FLAT ALMOST COMPLEX SURFACES IN $S^3 \times S^3$

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Abstract. We prove a Bonnet-type existence and uniqueness theorem for almost complex surfaces in the nearly Kähler manifold $S^3 \times S^3$. The proof uses a local correspondence between almost complex surfaces in $S^3 \times S^3$ and surfaces in $\mathbb{R}^3$ that satisfy the Wente H-equation. Furthermore we give a complete classification of flat almost complex surfaces.

1. Introduction

Nearly Kähler manifolds are almost Hermitian manifolds, with almost complex structure J, for which the tensor field $G = \nabla J$ is skew-symmetric. A nearly Kähler manifold is called strict nearly Kähler if furthermore $\nabla_X J \neq 0$ for every non-zero vector X. In particular, the almost complex structure of a strict nearly Kähler manifold is non-integrable and its fundamental 2-form is non-closed, in contrast with Kähler manifolds. They arise naturally as one to the sixteen Gray-Hervella classes of of almost Hermitian manifolds [?]. Probably the most known and simplest example of a nearly Kähler manifold is the 6-dimensional sphere, whose almost complex structure J can be defined in terms of the vector cross product on $\mathbb{R}^7$. In recent work of Butruille [?] it was proven that the only homogeneous 6-dimensional nearly Kähler manifolds are S^6 , $S^3 \times S^3$, $\mathbb{C}P^3$ and the flag manifold $SU(3)/U(1) \times U(1)$. These spaces are compact 3-symmetric spaces and they are the only known compact nearly Kähler manifolds in dimension 6.

A natural class of submanifolds of nearly Kähler manifolds are the almost complex submanifolds, also known as pseudoholomorphic curves. Almost complex submanifolds are submanifolds for which J sends tangent vectors on tangent one. Podestà and Spiro [?] proved that strict nearly Kähler manifolds of dimension 6 do not admit any 4-dimensional almost complex submanifolds. This fact was already proven by Gray for the nearly Kähler S^6 ([?]). Hence the only almost complex submanifolds are surfaces. Almost complex surfaces in the nearly Kähler manifold S^6 have been extensively studied by many authors (see e.g. [?], [?], [?], [?], [?], [?]) and more recently Xu [?] studied these surfaces in the nearly Kähler $\mathbb{C}P^3$.

Almost complex surfaces in $S^3 \times S^3$ were first studied in [?]. In this study an almost product structure P plays an important role. If M is an almost complex surface in $S^3 \times S^3$ with metric g and complex coordinate z on M, then $\Lambda\, dz^2 = g(P\partial_z, \partial_z)\, dz^2$ defines a holomorphic quadratic differential on the surface. There is a local correspondence between almost complex surfaces in $S^3 \times S^3$ and surfaces X in $\mathbb{R}^3$ satisfying the Wente H-equation ([?])

$$X_{uu} + X_{vv} = -\frac{4}{\sqrt{3}} X_u \times X_v.$$

The holomorphic differential $\Lambda\, dz^2$ on M vanishes if and only if the coordinate $z = u + iv$ is conformal on the H-surface X, that is, $|X_u| = |X_v|$ and $\langle X_u, X_v \rangle = 0$. (Note that z is a conformal coordinate for the almost complex surface but it is not necessarily conformal for the associated H-surface.) In particular if $\Lambda\, dz^2$ vanishes on the almost complex surface, the

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corresponding Wente H-surface is a surface of constant mean curvature (Theorem 3.10 in [?]). Using this fact and a well-known theorem of H. Hopf on constant mean curvature surfaces in $\mathbb{R}^3$ it was then proven that almost complex 2-spheres in $S^3 \times S^3$ are totally geodesic.

Now the main results of the present paper can be stated. Firstly, we have the following Bonnet-type existence and uniqueness theorems for almost complex surfaces in $S^3 \times S^3$.

