a_{1}+i b_{1}\right)\right) . \tag{50}
\end{align*}
$$

The first equation is obtained from applying integrability on $a_{3}$. Now we define

$$
w=\frac{a_{6}+i b_{6}}{\epsilon+a_{3}^{2}}
$$

which after derivation gives

$$
\begin{aligned}
X_{4}(w) & =-2 a_{3} \frac{\left(a_{6}+i b_{6}\right)}{\epsilon+a_{3}^{2}}+2 a_{3} \frac{a_{6}+i b_{6}}{\epsilon+a_{3}^{2}}=0 \\
S(w) & = \pm i \frac{5}{3} w
\end{aligned}
$$

The resulting differential equation for a function $G$ of $w$ will be

$$
S(G)= \pm G^{\prime} i \frac{5}{3} w=-1 \Leftrightarrow G^{\prime}= \pm i \frac{3}{5 w} .
$$

The solution is that $G$ is a logarithm of $w$. We find that $H$ defined by

$$
H=w^{2}\left(\epsilon+a_{3}^{2}\right)^{2} \rho^{5} r
$$

is a constant and hence can be used to express $\rho$. We can thus solve $w$ as

$$
w=e^{k_{1} \pm i \frac{5}{3} s}=e^{k_{1}}\left(\cos \left(\frac{5}{3} s\right) \pm i \sin \left(\frac{5}{3} s\right)\right)
$$

Applying (18),(46),(45),(12) and (11) when $\epsilon=0$ yields

$$
\begin{aligned}
a_{3} & =-\frac{1}{t} \\
a_{6} & =e^{k_{1}} \frac{\cos \left(\frac{5}{3} s\right)}{t^{2}} \\
b_{6} & = \pm e^{k_{1}} \frac{\sin \left(\frac{5}{3} s\right)}{t^{2}} \\
r & =\frac{e^{k_{2}}}{t}
\end{aligned}
$$

The equation (10) now gives $a_{1}+i b_{1}$ immediately without going through $\gamma_{1}$ because of (11). The final unknown, $\gamma_{1}$ can then be determined using (44). When we pick $b_{2}=a_{3}$,
we obtain

$$
\begin{aligned}
& \gamma_{1+}=\frac{-5 e^{k_{1}+\frac{5 s i}{3}}+e^{\frac{2 k_{1}+k_{2}}{5}+\frac{2 s i}{3}} t\left(\left(\frac{\partial k_{2}}{\partial v}-3 \frac{\partial k_{1}}{\partial v}\right)-i\left(\frac{\partial k_{2}}{\partial u}-3 \frac{\partial k_{1}}{\partial u}\right)\right)}{5 t^{2}}, \\
& a_{1+}=\frac{e^{k_{1}} \cos \left(\frac{5 s}{3}\right)+e^{\frac{2 k_{1}+k_{2}}{5}} t\left(\cos \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial v}+\sin \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial u}\right)}{3 t^{2}} \\
& b_{1+}=\frac{e^{k_{1}} \sin \left(\frac{5 s}{3}\right)-e^{\frac{2 k_{1}+k_{2}}{5}} t\left(\cos \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial u}-\sin \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial v}\right)}{3 t^{2}}
\end{aligned}
$$

and for $b_{2}=-a_{3}$ we obtain

$$
\begin{aligned}
& \gamma_{1-}=\frac{-5 e^{k_{1}-\frac{5 s i}{3}}+e^{\frac{2 k_{1}+k_{2}}{5}-\frac{2 s i}{3}} t\left(\left(3 \frac{\partial k_{1}}{\partial v}-\frac{\partial k_{2}}{\partial v}\right)-i\left(3 \frac{\partial k_{1}}{\partial u}-\frac{\partial k_{2}}{\partial u}\right)\right)}{5 t^{2}} \\
& a_{1-}=\frac{-e^{k_{1}} \cos \left(\frac{5 s}{3}\right)+e^{\frac{2 k_{1}+k_{2}}{5}} t\left(\cos \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial v}-\sin \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial u}\right)}{3 t^{2}} \\
& b_{1-}=\frac{e^{k_{1}} \sin \left(\frac{5 s}{3}\right)-e^{\frac{2 k_{1}+k_{2}}{5}} t\left(\cos \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial u}+\sin \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial v}\right)}{3 t^{2}}
\end{aligned}
$$

Equations (47) and (39) result in restrictions on the functions $k_{1}$ and $k_{2}$ of $(u, v)$ given by

$$
\begin{align*}
& \Delta k_{2}=e^{-\frac{2}{5}\left(2 k_{1}+k_{2}\right)}\left(8-6 e^{2 k_{2}}\right), \\
& \Delta k_{1}=e^{-\frac{2}{5}\left(2 k_{1}+k_{2}\right)}\left(6-2 e^{2 k_{2}}\right) \tag{51}
\end{align*}
$$

These equations are valid for both $b_{2}= \pm a_{3}$. Using the constructed functions, the rest of the Gauss equations don't impose further conditions. We can summarize this result in the following theorem.

Theorem 3.4. Each special Lagrangian submanifold of $\mathbb{C}^{4}$ with $S O(2) \rtimes S_{3}$-symmetry where the only integral distribution containing $\mathcal{N}_{1}$ is the tangent bundle, can be constructed in the way above using either functions $\left\{k_{1}, k_{2}, k_{3}, k_{4}\right\}$ subject to (43) or functions $\left\{k_{1}, k_{2}\right\}$ subject to (51). Conversely, each such a construction results in such a submanifold, unique up to local isometry.

In the upcoming sections we will consider the construction for $\epsilon= \pm 1$.

## 4. Submanifolds in $\mathbb{C} P^{4}$.

4.1. The case where $b_{2}=0$. This means that both $\mathcal{N}_{1}$ and $\mathcal{N}_{2}$ are integrable distributions. We can assume $a_{3}=0$. However, the Gauss equation

$$
\begin{equation*}
X_{3}\left(a_{2}\right)=1+a_{2}^{2} \tag{52}
\end{equation*}
$$

no longer allows for $a_{2}$ being a constant. The following result is obtained:
Theorem 4.1. Suppose $M$ is a special Lagrangian submanifold in $\mathbb{C} P^{4}$ having $S O(2) \rtimes S_{3^{-}}$ symmetry. Suppose $\mathcal{N}_{1}$ is integrable. Then $M$ can be lifted horizontally to a submanifold in the unit sphere of $\mathbb{C}^{5}$ through $F$ and this lift is congruent to

$$
\begin{equation*}
F(t, s, u, v)=(\phi(u, v) \cos (s), \sin (s) \cos (t), \sin (s) \sin (t)), \tag{53}
\end{equation*}
$$

where $\phi$ is the horizontal lift of a special Lagrangian submanifold in $\mathbb{C} P^{2}$ to the unit sphere in $\mathbb{C}^{3}$.

Proof. Equations (7) reduce $\nabla$ to

$$
\begin{array}{c|c}
\nabla_{X_{1}} X_{1}=a_{1} X_{2}+a_{2} X_{3} & \nabla_{X_{1}} X_{2}=-a_{1} X_{1} \\
\nabla_{X_{2}} X_{1}=b_{1} X_{2} & \nabla_{X_{2}} X_{2}=-b_{1} X_{1}+a_{2} X_{3} \\
\nabla_{X_{3}} X_{1}=0 & \nabla_{X_{3}} X_{2}=0 \\
\nabla_{X_{4}} X_{1}=0 & \nabla_{X_{4}} X_{2}=0 \\
\nabla_{X_{1}} X_{3}=-a_{2} X_{1} & \nabla_{X_{1} X_{4}=0}^{\nabla_{X_{2}} X_{3}=-a_{2} X_{2}} \\
\nabla_{X_{3}} X_{3}=0 & \nabla_{X_{2}} X_{4}=0 \\
\nabla_{X_{4}} X_{3}=\frac{X_{4}}{a_{2}} & \nabla_{X_{3}} X_{4}=0 \\
\nabla_{X_{4}} X_{4}=-\frac{X_{3}}{a_{2}}
\end{array}
$$

The distributions $\mathcal{N}_{1}$ and $\mathcal{N}_{2}$ satisfy the conditions for a warped product $N_{2} \times{ }_{e f} N_{1}$. Furthermore, the distributions span $\left\{X_{3}\right\}$ and $\operatorname{span}\left\{X_{4}\right\}$ satisfy those of a warped product and we can write $M=\mathbb{R} \times_{e^{g}} \mathbb{R} \times{ }_{e} N_{1}$. The functions $f$ and $g$ depend solely on the parameter corresponding to $X_{3}$ and are given by $X_{3}(f)=-a_{2}$ and $X_{3}(g)=\frac{1}{a_{2}}$. We can assume $X_{3}=\frac{\partial}{\partial s}$ on the submanifold. We can also find a function $\mu(s)$ such that $\mu X_{4}=\frac{\partial}{\partial t}$. To find a suitable $\mu$, we solve

$$
\left[X_{3}, \mu X_{4}\right]=\left(X_{3}(\mu)-\frac{\mu}{a_{2}}\right) X_{4}=\left(\mu^{\prime}\left(1+a_{2}^{2}\right)-\frac{\mu}{a_{2}}\right) X_{4}=0
$$

The function $\mu=\frac{a_{2}}{\sqrt{1+a_{2}^{2}}}$ satisfies this equation. We can find $a_{2}(s)$ by solving

$$
\frac{\partial a_{2}}{\partial s}=1+a_{2}^{2} \Rightarrow a_{2}=\tan (s)
$$

Hence $\mu(s)=\sin (s)$ and we calculate for $i \in\{1,2\}$ that

$$
\begin{aligned}
D_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} & =\frac{\partial^{2} F}{\partial s^{2}}=-F \\
& \Rightarrow F=A \cos (s)+B \sin (s), \\
D_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s} & =\frac{\partial^{2} F}{\partial t \partial s}=-\frac{\partial A}{\partial t} \sin (s)+\frac{\partial B}{\partial t} \cos (s) \\
& =\cot (s) \frac{\partial F}{\partial t}=\frac{\cos (s)^{2}}{\sin (s)} \frac{\partial A}{\partial t}+\cos (s) \frac{\partial B}{\partial t} \\
& \Rightarrow \frac{\partial A}{\partial t}=0, \\
D_{X_{i}} \frac{\partial}{\partial s} & =\frac{\partial F_{*} X_{i}}{\partial s}=-A_{*} X_{i} \sin (s)+B_{*} X_{i} \cos (s) \\
& =-\tan (s) X_{i}=-A_{*} X_{i} \sin (s)-\frac{\sin (s)^{2}}{\cos (s)} B_{*} X_{i} \\
& \Rightarrow B_{*} X_{i}=0 .
\end{aligned}
$$

So $A$ is the immersion of $N_{1}$ and $B$ is a curve tangent to $X_{4}$. Because $F$ lies in the unit sphere, one has

$$
\langle F, F\rangle=\cos (s)^{2}\langle A, A\rangle+\sin (s)^{2}\langle B, B\rangle+\sin (2 s)\langle A, B\rangle
$$

which implies that $A$ and $B$ have both unit length and are orthogonal. We can also calculate

$$
\begin{aligned}
D_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} & =-\cos (s) \sin (s) \frac{\partial F}{\partial s}-\sin (s)^{2} F=-\sin (s) B \\
& =\frac{\partial^{2} F}{\partial t^{2}}=\sin (s) \frac{\partial^{2} B}{\partial t^{2}} \\
& \Rightarrow B=B_{1} \cos (t)+B_{2} \sin (t)
\end{aligned}
$$

Vector fields $B_{1}$ and $B_{2}$ are constant, normalized and orthogonal. This follows from the fact that $\langle B, B\rangle=1$. Finally similar to (28), $A$ can be shown to be any special Lagrangian submanifold in $\mathbb{C} P^{2}$ lifted to the unit sphere in $\mathbb{C}^{3}$ orthogonal to $B_{1}$ and $B_{2}$ directions. Fixing $B_{1}$ and $B_{2}$ by an isometry leads to (53).
4.2. The case where $b_{2} \neq 0$. When $\mathcal{N}_{+}$is integrable, so when $a_{6}=b_{6}=0$, the equations for $\nabla$ are given by (30). We have:

Theorem 4.2. Suppose $M$ is a special Lagrangian submanifold in $\mathbb{C} P^{4}$ having a local $S O(2) \rtimes S_{3}$ symmetry. Suppose $\mathcal{N}_{+}$is integrable. Then $M$ can be lifted horizontally to a submanifold in the unit sphere of $\mathbb{C}^{5}$ through $F$ and is locally isometric to

$$
\begin{equation*}
F(t, s, u, v)=(\phi(s, u, v) \cos (t), \sin (t)) \tag{54}
\end{equation*}
$$

where $\phi$ is the horizontal lift of a special Lagrangian submanifold in $\mathbb{C} P^{3}$ with $S_{3}$-symmetry to the unit sphere in $\mathbb{C}^{4}$.