Theorem 1.1. *Let $U \subset \mathbb{R}^2$ be a simply connected open set and z a complex coordinate on U. Suppose we have a metric g and two functions $\omega\colon U \to \mathbb{R}$ and $\mu\colon U \to \mathbb{C}$ such that $g(\partial_z, \partial_{\bar z}) = e^\omega$ and $\omega > 0$. Moreover suppose that ω and μ satisfy*

$$\omega_{z\bar z} = \frac{e^{-\omega}}{2\sinh\omega}|\omega_z|^2 + \frac{4}{3}\sinh\omega(1-|\mu|^2), \tag{1}$$

$$\mu_{\bar z} = \frac{\omega_z\bar\mu - e^\omega\mu\omega_{\bar z}}{2\sinh\omega}. \tag{2}$$

Then there is an almost complex immersion $\phi\colon U \to S^3 \times S^3$ with $\Lambda = 1$ and with an adapted frame

$$\partial_z, \partial_{\bar z}, N, \bar N, G(\partial_z, P\partial_{\bar z}), G(\partial_{\bar z}, P\partial_z)$$

such that its second fundamental form $h(\partial_z, \partial_z)$ is

$$h(\partial_z, \partial_z) = \frac{-\omega_z}{e^\omega - e^{-\omega}}\bar N + \mu G(\partial_{\bar z}, P\partial_z).$$

Moreover, two such almost complex immersions are the same up to an isometry of $S^3 \times S^3$.

Theorem 1.2. *Let $U \subset \mathbb{R}^2$ be a simply connected open set and z a complex coordinate on U. Suppose we have a metric g and two functions $\omega\colon U \to \mathbb{R}$ and $\mu\colon U \to \mathbb{C}$ such that $g(\partial_z, \partial_{\bar z}) = e^\omega$. Moreover suppose that ω and μ satisfy the equations*

$$\omega_{z\bar z} = \frac{2}{3}e^\omega(|\mu|^2 - 1),$$
$$\mu_{\bar z} = -\omega_{\bar z}\mu.$$

Then there is an almost complex immersion $\phi\colon U \to S^3 \times S^3$ with vanishing differential $\Lambda\, dz^2$ and an adapted frame $\partial_z, \partial_{\bar z}, P\partial_z, P\partial_{\bar z}, G(\partial_z, P\partial_{\bar z}), G(\partial_{\bar z}, P\partial_z)$ such that the second fundamental form is

$$h(\partial_z, \partial_z) = \mu G(\partial_{\bar z}, P\partial_z).$$

Moreover, two such almost complex immersions are the same up to an isometry of $S^3 \times S^3$.

It is interesting to note that the equations in Theorem 1.2 can be rewritten, by choosing a suitable coordinate z, as the sinh-Gordon equation $\omega_{z\bar z} + \frac{4}{3}\sinh\omega = 0$. This is in accordance with the correspondence between almost complex surfaces with vanishing holomorphic differential $\Lambda\, dz^2$ and constant mean curvature surfaces in $\mathbb{R}^3$ (Theorem 3.10 in [?]).

Secondly we obtain a complete classification of flat almost complex surfaces in $S^3 \times S^3$, which consists of a 2-parameter family of flat almost complex surfaces. Precisely, we prove

Theorem 1.3. *Any flat almost complex surface in $S^3 \times S^3$ must be an open part of the following 2-parameter family of homogenous tori $f(u, v) = (p(u, v), q(u, v))$ with*

$$p(u,v) = \left(\sqrt{\frac{1+a}{2}}e^{i(c_1u+c_2v)}, \sqrt{\frac{1-a}{2}}e^{i(c_3u+c_4v)}\right),$$

and

$$q(u,v) = \left(\sqrt{\frac{1+b}{2}}e^{i(b_1u+b_2v)}, \sqrt{\frac{1-a}{2}}e^{i(b_3u+b_4v)}\right) * i,$$

where $*$ denotes the quaternion multiplication, $(a, b) \in]-1, 1[\times]-1, 1[$ and $c_1, c_2, c_3, c_4, b_1, b_2, b_3, b_4$ are constants determined by a, b (see (44) - (51)).

We conclude the introduction with an outline of the article. In Section 2 we set notations and recall the basic properties of the nearly Kähler structure on $S^3 \times S^3$. We discuss the nearly Kähler structure, the almost product structure P and their relation to $S^3 \times S^3$ equipped with the canonical product metric structure. In Section 3 we provide some properties on almost complex surfaces and recall the the correspondence between almost complex surfaces in $S^3 \times S^3$ and solutions of the Wente H-equation. In Section 4 we introduce the adapted frame on almost complex surfaces in $S^3 \times S^3$ and discuss its existence. In the final two sections we prove our results stated above.

2. The nearly Kähler $S^3 \times S^3$

The 4-dimensional Euclidean space $\mathbb{R}^4$ can naturally be identified to the set of quaternions. The 3-sphere S^3 then is the set of all unit quaternions. The vector fields given by $X_1(p) = pi$, $X_2(p) = pj$ and $X_3(p) = pk$ at $p \in \mathbb{S}^3$, where i, j, k are the standard basis of unit quaternions, form a basis of tangent vector fields. Therefore any tangent vector in T_pS^3 can be expressed as $p\alpha$ where α is an imaginary quaternion. In particular, if $X \in T_pS^3$, then $qp^{-1}X$ is a vector in T_qS^3.

We identify $T_{(p,q)}(S^3 \times S^3)$ with $T_pS^3 \oplus T_qS^3$ to write a tangent vector at (p, q) as $Z(p, q) = \big(U(p, q), V(p, q)\big)$ or just $Z = (U, V)$. The almost complex structure J on $S^3 \times S^3$ is given by

$$JZ_{(p,q)} = \frac{1}{\sqrt{3}}\left(2pq^{-1}V - U, -2qp^{-1}U + V\right)$$

for a vector Z in $T_{(p,q)}(S^3 \times S^3)$ (see [?]); it is straightforward to check that $J^2 = -\mathrm{Id}$. The nearly Kähler metric g on $S^3 \times S^3$ is then defined as the Hermitian metric associated to the standard Euclidean product metric:

$$\begin{aligned} g(Z, Z') &= \frac{1}{2}\left(\langle Z, Z'\rangle + \langle JZ, JZ'\rangle\right) \\ &= \frac{4}{3}\left(\langle U, U'\rangle + \langle V, V'\rangle\right) - \frac{2}{3}\left(\langle p^{-1}U, q^{-1}V'\rangle + \langle p^{-1}U', q^{-1}V\rangle\right). \end{aligned}$$

In the first equation $\langle\cdot,\cdot\rangle$ stands for the product metric on $S^3 \times S^3$ and in the second one it stands for the Euclidean metric on S^3. By definition J is compatible with g: $g(JZ, JZ') = g(Z, Z')$. The metric g only differs up to a constant factor from the one introduced in [?]. In [?] the factor was chosen such that the standard basis on $S^3 \times S^3$ has unit volume. Here and in [?] we have chosen the factor $\frac{1}{2}$ for the sake of simplicity. We mention that the isometries of $(S^3 \times S^3, g)$ are given by

$$F\colon S^3 \times S^3 \to S^3 \times S^3 : (p, q) \to (apc^{-1}, bqc^{-1}),$$

where a, b and c are unit quaternions.

We write the Levi-Civita connection of $(S^3 \times S^3, g)$ as $\tilde\nabla$ and the tensor field $\tilde\nabla J$ as G. Endowed with the almost Hermitian structure (g, J), $S^3 \times S^3$ is nearly Kähler, which means that the tensor field G is skew-symmetric. The following properties can be deduced from the nearly Kähler property.

$$G(X, Y) + G(Y, X) = 0, \tag{3}$$

$$G(X, JY) + JG(X, Y) = 0, \tag{4}$$

$$g(G(X, Y), Z) + g(G(X, Z), Y) = 0. \tag{5}$$

The almost product structure P on $S^3 \times S^3$ is an endomorphism defined by

$$PZ = (pq^{-1}V, qp^{-1}U).$$

If $U = p\alpha$ and $V = q\beta$, then $PZ = (p\beta, q\alpha)$. The almost product structure can be interpreted in the following way. We start with the vector (U, V) at (p, q). First left translate it to the point $(1, 1)$ to get the vector $(p^{-1}U, q^{-1}V)$. Next we switch the components and finally we left translate this vector back to the point (p, q) and we obtain $(pq^{-1}V, qp^{-1}U)$. It is straightforward to show that

$$P^2 = \mathrm{Id}, \qquad PJ = -JP, \qquad g(PZ, PZ') = g(Z, Z'). \tag{6}$$

Thus P is also symmetric with respect to g. The tensor field $\tilde{\nabla}P$ does not vanish identically, so P is not a product structure, but it satisfies the following properties.