Proof. The manifold is a warped product $\mathbb{R} \times{ }_{e f} N^{3}$. Solving the Gauss equation

$$
X_{4}\left(a_{3}\right)=\frac{\partial a_{3}}{\partial t}=1+a_{3}^{2}
$$

yields $a_{3}=\tan (t)$. For $i \in\{1,2,3\}$ this implies

$$
\begin{aligned}
D_{X_{4}} X_{4} & =\frac{\partial^{2} F}{\partial t^{2}}=-F \\
& \Rightarrow F=A \cos (t)+B \sin (t) \\
D_{X_{i}} X_{4} & =-A_{*} X_{i} \sin (t)+B_{*} X_{i} \cos (t) \\
& =-\tan (t) X_{i}=-A_{*} X_{i} \sin (t)-B_{*} X_{i} \frac{\sin (t)^{2}}{\cos (t)} \\
& \Rightarrow B_{*}=0
\end{aligned}
$$

Thus $B$ is a constant vector field along the submanifold and $A$ is an immersion of a 3 -fold $N^{3}$. Using the fact that $F$ is of unit length, $A$ and $B$ are orthogonal and of unit length. Using calculations similar to (28) $A$ is a C-totally real submanifold in $S^{7}$ having local $S_{3}$-symmetry, where $S^{7}$ lies in the subspace orthogonal to $B$ and $J B$. Applying a suitable isometry results in (54).

The method to solve the case where the only integrable distribution containing $\mathcal{N}_{1}$ is the tangent bundle, has been analyzed earlier for a non-specific complex space form. We
can now fill in $\epsilon=1$ and we find for $z^{2} \neq-1$ that

$$
\begin{aligned}
a_{3} & =\tan (t), \\
a_{2} & =\frac{\sin (2 s)}{\cos (t)\left(\cos (2 s)+\cosh \left(2 k_{1}\right)\right)}, \\
b_{2} & =\frac{\sinh \left(2 k_{1}\right)}{\cos (t)\left(\cos (2 s)+\cosh \left(2 k_{1}\right)\right)}, \\
r & =\frac{e^{k_{2}}}{\cos (t) \sqrt{\cos (2 s)+\cosh \left(2 k_{1}\right)}}, \\
a_{6}+i b_{6} & =\frac{k_{3}+i k_{4}}{\rho} \sqrt{1+a_{3}^{2}}\left(1+\bar{z}^{2}\right)^{-\frac{1}{2}},
\end{aligned}
$$

where the functions $k_{i}$ depend only on $(u, v)$. Solving for (7), we obtain furthermore that

$$
\begin{aligned}
\gamma_{1}= & \frac{-\tan (t)\left(k_{3}+i k_{4}\right) \cos \left(s-i k_{1}\right)+\left(\frac{\partial k_{1}}{\partial v}-i \frac{\partial k_{1}}{\partial u}\right)}{\rho} \\
a_{1}= & \frac{2^{\frac{2}{3}} e^{\frac{2}{3} k_{2}}}{3 \cos (t)^{2}\left(\cos (2 s)+\cosh \left(2 k_{1}\right)\right)^{2}}\left(\left(\cos (2 s)+\cosh \left(2 k_{1}\right)\right)\left(\rho_{1} \frac{\partial k_{2}}{\partial v}+\rho_{2} \frac{\partial k_{2}}{\partial u}\right)\right. \\
& +\sin (2 s)\left(\rho_{1} \frac{\partial k_{1}}{\partial u}-\rho_{2} \frac{\partial k_{1}}{\partial v}\right)-\sinh \left(2 k_{1}\right)\left(\rho_{1} \frac{\partial k_{1}}{\partial v}+\rho_{2} \frac{\partial k_{1}}{\partial u}\right) \\
& \left.-\tan (t) \sinh \left(2 k_{1}\right)\left(\cos (s) \cosh \left(k_{1}\right)\left(k_{4} \rho_{2}-k_{3} \rho_{1}\right)+\sin (s) \sinh \left(k_{1}\right)\left(k_{4} \rho_{1}+k_{3} \rho_{2}\right)\right)\right), \\
b_{1}= & \frac{2^{\frac{2}{3}} e^{\frac{2}{3} k_{2}}}{3 \cos (t)^{2}\left(\cos (2 s)+\cosh \left(2 k_{1}\right)\right)^{2}}\left(\left(\cos (2 s)+\cosh \left(2 k_{1}\right)\right)\left(\rho_{2} \frac{\partial k_{2}}{\partial v}-\rho_{1} \frac{\partial k_{2}}{\partial u}\right)\right. \\
& +\sin (2 s)\left(\rho_{2} \frac{\partial k_{1}}{\partial u}+\rho_{1} \frac{\partial k_{1}}{\partial v}\right)+\sinh \left(2 k_{1}\right)\left(\rho_{1} \frac{\partial k_{1}}{\partial u}-\rho_{2} \frac{\partial k_{1}}{\partial v}\right) \\
& \left.-\sinh \left(2 k_{1}\right) \tan (t)\left(\sin (s) \sinh \left(k_{1}\right)\left(k_{4} \rho_{2}-k_{3} \rho_{1}\right)-\cos (s) \cosh \left(k_{1}\right)\left(k_{4} \rho_{1}+k_{3} \rho_{2}\right)\right)\right) .
\end{aligned}
$$

The other equations in (7) impose restrictions on $\left\{k_{1}, k_{2}, k_{3}, k_{4}\right\}$ given by

$$
\begin{aligned}
\frac{\partial k_{4}}{\partial u}-\frac{\partial k_{3}}{\partial v} & =2 \tanh \left(k_{1}\right)\left(k_{3} \frac{\partial k_{1}}{\partial v}-k_{4} \frac{\partial k_{1}}{\partial u}\right) \\
\frac{\partial k_{4}}{\partial v}+\frac{\partial k_{3}}{\partial u} & =-2 \operatorname{coth}\left(k_{1}\right)\left(k_{3} \frac{\partial k_{1}}{\partial u}+k_{4} \frac{\partial k_{1}}{\partial v}\right) \\
\Delta k_{1} & =e^{-\frac{2 k_{2}}{3}} \frac{\sinh \left(2 k_{1}\right)}{2}\left(-2^{\frac{4}{3}}+e^{\frac{2 k_{2}}{3}}\left(k_{3}^{2}+k_{4}^{2}\right)\right), \\
\Delta k_{2} & =3 * 2^{\frac{1}{3}} e^{-\frac{2 k_{2}}{3}}\left(\cosh \left(2 k_{1}\right)-e^{2 k_{2}}\right)
\end{aligned}
$$

When $a_{2}=0$ and $b_{2}= \pm \sqrt{1+a_{3}^{2}}$, we find

$$
\begin{aligned}
a_{6} & =\frac{e^{k_{1}} \cos \left(\frac{5}{3} s\right)}{\cos (t)^{2}} \\
b_{6} & = \pm \frac{e^{k_{1}} \sin \left(\frac{5}{3} s\right)}{\cos (t)^{2}} \\
r & =\frac{e^{k_{2}}}{\cos (t)}
\end{aligned}
$$