Lemma 2.1. *For tangent vectors X, $Y \in T(S^3 \times S^3)$ one has*

$$PG(X, Y) + G(PX, PY) = 0, \tag{7}$$

$$G(X, PY) + PG(X, Y) = -2J(\tilde{\nabla}_X P)Y, \tag{8}$$

$$(\tilde{\nabla}_X P)JY = J(\tilde{\nabla}_X P)Y. \tag{9}$$

The Riemann curvature tensor $\tilde{R}$ on $(S^3 \times S^3, g)$ is given by

$$\begin{aligned}\tilde{R}(X, Y)Z = &\frac{5}{12}\big(g(Y, Z)X - g(X, Z)Y\big)\\ &+ \frac{1}{12}\big(g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ\big)\\ &+ \frac{1}{3}\big(g(PY, Z)PX - g(PX, Z)PY\\ &\qquad + g(JPY, Z)JPX - g(JPX, Z)JPY\big),\end{aligned} \tag{10}$$

and from this one can show that the tensors $\tilde{\nabla}G$ and G satisfy

$$(\tilde{\nabla}G)(X, Y, Z) = \frac{1}{3}(g(X, Z)JY - g(X, Y)JZ - g(JY, Z)X), \tag{11}$$

$$\begin{aligned}g\big(G(X, Y), G(Z, W)\big) = &\frac{1}{3}\big(g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\\ &+ g(JX, Z)g(JW, Y) - g(JX, W)g(JZ, Y)\big),\end{aligned} \tag{12}$$

$$G\big(X, G(Y, Z)\big) = \frac{1}{3}\big(g(X, Z)Y - g(X, Y)Z + g(JX, Z)JY - g(JX, Y)JZ\big). \tag{13}$$

In particular, it follows that $S^3 \times S^3$ is of constant type $1/3$ ([**?**]). We note that similar expressions hold on any 6-dimensional strictly nearly Kähler manifold.

Now we will give the relation between the geometry of the nearly Kähler manifold $(S^3 \times S^3, g)$ and the product manifold $(S^3 \times S^3, \langle\cdot,\cdot\rangle)$ endowed with the Euclidean product metric $\langle\cdot,\cdot\rangle$. The equations in this paragraph will be used in Section 6.2.

The almost product structure P can be expressed in terms of the usual product structure $QZ = Q(U, V) = (-U, V)$ and vice versa:

$$QZ = \frac{1}{\sqrt{3}}(2PJZ - JZ), \tag{14}$$

$$PZ = \frac{1}{2}(Z - \sqrt{3}QJZ). \tag{15}$$

Using these equations the Euclidean metric $\langle\cdot,\cdot\rangle$ can be expressed in terms of g and P:

$$\langle Z, Z'\rangle = \frac{3}{8}\big(g(Z, Z') + g(QZ, QZ')\big) = g(Z, Z') + \frac{1}{2}g(Z, PZ'), \tag{16}$$

and consequently

$$\langle Z, QZ' \rangle = \frac{\sqrt{3}}{2} g(Z, PJZ'). \tag{17}$$

We can now show the relation between the Levi-Civita connections $\tilde{\nabla}$ of g and ∇^E of the Euclidean product metric $\langle \cdot, \cdot \rangle$ on $S^3 \times S^3$.

Lemma 2.2. *The relation between the nearly Kähler connections $\tilde{\nabla}$ and the Euclidean connection ∇^E is*

$$\nabla^E_X Y = \tilde{\nabla}_X Y + \frac{1}{2}\big(JG(X, PY) + JG(Y, PX)\big). \tag{18}$$

Proof. It is easy to check that the right hand side of (18) defines a connection and we denote it by $\widehat{\nabla}$. The connection $\widehat{\nabla}$ is symmetric, since $\frac{1}{2}\big(JG(X, PY) + JG(Y, PX)\big)$ is symmetric. We now only have to show that $\widehat{\nabla}$ is compatible with the Euclidean metric $\langle \cdot, \cdot \rangle$. Equations (6), (7) and (16) give

$$\begin{aligned}
\langle \widehat{\nabla}_X Y, Z \rangle &+ \langle \widehat{\nabla}_X Z, Y \rangle \\
&= g(\tilde{\nabla}_X Y, Z) + g(\tilde{\nabla}_X Z, Y) \\
&\quad + \frac{1}{2}\big(g(\tilde{\nabla}_X Y, PZ) + g(\tilde{\nabla}_X Z, PY) + g(JG(X, PY), Z) \\
&\qquad + g(JG(Y, PX), Z) + g(JG(X, PZ), Y) + g(JG(Z, PX), Y)\big) \\
&\quad + \frac{1}{4}\big(g(JG(PX, Y), Z) + g(JG(PY, X), Z) \\
&\qquad + g(JG(PX, Z), Y) + g(JG(PZ, X), Y)\big) \\
&= Xg(Y, Z) + \frac{1}{2}\big(g(\tilde{\nabla}_X Y, PZ) + g(\tilde{\nabla}_X Z, PY)\big) \\
&\quad + \frac{1}{4}\big(g(JG(X, PY), Z) + g(JG(X, PZ), Y)\big),
\end{aligned}$$

and then by (8) and (16) we obtain

$$\begin{aligned}
\langle \widehat{\nabla}_X Y, Z \rangle + \langle \widehat{\nabla}_X Z, Y \rangle &= Xg(Y, Z) + \frac{1}{2}\big(g(\tilde{\nabla}_X Y, PZ) + g(\tilde{\nabla}_X Z, PY) \\
&\quad + g(\tilde{\nabla}_X PY, Z) - g(\tilde{\nabla}_X Y, PZ)\big) \\
&= Xg(Y, Z) + \frac{1}{2} Xg(PY, Z) \\
&= X\langle Y, Z \rangle.
\end{aligned}$$

This finishes the proof of the lemma. □

Remark. Using this lemma and (15) one can show that $(\nabla^E_X Q)Y = 0$ implies equation (9) and vice versa. In this sense P really is the “nearly Kähler analogue” of the Euclidean product structure Q.

3. Almost complex surfaces in $S^3 \times S^3$

An almost complex submanifold M in a nearly Kähler manifold $\tilde{M}$ is a submanifold such that J maps tangent vectors to tangent vectors, i.e., the tangent bundle TM is J-invariant. Consequently J maps normal vectors on normal ones. It was proven by Podestà and Spiro that 6-dimensional strictly nearly Kähler manifolds do not admit 4-dimensional almost complex submanifolds [**?**], hence the almost complex submanifolds of $S^3 \times S^3$ are surfaces.

Denote the Levi-Civita connection on M by ∇ and the normal connection on the normal bundle $T^\perp M$ by $\nabla^\perp$. The Gauss and Weingarten formulas and the basic properties of G imply

$$\nabla_X JX = J\nabla_X X, \qquad h(X,JY) = Jh(X,Y), \tag{19}$$

$$A_{J\xi}X = JA_\xi X = -A_\xi JX, \qquad G(X,\xi) = \nabla^\perp_X J\xi - J\nabla^\perp_X \xi, \tag{20}$$

for X, $Y \in TM$ and $\xi \in T^\perp M$, where h denotes the second fundamental form and A_ξ denotes the shape operator (see e.g. [?] or [?]). Hence M is minimal and nearly Kähler. Furthermore, since M is an almost complex surface, the tangent space is spanned by a unit vector X and JX. Hence $\tilde\nabla J$ vanishes on M, so M is in fact Kähler. So if X, Y are tangent vectors on an almost complex surface in $S^3 \times S^3$, then $G(X,Y) = 0$ and thus equation (8) simplifies to $(\tilde\nabla_X P)Y = \frac{1}{2}JG(X,PY)$.