Furthermore, we obtain for $b_{2}=\sqrt{1+a_{3}^{2}}$ that

$$
\begin{aligned}
& \gamma_{1+}=\frac{-5 e^{k_{1}+\frac{5 s i}{3}} \tan (t)+e^{\frac{2 k_{1}+k_{2}}{5}+\frac{2 s i}{3}}\left(\left(\frac{\partial k_{2}}{\partial v}-3 \frac{\partial k_{1}}{\partial v}\right)-i\left(\frac{\partial k_{2}}{\partial u}-3 \frac{\partial k_{1}}{\partial u}\right)\right)}{5 \cos (t)} \\
& a_{1+}=\frac{e^{k_{1}} \cos \left(\frac{5 s}{3}\right) \tan (t)+e^{\frac{2 k_{1}+k_{2}}{5}}\left(\cos \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial v}+\sin \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial u}\right)}{3 \cos (t)} \\
& b_{1+}=\frac{e^{k_{1}} \sin \left(\frac{5 s}{3}\right) \tan (t)-e^{\frac{2 k_{1}+k_{2}}{5}}\left(\cos \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial u}-\sin \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial v}\right)}{3 \cos (t)}
\end{aligned}
$$

and for $b_{2}=-\sqrt{1+a_{3}^{2}}$ we obtain

$$
\begin{aligned}
& \gamma_{1-}=\frac{-5 e^{k_{1}-\frac{5 s i}{3}} \tan (t)+e^{\frac{2 k_{1}+k_{2}}{5}-\frac{2 s i}{3}}\left(\left(3 \frac{\partial k_{1}}{\partial v}-\frac{\partial k_{2}}{\partial v}\right)-i\left(3 \frac{\partial k_{1}}{\partial u}-\frac{\partial k_{2}}{\partial u}\right)\right)}{5 \cos (t)}, \\
& a_{1-}=\frac{-e^{k_{1}} \cos \left(\frac{5 s}{3}\right) \tan (t)+e^{\frac{2 k_{1}+k_{2}}{5}}\left(\cos \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial v}-\sin \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial u}\right)}{3 \cos (t)} \\
& b_{1-}=\frac{e^{k_{1}} \sin \left(\frac{5 s}{3}\right) \tan (t)-e^{\frac{2 k_{1}+k_{2}}{5}}\left(\cos \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial u}+\sin \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial v}\right)}{3 \cos (t)}
\end{aligned}
$$

Solving the last equations in (7) implies restrictions on the functions $k_{1}(u, v)$ and $k_{2}(u, v)$ given by

$$
\begin{align*}
& \Delta k_{1}=2 e^{-\frac{2\left(2 k_{1}+k_{2}\right)}{5}}\left(3-e^{2 k_{1}}-e^{2 k_{2}}\right) \\
& \Delta k_{2}=e^{-\frac{2\left(2 k_{1}+k_{2}\right)}{5}}\left(8-e^{2 k_{1}}-6 e^{2 k_{2}}\right) . \tag{56}
\end{align*}
$$

These equations are valid for both $b_{2}= \pm \sqrt{1+a_{3}^{2}}$. We summarize this in the following theorem.

Theorem 4.3. Each special Lagrangian submanifold of $\mathbb{C} P^{4}$ with $S O(2) \rtimes S_{3}$-symmetry where the only integral distribution containing $\mathcal{N}_{1}$ is the tangent bundle, can be constructed in the way above using either functions $\left\{k_{1}, k_{2}, k_{3}, k_{4}\right\}$ subject to (55) or functions $\left\{k_{1}, k_{2}\right\}$ subject to (56). Conversely, each such a construction results in such a submanifold, unique up to local isometry.
5. Submanifolds in $\mathbb{C} H^{4}$.
5.1. The case where $b_{2}=0$. This is the case where $\mathcal{N}_{1}$ is an integrable distribution. We assume that $a_{3}=0$. Similar to the case in $\mathbb{C} P^{4}$ we have that $M$ is a double warped product $\mathbb{R} \times_{e^{g}} \mathbb{R} \times_{e^{f}} N^{2}$. the function $a_{2}$ depends only on the coordinate $s$ and is given by

$$
\frac{\partial a_{2}}{\partial s}=a_{2}^{2}-1
$$

This equation has 3 possible solutions, depending on the initial conditions. For $a_{2}(0)=1$, it is a constant. For $a_{2}(0)>1$ it is given as $a_{2}=-\operatorname{coth}(s)$. Finally for $a_{2}(0)<1$, it is given as $a_{2}(s)=-\tanh (s)$. The connection $\nabla$ is given by

$$
\begin{array}{c|c}
\nabla_{X_{1}} X_{1}=a_{1} X_{2}+a_{2} X_{3} & \nabla_{X_{1}} X_{2}=-a_{1} X_{1} \\
\nabla_{X_{2}} X_{1}=b_{1} X_{2} & \nabla_{X_{2}} X_{2}=-b_{1} X_{1}+a_{2} X_{3} \\
\nabla_{X_{3}} X_{1}=0 & \nabla_{X_{3}} X_{2}=0 \\
\nabla_{X_{4}} X_{1}=0 & \nabla_{X_{4}} X_{2}=0 \\
\nabla_{X_{1}} X_{3}=-a_{2} X_{1} & \nabla_{X_{1}} X_{4}=0 \\
\nabla_{X_{2}} X_{3}=-a_{2} X_{2} & \nabla_{X_{2}} X_{4}=0 \\
\nabla_{X_{3}} X_{3}=0 & \nabla_{X_{3}} X_{4}=0 \\
\nabla_{X_{4}} X_{3}=-\frac{X_{4}}{a_{2}} & \nabla_{X_{4}} X_{4}=\frac{X_{3}}{a_{2}}
\end{array}
$$

We have the following result.
Theorem 5.1. Suppose $M$ is a special Lagrangian submanifold in $\mathbb{C} H^{4}$ having a local $S O(2) \rtimes S_{3}$-symmetry. Suppose $\mathcal{N}_{1}$ is integrable. Then $M$ can be lifted horizontally to a submanifold in $H^{9}$ through $F$ and is locally isometric to either

$$
\begin{equation*}
F(t, s, u, v)=(\sin (t) \sinh (s), \cos (t) \sinh (s), \phi(u, v) \cosh (s)) \tag{57}
\end{equation*}
$$

where $\phi$ is the horizontal lift of a special Lagrangian submanifold of $\mathbb{C} H^{2}$ to $H^{5}$ in case $a_{2}^{2}<1$, or

$$
\begin{equation*}
F(t, s, u, v)=(\phi(u, v) \sinh (s), \cos (t) \cosh (s), \sin (t) \cosh (s)) \tag{58}
\end{equation*}
$$

where $\phi$ is the horizontal lift of a special Lagrangian submanifold of $\mathbb{C} P^{2}$ to $S^{5}$ in case $a_{2}^{2}>1$, or