Next we recall the correspondence theorem from [?]. Let $\phi\colon M \to S^3 \times S^3\colon (u,v) \mapsto \big(p(u,v), q(u,v)\big)$ be an almost complex surface, where $z = u + iv$ are isothermal coordinates on M. We may assume that $\phi_v = J\phi_u$ by interchanging u and v, if necessary. Furthermore, as p and q are maps into S^3, there are well defined local functions $\tilde\alpha$, $\tilde\beta$, $\tilde\gamma$ and $\tilde\delta$ from M to $\operatorname{Im}\mathbb{H}$ such that

$$p_u = p\tilde\alpha, \qquad p_v = p\tilde\beta, \qquad q_u = q\tilde\gamma, \qquad q_v = q\tilde\delta. \tag{21}$$

Then $\phi_v = J\phi_u$ gives

$$\tilde\gamma = \frac{\sqrt{3}}{2}\tilde\beta + \frac{1}{2}\tilde\alpha, \qquad \tilde\delta = \frac{1}{2}\tilde\beta - \frac{\sqrt{3}}{2}\tilde\alpha. \tag{22}$$

The integrability conditions $p_{uv} = p_{vu}$ and $q_{uv} = q_{vu}$ written in terms of $\tilde\alpha$ and $\tilde\beta$ become

$$\tilde\alpha_v - \tilde\beta_u = 2\tilde\alpha \times \tilde\beta, \qquad \tilde\alpha_u + \tilde\beta_v = \frac{2}{\sqrt{3}}\tilde\alpha \times \tilde\beta. \tag{23}$$

Setting $\alpha = \cos\theta\tilde\alpha + \sin\theta\tilde\beta$ and $\beta = -\sin\theta\tilde\alpha + \cos\theta\tilde\beta$ with $\theta = 2\pi/3$, these two equations become

$$\alpha_v = \beta_u, \qquad \alpha_u + \beta_v = -\frac{4}{\sqrt{3}}\alpha \times \beta. \tag{24}$$

Assume now that we are working on a simply connected neighborhood. Then any closed 1-form is exact and hence there exists a $\mathbb{R}^3$-valued function X such that $X_u = \alpha$, $X_v = \beta$ and

$$X_{uu} + X_{vv} = -\frac{4}{\sqrt{3}}X_u \times X_v. \tag{25}$$

This equation is known as the H-surface equation (cf. [?]). The correspondence theorem can now be formulated as

Theorem 3.1. *There is locally a one-to-one correspondence between almost complex surfaces in $S^3\times S^3$ and solutions of the general H-system equation. Moreover, two solutions are congruent in $\mathbb{R}^3$ if and only if the associated solutions in $S^3 \times S^3$ are congruent.*

On an almost complex surface, a quadratic differential $\Lambda\, dz^2$ is defined by $g(P\phi_z, \phi_z)\, dz^2$. The Cauchy-Riemann-like equations (24) imply that $\Lambda\, dz^2$ is a holomorphic differential. In Section 5 we show this fact using a straightforward calculation (Lemma 5.2).

4. An adapted frame for almost complex surfaces

Consider an almost complex surface M in $S^3 \times S^3$ and let $z = u + iv$ be a complex coordinate on M. We will define a suitable frame adapted to the almost complex surface M in terms of the almost complex product structure P and the tensor field $G = \tilde\nabla J$. This frame will be used to prove the existence and uniqueness theorems (Theorem 1.1 and 1.2) in the next section. The frame can only be defined at points where a certain condition is satisfied.

Define $\partial_z = \frac{1}{2}(\partial_u - i\partial_v)$ and $\partial_{\bar z} = \frac{1}{2}(\partial_u + i\partial_v)$ as usual. Then the metric g on the surface M is given by

$$g(\partial_z, \partial_z) = g(\partial_{\bar z}, \partial_{\bar z}) = 0, \qquad g(\partial_z, \partial_{\bar z}) = e^{\omega},$$

for some local function ω. It follows from the well-known Koszul formula that the Levi-Civita connection on the surface is

$$\nabla_{\partial_z}\partial_{\bar z} = \nabla_{\partial_{\bar z}}\partial_z = 0, \qquad \nabla_{\partial_z}\partial_z = \omega_z\partial_z, \qquad \nabla_{\partial_{\bar z}}\partial_{\bar z} = \omega_{\bar z}\partial_{\bar z}.$$