$$
\begin{equation*}
F(t, s, u, v)=\left((\phi(u, v), t) e^{-s},-\frac{1}{2} e^{-s},\left(\|(\phi(u, v), t)\|^{2}+i f(u, v)\right) e^{-s}+e^{s}\right) \tag{59}
\end{equation*}
$$

where $\phi$ is a special Lagrangian surface in $\mathbb{C}^{2}$ and $f$ is the integral of the differential form

$$
2 \sum_{i=1}^{2}\left(x^{i} \mathrm{~d} y^{i}-y^{i} \mathrm{~d} x^{i}\right)
$$

on $\mathbb{C}^{2}$ in case $a_{2}^{2}=1$.
Proof. We can check similar to the case in $\mathbb{C} P^{4}$ that $M=\mathbb{R} \times{ }_{e^{g}} \mathbb{R} \times{ }_{e} \times N^{2}$, where $f$ and $g$ are functions on the first factor, determined by $X_{3}(g)=-\frac{1}{a_{2}}$ and $X_{3}(f)=-a_{2}$. We can treat the cases separately for each solution to $a_{2}(s)$.

Assume $a_{2}=-\tanh (s)$, then it is easy to see that $\frac{\partial}{\partial t}=-\sinh (s) X_{4}$ commutes with $\frac{\partial}{\partial s}$. The Gauss identity now implies for $i \in\{1,2\}$ that

$$
\begin{aligned}
D_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} & =\frac{\partial^{2} F}{\partial s^{2}}=F \\
& \Rightarrow F=A \sinh (s)+B \cosh (s) \\
D_{\frac{\partial}{\partial T}} \frac{\partial}{\partial s} & =\frac{\partial^{2} F}{\partial t \partial s}=\frac{\partial A}{\partial t} \cosh (s)+\frac{\partial B}{\partial t} \sinh (s) \\
& =\operatorname{coth}(s) \frac{\partial F}{\partial t}=\frac{\partial A}{\partial t} \cosh (s)+\frac{\partial B}{\partial t} \frac{\cosh (s)^{2}}{\sinh (s)} \\
& \Rightarrow \frac{\partial B}{\partial t}=0 \\
D_{X_{i}} \frac{\partial}{\partial s} & =\frac{\partial F_{*} X_{i}}{\partial s}=A_{*} X_{i} \cosh (s)+B_{*} X_{i} \sinh (s) \\
& =\tanh (s) X_{i}=A_{*} X_{i} \frac{\sinh (s)^{2}}{\cosh (s)}+B_{*} X_{i} \sinh (s) \\
& \Rightarrow A_{*} X_{i}=0
\end{aligned}
$$

Using the fact that $\langle F, F\rangle_{1}=-1$, we get that $\langle B, B\rangle_{1}=-\langle A, A\rangle_{1}=-1$ and $\langle A, B\rangle_{1}=0$. Furthermore, we find

$$
\begin{aligned}
D_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} & =\frac{\partial^{2} F}{\partial t^{2}}=\frac{\partial^{2} A}{\partial t^{2}} \sinh (s) \\
& =-\cosh (s) \sinh (s) \frac{\partial F}{\partial s}+\sinh (s)^{2} F=-A \sinh (s) \\
& \Rightarrow A=A_{1} \cos (t)+A_{2} \sin (t)
\end{aligned}
$$

Because $A$ has unit length, so do $A_{1}$ and $A_{2}$ and they are both orthogonal. Calculations similar to (28) show that $B$ can be taken as the horizontal lift of any special Lagrangian submanifold in $\mathbb{C} H^{2}$ and applying a suitable isometry gives (57).
For $a_{2}=-\operatorname{coth}(s)$ calculations similar to the previous case result in (58).
Finally we assume $a_{2}=1$. Then the vector field given by $\frac{\partial}{\partial t}=e^{-s} X_{4}$ commutes with $\frac{\partial}{\partial s}$. We can calculate for $i \in\{1,2\}$ that

$$
\begin{aligned}
D_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} & =\frac{\partial^{2} F}{\partial s^{2}}=F \Rightarrow F=A e^{s}+B e^{-s} \\
D_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s} & =\frac{\partial^{2} F}{\partial t \partial s}=\frac{\partial A}{\partial t} e^{s}-\frac{\partial B}{\partial t} e^{-s} \\
& =-\frac{\partial F}{\partial t}=-\frac{\partial A}{\partial t} e^{s}-\frac{\partial B}{\partial t} e^{-s} \\
& \Rightarrow \frac{\partial A}{\partial t}=0 \\
D_{X_{i}} \frac{\partial}{\partial s} & =\frac{\partial F_{*} X_{i}}{\partial s}=A_{*} X_{i} e^{s}-B_{*} X_{i} e^{-s} \\
& =-F_{*} X_{i}=-\left(A_{*} X_{i} e^{s}+B_{*} X_{i} e^{-s}\right) \\
& \Rightarrow A_{*}=0
\end{aligned}
$$

Using the fact that $\langle F, F\rangle_{1}=-1$, we obtain that $A$ and $B$ are vector fields with 0 length and they satisfy $\langle A, B\rangle_{1}=-\frac{1}{2}$. Further calculations show

$$
\begin{aligned}
D_{\frac{\partial}{\partial t}} \frac{\partial}{\partial t} & =\frac{\partial^{2} F}{\partial t^{2}}=e^{-s} \frac{\partial^{2} B}{\partial t^{2}} \\
& =e^{-2 s}\left(\frac{\partial F}{\partial s}+F\right)=2 A e^{-s} \\
& \Rightarrow B=A t^{2}+B_{1} t+B_{2}, \\
D_{X_{i}} \frac{\partial}{\partial t} & =\frac{\partial F_{*} X_{i}}{\partial t}=B_{1 *} X_{i} e^{-s}=0 \\
& \Rightarrow B_{1 *}=0 .
\end{aligned}
$$

We can conclude that $F$ has the form

$$
F=\left(A t^{2}+B_{1} t+\phi\right) e^{-s}+A e^{s}
$$

Here, $\phi$ is an immersion of a 2-fold in $\mathbb{C}_{1}^{5}$ tangent to $\mathcal{N}_{1}$. Calculating the scalar products of $B$ and $A$, we get

$$
\begin{align*}
\langle A, B\rangle_{1} & =t\left\langle A, B_{1}\right\rangle_{1}+\langle A, \phi\rangle_{1}=-\frac{1}{2} \\
& \Rightarrow\left\langle A, B_{1}\right\rangle_{1}=0 \bigwedge\langle A, \phi\rangle_{1}=-\frac{1}{2}  \tag{60}\\
\langle B, B\rangle_{1} & =t^{2}\left(\left\langle B_{1}, B_{1}\right\rangle_{1}-1\right)+2 t\left\langle B_{1}, \phi\right\rangle_{1}+\langle\phi, \phi\rangle_{1}=0 \\
& \Rightarrow\left\langle B_{1}, B_{1}\right\rangle_{1}=1 \bigwedge\left\langle B_{1}, \phi\right\rangle_{1}=0 \bigwedge\langle\phi, \phi\rangle_{1}=0 .
\end{align*}
$$