The Gauss curvature is $K = -e^{-\omega}\omega_{z\bar z}$ and the minimality of M implies that $h(\partial_z, \partial_{\bar z}) = 0$. We will write $\Lambda = g(P\partial_z, \partial_z)$. Moreover

$$\begin{aligned} g(P\partial_z, \partial_{\bar z}) &= \frac{1}{4}\big(g(P\partial_u, \partial_u) + g(P\partial_v, \partial_v)\big) \\ &= \frac{1}{4}\big(g(P\partial_u, \partial_u) - g(JP\partial_u, J\partial_u)\big) \\ &= 0. \end{aligned}$$

From this it follows that

$$P\partial_z = e^{-\omega}\Lambda\partial_{\bar z} + N, \qquad P\partial_{\bar z} = e^{-\omega}\bar\Lambda\partial_z + \bar N,$$

where N is the normal part of $P\partial_z$. The almost product structure P preserves the metric, so $g(P\partial_z, P\partial_z) = 0$ and $g(P\partial_z, P\partial_{\bar z}) = e^{\omega}$, which gives

$$g(N, N) = 0, \qquad g(N, \bar N) = e^{\omega} - |\Lambda|^2 e^{-\omega}. \tag{26}$$

The vectors $G(\partial_z, P\partial_{\bar z})$ and $G(\partial_{\bar z}, P\partial_z)$ are orthogonal to ∂_z, $\partial_{\bar z}$, $P\partial_z$ and $P\partial_{\bar z}$, which is easily seen using equation (5), and thus they are also orthogonal to N and $\bar N$. Writing out $G(\partial_z, P\partial_z)$ and using $\partial_v = J\partial_u$ and the basic properties of G and P, one gets

$$\begin{aligned} G(\partial_z, P\partial_z) &= \frac{1}{4}\big(G(\partial_u, P\partial_u) - G(\partial_v, P\partial_v)\big) - \frac{i}{4}\big(G(\partial_u, P\partial_v) + G(\partial_v, P\partial_u)\big) \\ &= \frac{1}{4}\big(G(\partial_u, P\partial_u) - G(\partial_u, P\partial_u)\big) + \frac{i}{4}\big(G(\partial_u, JP\partial_u) - G(J\partial_u, P\partial_u)\big) \\ &= 0, \end{aligned}$$

and a similar calculation gives

$$G(\partial_z, P\partial_{\bar z}) = \frac{1}{2}\big(G(\partial_u, P\partial_u) + iG(\partial_u, PJ\partial_u)\big).$$

From equation (12) we get

$$g(G(\partial_z, P\partial_{\bar z}), G(\partial_{\bar z}, P\partial_z)) = \frac{2}{3}(e^{2\omega} - |\Lambda|^2) \tag{27}$$

and

$$g(G(\partial_z, P\partial_{\bar z}), G(\partial_z, P\partial_{\bar z})) = 0. \tag{28}$$

So summing up, we have found that

$$\mathcal{F} = \big\{\partial_z, \partial_{\bar z}, N, \bar N, G(\partial_z, P\partial_{\bar z}), G(\partial_{\bar z}, P\partial_z)\big\}$$

gives an adapted frame to the almost complex surface. The vectors are orthogonal, but do not have the same length. Also note that the frame only exists at points where $|\Lambda|^2 \neq e^{2\omega}$. At points where the equality $|\Lambda|^2 = e^{2\omega}$ holds the normal N is zero. At these points P preserves the tangent space: $PTM = TM$. In Theorem 4.2 of [?] the almost complex surfaces with $PTM = TM$ have been classified; such a surface is locally congruent to the torus $f(u, v) = (\cos u + i \sin u, \cos v +$

$i \sin v$). Therefore we will restrict ourselves to the open subset of M on which $|\Lambda|^2 \neq e^{2\omega}$ and we may assume that it is dense in M.