We can shift to a different standard basis of $\mathbb{C}_{1}^{5}$ such that

$$
\langle\vec{z}, \vec{w}\rangle=\Re\left(\sum_{j=1}^{3} z_{j} \bar{w}_{j}+z_{4} \bar{w}_{5}+z_{5} \bar{w}_{4}\right) .
$$

In this case the constant light-like vector $A$ and time-like $B_{1}$, after applying a suitable isometry, can be chosen to be

$$
\begin{aligned}
A & =(0,0,0,0,1) \\
B_{1} & =(0,0,1,0,0)
\end{aligned}
$$

We can write $\phi=\left(\phi_{1}, \phi_{2}, \phi_{3}, \phi_{4}, \phi_{5}\right)$ where $\phi_{j}=x_{j}+i y_{j}$. Then (60) implies

$$
\begin{aligned}
x_{4} & =-\frac{1}{2}, \\
x_{3} & =0 \\
x_{5}-2 y_{4} y_{5} & =\sum_{j=1}^{3}\left|\phi_{j}\right|^{2} .
\end{aligned}
$$

We can use the fact that both $F$ and $i F$ are orthogonal to the tangent space in $\mathbb{C}_{1}^{5}$ and this results in

$$
\begin{aligned}
\phi_{4} & =-\frac{1}{2} \\
\phi_{3} & =0 \\
\mathrm{~d} y_{5} & =2 \sum_{i=1}^{2}\left(x_{i} \mathrm{~d} y_{i}-y_{i} \mathrm{~d} x_{i}\right)
\end{aligned}
$$

This last equation is integrable if and only its derivative equals 0 along the submanifold. But this derivative is nothing more than a multiple of the Kähler form on $\mathbb{C}^{2}$ spanned by the first 2 complex coordinates. In other words, for such a submanifold to exist, the projection of $\phi$ onto the first 2 coordinates should be a Lagrangian submanifold in $\mathbb{C}^{2}$. Calculating the Gauss identity on $D_{X_{i}} X_{j}$ we find that the metric on this immersion is given by

$$
\left\langle\phi_{*} X_{i}, \phi_{*} X_{j}\right\rangle=e^{2 s} \delta_{i j}
$$

where $\langle a, b\rangle$ is the standard scalar product on $\mathbb{C}^{2}$ and $\phi$ here is the restriction to the first 2 complex coordinates. Because $\left\langle F_{*} X_{i}, F_{*} X_{j}\right\rangle=\delta_{i j}$ and because $\phi_{3 *}=0$ and $\phi_{4 *}=0$ this condition is included in the warped product structure. Using calculations like (28) we conclude that $\left(\phi_{1}, \phi_{2}\right)$ can be any special Lagrangian 2 -fold in $\mathbb{C}^{2}$. The result is summarized in (59).
5.2. The case where $b_{2} \neq 0$. First we assume that $\mathcal{N}_{+}$is an integrable distribution. This is equivalent to $a_{6}+i b_{6}=0$. The connection is given by (30), resulting in a warped product structure $\mathbb{R} \times{ }_{e f} N^{3}$. The equation

$$
X_{4}\left(a_{3}\right)=\frac{\partial a_{3}}{\partial t}=a_{3}^{2}-1
$$

has a solution given as $\left|a_{3}\right|=1, a_{3}=-\tanh (t)$ or $a_{3}=-\operatorname{coth}(t)$, depending on the initial value of $a_{3}$. Using an analysis similar to the case of $\mathbb{C} P^{4}$ and the case above gives the following result.

Theorem 5.2. Suppose $M$ is a special Lagrangian submanifold in $\mathbb{C} H^{4}$ having a local $S O(2) \rtimes S_{3}$-symmetry. Suppose $\mathcal{N}_{1}$ is non-integrable, but is contained in the integrable $\mathcal{N}_{+}$distribution. Then $M$ can be lifted horizontally to a submanifold in $H^{9}$ through $F$ and is locally isometric to either

$$
\begin{equation*}
F(t, s, u, v)=(\sinh (t), \phi(s, u, v) \cosh (t)) \tag{61}
\end{equation*}
$$

where $\phi$ is the horizontal lift of a special Lagrangian submanifold of $\mathbb{C} H^{3}$ with a local $S_{3}$-symmetry to $H^{7}$ in case $a_{3}^{2}<1$, or

$$
\begin{equation*}
F(t, s, u, v)=(\phi(s, u, v) \sinh (t), \cosh (t)) \tag{62}
\end{equation*}
$$

where $\phi$ is the horizontal lift of a special Lagrangian submanifold of $\mathbb{C} P^{3}$ with a local $S_{3}$-symmetry to $S^{7}$ in case $a_{3}^{2}>1$, or

$$
\begin{equation*}
F(t, s, u, v)=\left(\phi(s, u, v) e^{-t},-e^{-t} / 2,\left(\|\phi(s, u, v)\|^{2}+i f(s, u, v)\right) e^{-t}+e^{t}\right) \tag{63}
\end{equation*}
$$

where $\phi$ is a special Lagrangian submanifold with a local $S_{3}$-symmetry in $\mathbb{C}^{3}$ and $f$ is the integral of the differential form

$$
2 \sum_{i=1}^{3}\left(x^{i} \mathrm{~d} y^{i}-y^{i} \mathrm{~d} x^{i}\right)
$$

Finally we assume that there is no integrable distribution that contains $\mathcal{N}_{1}$ except for the tangent bundle. We return to the analysis as done for $\mathbb{C}^{4}$, but set $\epsilon=-1$. The result will depend on the initial value of $a_{3}$. First assume that $a_{3}^{2}<1$, then $\tilde{\epsilon}=-1$. We find functions $\left\{k_{1}, k_{2}, k_{3}, k_{4}\right\}$ of $(u, v)$ such that

$$
\begin{aligned}
a_{3} & =-\tanh (t), \\
a_{2} & =-\frac{\sinh (2 s)}{\cosh (t)\left(\cosh (2 s)+\cos \left(2 k_{1}\right)\right)}, \\
b_{2} & =-\frac{\sin \left(2 k_{1}\right)}{\cosh (t)\left(\cosh (2 s)+\cos \left(2 k_{1}\right)\right)}, \\
r & =\frac{e^{k_{2}}}{\cosh (t) \sqrt{\cosh (2 s)+\cos \left(2 k_{1}\right)}}, \\
a_{6}+i b_{6} & =\frac{k_{3}+i k_{4}}{\rho} \sqrt{1-a_{3}^{2}}\left(1-\bar{z}^{2}\right)^{-\frac{1}{2}}
\end{aligned}
$$

Using (7) as earlier, we obtain $a_{1}, b_{1}, \gamma_{1}$ as

$$
\begin{aligned}
\gamma_{1}= & \frac{-\tanh (t)\left(k_{3}+i k_{4}\right) \cosh \left(s-i k_{1}\right)+\left(\frac{\partial k_{1}}{\partial v}-i \frac{\partial k_{1}}{\partial u}\right)}{\rho}, \\
a_{1}= & \frac{2^{\frac{2}{3}} e^{\frac{2}{3} k_{2}}}{3 \cosh (t)^{2}\left(\cosh (2 s)+\cos \left(2 k_{1}\right)\right)^{2}}\left(\left(\cosh (2 s)+\cos \left(2 k_{1}\right)\right)\left(\rho_{1} \frac{\partial k_{2}}{\partial v}+\rho_{2} \frac{\partial k_{2}}{\partial u}\right)\right. \\
& +\sinh (2 s)\left(\rho_{2} \frac{\partial k_{1}}{\partial v}-\rho_{1} \frac{\partial k_{1}}{\partial u}\right)+\sin \left(2 k_{1}\right)\left(\rho_{1} \frac{\partial k_{1}}{\partial v}+\rho_{2} \frac{\partial k_{1}}{\partial u}\right) \\
& \left.+\sin \left(2 k_{1}\right) \tanh (t)\left(\cosh (s) \cos \left(k_{1}\right)\left(k_{4} \rho_{2}-k_{3} \rho_{1}\right)-\sinh (s) \sin \left(k_{1}\right)\left(k_{4} \rho_{1}+k_{3} \rho_{2}\right)\right)\right), \\
b_{1}= & \frac{2^{\frac{2}{3}} e^{\frac{2}{3} k_{2}}}{3 \cosh (t)^{2}\left(\cosh (2 s)+\cos \left(2 k_{1}\right)\right)^{2}}\left(\left(\cosh (2 s)+\cos \left(2 k_{1}\right)\right)\left(\rho_{2} \frac{\partial k_{2}}{\partial v}-\rho_{1} \frac{\partial k_{2}}{\partial u}\right)\right. \\
& \left.-\sinh (2 s)\left(\rho_{2} \frac{\partial k_{1}}{\partial u}+\rho_{1} \frac{\partial k_{1}}{\partial v}\right)-\sin \left(2 k_{1}\right)\left(\rho_{1} \frac{\partial k_{1}}{\partial u}-\rho_{2} \frac{\partial k_{1}}{\partial v}\right)\right) \\
& \left.-\sin \left(2 k_{1}\right) \tanh (t)\left(\sinh (s) \sin \left(k_{1}\right)\left(k_{4} \rho_{2}-k_{3} \rho_{1}\right)+\cosh (s) \cos \left(k_{1}\right)\left(k_{4} \rho_{1}+k_{3} \rho_{2}\right)\right)\right) .
\end{aligned}
$$

The other equations in (7) put restrictions on $\left\{k_{1}, k_{2}, k_{3}, k_{4}\right\}$ given by

$$
\begin{align*}
\frac{\partial k_{4}}{\partial u}-\frac{\partial k_{3}}{\partial v} & =2 \tan \left(k_{1}\right)\left(k_{4} \frac{\partial k_{1}}{\partial u}-k_{3} \frac{\partial k_{1}}{\partial v}\right) \\
\frac{\partial k_{4}}{\partial v}+\frac{\partial k_{3}}{\partial u} & =-2 \cot \left(k_{1}\right)\left(k_{3} \frac{\partial k_{1}}{\partial u}+k_{4} \frac{\partial k_{1}}{\partial v}\right)  \tag{64}\\
\Delta k_{1} & =\frac{\sin \left(2 k_{1}\right)}{2}\left(2^{\frac{4}{3}} e^{-\frac{2}{3} k_{2}}+k_{3}^{2}+k_{4}^{2}\right) \\
\Delta k_{2} & =-3 * 2^{\frac{1}{3}} e^{-\frac{2}{3} k_{2}}\left(e^{2 k_{2}}+\cos \left(2 k_{1}\right)\right) .
\end{align*}
$$

Then we set $a_{3}^{2}>1$ and assume $z^{2} \neq-1$. We then find

$$
\begin{aligned}
a_{3} & =-\operatorname{coth}(t), \\
a_{2} & =\frac{\sin (2 s)}{\sinh (t)\left(\cos (2 s)+\cosh \left(2 k_{1}\right)\right)}, \\
b_{2} & =\frac{\sinh \left(2 k_{1}\right)}{\sinh (t)\left(\cos (2 s)+\cosh \left(2 k_{1}\right)\right)}, \\
r & =\frac{e^{k_{2}}}{\sinh (t) \sqrt{\cos (2 s)+\cosh \left(2 k_{1}\right)}}, \\
a_{6}+i b_{6} & =\frac{k_{3}+i k_{4}}{\rho} \sqrt{a_{3}^{2}-1}\left(1+\bar{z}^{2}\right)^{-\frac{1}{2}}
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\gamma_{1}= & \frac{\operatorname{coth}(t)\left(k_{3}+i k_{4}\right) \cos \left(s-i k_{1}\right)+\left(\frac{\partial k_{1}}{\partial v}-i \frac{\partial k_{1}}{\partial u}\right)}{\rho} \\
a_{1}= & \frac{2^{\frac{2}{3}} e^{\frac{2}{3} k_{2}}}{3 \sinh (t)^{2}\left(\cos (2 s)+\cosh \left(2 k_{1}\right)\right)^{2}}\left(\left(\cos (2 s)+\cosh \left(2 k_{1}\right)\right)\left(\rho_{1} \frac{\partial k_{2}}{\partial v}+\rho_{2} \frac{\partial k_{2}}{\partial u}\right)\right. \\
& +\sin (2 s)\left(\rho_{1} \frac{\partial k_{1}}{\partial u}-\rho_{2} \frac{\partial k_{1}}{\partial v}\right)-\sinh \left(2 k_{1}\right)\left(\rho_{1} \frac{\partial k_{1}}{\partial v}+\rho_{2} \frac{\partial k_{1}}{\partial u}\right) \\
& \left.+\sinh \left(2 k_{1}\right) \operatorname{coth}(t)\left(\cos (s) \cosh \left(k_{1}\right)\left(k_{4} \rho_{2}-k_{3} \rho_{1}\right)+\sin (s) \sinh \left(k_{1}\right)\left(k_{4} \rho_{1}+k_{3} \rho_{2}\right)\right)\right), \\
b_{1}= & \frac{2^{\frac{2}{3}} e^{\frac{2}{3} k_{2}}}{3 \sinh (t)^{2}\left(\cosh (2 s)+\cos \left(2 k_{1}\right)\right)^{2}}\left(\left(\cosh (2 s)+\cos \left(2 k_{1}\right)\right)\left(\rho_{2} \frac{\partial k_{2}}{\partial v}-\rho_{1} \frac{\partial k_{2}}{\partial u}\right)\right. \\
& \left.+\sin (2 s)\left(\rho_{2} \frac{\partial k_{1}}{\partial u}+\rho_{1} \frac{\partial k_{1}}{\partial v}\right)+\sinh \left(2 k_{1}\right)\left(\rho_{1} \frac{\partial k_{1}}{\partial u}-\rho_{2} \frac{\partial k_{1}}{\partial v}\right)\right) \\
& \left.\quad-\sinh \left(2 k_{1}\right) \operatorname{coth}(t)\left(\sin (s) \sinh \left(k_{1}\right)\left(k_{3} \rho_{1}-k_{4} \rho_{2}\right)+\cos (s) \cosh \left(k_{1}\right)\left(k_{4} \rho_{1}+k_{3} \rho_{2}\right)\right)\right) .
\end{aligned}
$$

The functions $\left\{k_{1}, k_{2}, k_{3}, k_{4}\right\}$ have to satisfy

$$
\begin{align*}
\frac{\partial k_{4}}{\partial u}-\frac{\partial k_{3}}{\partial v} & =2 \tanh \left(k_{1}\right)\left(k_{3} \frac{\partial k_{1}}{\partial v}-k_{4} \frac{\partial k_{1}}{\partial u}\right) \\
\frac{\partial k_{4}}{\partial v}+\frac{\partial k_{3}}{\partial u} & =-2 \operatorname{coth}\left(k_{1}\right)\left(k_{3} \frac{\partial k_{1}}{\partial u}+k_{4} \frac{\partial k_{1}}{\partial v}\right)  \tag{65}\\
\Delta k_{1} & =-\sinh \left(2 k_{1}\right)\left(2^{\frac{1}{3}} e^{-\frac{2}{3} k_{2}}+\frac{k_{3}^{2}+k_{4}^{2}}{2}\right) \\
\Delta k_{2} & =3 * 2^{\frac{1}{3}} e^{-\frac{2}{3} k_{2}}\left(\cosh \left(2 k_{1}\right)-e^{2 k_{2}}\right)
\end{align*}
$$

Finally, assume $a_{3}^{2}>1$ and $z^{2}=-1$. Then we find

$$
\begin{aligned}
a_{3} & =-\operatorname{coth}(t) \\
a_{6} & =\frac{e^{k_{1}} \cos \left(\frac{5}{3} s\right)}{\sinh (t)^{2}} \\
b_{6} & = \pm \frac{e^{k_{1}} \sin \left(\frac{5}{3} s\right)}{\sinh (t)^{2}} \\
r & =\frac{e^{k_{2}}}{\sinh (t)}
\end{aligned}
$$

We obtain for $b_{2}=\sqrt{a_{3}^{2}-1}$ that

$$
\begin{aligned}
& \gamma_{1+}=\frac{5 e^{k_{1}+\frac{5 s i}{3}} \operatorname{coth}(t)+e^{\frac{2 k_{1}+k_{2}}{5}+\frac{2 s i}{3}}\left(\left(\frac{\partial k_{2}}{\partial v}-3 \frac{\partial k_{1}}{\partial v}\right)-i\left(\frac{\partial k_{2}}{\partial u}-3 \frac{\partial k_{1}}{\partial u}\right)\right)}{5 \sinh (t)} \\
& a_{1+}=\frac{-e^{k_{1}} \cos \left(\frac{5 s}{3}\right) \operatorname{coth}(t)+e^{\frac{2 k_{1}+k_{2}}{5}}\left(\cos \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial v}+\sin \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial u}\right)}{3 \sinh (t)} \\
& b_{1+}=\frac{-e^{k_{1}} \sin \left(\frac{5 s}{3}\right) \operatorname{coth}(t)-e^{\frac{2 k_{1}+k_{2}}{5}}\left(\cos \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial u}-\sin \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial v}\right)}{3 \sinh (t)}
\end{aligned}
$$

and for $b_{2}=-\sqrt{a_{3}^{2}-1}$ we obtain

$$
\begin{aligned}
& \gamma_{1-}=\frac{5 e^{k_{1}-\frac{5 s i}{3}} \operatorname{coth}(t)+e^{\frac{2 k_{1}+k_{2}}{5}-\frac{2 s i}{3}}\left(\left(3 \frac{\partial k_{1}}{\partial v}-\frac{\partial k_{2}}{\partial v}\right)-i\left(3 \frac{\partial k_{1}}{\partial u}-\frac{\partial k_{2}}{\partial u}\right)\right)}{5 \sinh (t)} \\
& a_{1-}=\frac{e^{k_{1}} \cos \left(\frac{5 s}{3}\right) \operatorname{coth}(t)+e^{\frac{2 k_{1}+k_{2}}{5}}\left(\cos \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial v}-\sin \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial u}\right)}{3 \sinh (t)} \\
& b_{1-}=\frac{-e^{k_{1}} \sin \left(\frac{5 s}{3}\right) \operatorname{coth}(t)-e^{\frac{2 k_{1}+k_{2}}{5}}\left(\cos \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial u}+\sin \left(\frac{2 s}{3}\right) \frac{\partial k_{2}}{\partial v}\right)}{3 \sinh (t)} .
\end{aligned}
$$

Solving the other Gauss equations results in the relations

$$
\begin{align*}
& \Delta k_{1}=e^{-\frac{2}{5}\left(2 k_{1}+k_{2}\right)}\left(6+2 e^{2 k_{1}}-2 e^{2 k_{2}}\right), \\
& \Delta k_{2}=e^{-\frac{2}{5}\left(2 k_{1}+k_{2}\right)}\left(8+e^{2 k_{1}}-6 e^{2 k_{2}}\right) \tag{66}
\end{align*}
$$

These equations are valid for both $b_{2}= \pm \sqrt{a_{3}^{2}-1}$. We can conclude with the following proposition.
Theorem 5.3. Each special Lagrangian submanifold of $\mathbb{C} H^{4}$ with $S O(2) \rtimes S_{3}$-symmetry where the only integral distribution containing $\mathcal{N}_{1}$ is the whole tangent bundle, can be constructed in the way above using functions $\left\{k_{1}, k_{2}, k_{3}, k_{4}\right\}$ subject to (64) in case $a_{3}^{2}(0)<$ 1 , subject to (65) in case $a_{3}^{2}(0)>1$, or functions $\left\{k_{1}, k_{2}\right\}$ subject to (66) when $a_{3}^{2}(0)>1$ and $z^{2}=-1$. Conversely, each such a construction results in such a submanifold, unique up to local isometry.

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