5. Proofs of Theorem 1.1 and Theorem 1.2

In this section we will prove Theorem 1.1 and Theorem 1.2. The proof consists of three steps. First we write the second fundamental form h in terms of the normal vectors $N, \dots, G(\partial_{\bar{z}}, P\partial_z)$. Next we deduce the compatibility equations for almost complex surfaces. In this second step the proof will be split into two cases: (a) Λ is nowhere zero and (b) Λ vanishes. Case (a) gives Theorem 1.1 and case (b) gives Theorem 1.2. In the last step we use the correspondence theorem (Theorem 3.1) and verify that the compatibility conditions for a H-surface are the same as the compatibility equations for almost complex surfaces. The proof then follows from the existence and uniqueness theorem for differential equations and the correspondence theorem.

The equations (7) and (8) give two useful expressions that will be used frequently in this section.

$$PG(\partial_z, P\partial_{\bar{z}}) = G(\partial_{\bar{z}}, P\partial_z), \qquad PG(\partial_{\bar{z}}, P\partial_z) = G(\partial_z, P\partial_{\bar{z}}), \tag{29}$$

$$\tilde{\nabla}_{\partial_z} P\partial_z = P\tilde{\nabla}_{\partial_z}\partial_z, \qquad \tilde{\nabla}_{\partial_{\bar{z}}} P\partial_{\bar{z}} = P\tilde{\nabla}_{\partial_{\bar{z}}}\partial_{\bar{z}}. \tag{30}$$

Consider the adapted frame $\mathcal{F}$ on the open subset of M where $|\Lambda|^2 \neq e^{2\omega}$.

Proposition 5.1. *The second fundamental form is given by*

$$h(\partial_z, \partial_z) = \lambda \bar{N} + \mu G(\partial_{\bar{z}}, P\partial_z) \tag{31}$$

with λ and μ local functions satisfying

$$\lambda = \frac{1}{2}\frac{\Lambda_z - 2\omega_z\Lambda}{e^{\omega} - |\Lambda|^2 e^{-\omega}}, \qquad \frac{2}{3}(e^{2\omega} - |\Lambda|^2)\mu = g(h(\partial_z, \partial_z), G(\partial_z, P\partial_{\bar{z}})). \tag{32}$$

Proof. Differentiating Λ with respect to z gives

$$\begin{aligned} \Lambda_z &= g(\tilde{\nabla}_{\partial_z} P\partial_z, \partial_z) + g(P\partial_z, \tilde{\nabla}_{\partial_z}\partial_z) \\ &= g(P\tilde{\nabla}_{\partial_z}\partial_z, \partial_z) + g(P\partial_z, \omega_z\partial_z) + g(P\partial_z, h(\partial_z, \partial_z)) \\ &= 2\omega_z\Lambda + 2g(h(\partial_z, \partial_z), N), \end{aligned} \tag{33}$$

where we used (30). Similarly, $\partial_z g(P\partial_z, \partial_{\bar{z}}) = 0$ gives us

$$\begin{aligned} 0 &= g(\tilde{\nabla}_{\partial_z} P\partial_z, \partial_{\bar{z}}) + g(P\partial_z, \tilde{\nabla}_{\partial_z}\partial_{\bar{z}}) \\ &= g((\tilde{\nabla}_{\partial_z} P)\partial_z, \partial_{\bar{z}}) + g(P\tilde{\nabla}_{\partial_z}\partial_z, \partial_{\bar{z}}) \\ &= \omega_z g(P\partial_z, \partial_{\bar{z}}) + g(Ph(\partial_z, \partial_z), \partial_{\bar{z}}) \\ &= g(h(\partial_z, \partial_z), \bar{N}). \end{aligned}$$

Now we calculate the inner product of $h(\partial_z, \partial_z)$ and $G(\partial_{\bar{z}}, P\partial_z)$. Since the surface is almost complex, $Jh(\partial_z, \partial_z)$ is equal to $h(\partial_z, J\partial_z) = ih(\partial_z, \partial_z)$, so we have

$$\begin{aligned} g(h(\partial_z, \partial_z), G(\partial_{\bar{z}}, P\partial_z)) &= -g(G(\partial_{\bar{z}}, h(\partial_z, \partial_z)), P\partial_z) \\ &= -g(\tilde{\nabla}_{\partial_{\bar{z}}} Jh(\partial_z, \partial_z), P\partial_z) - g(\tilde{\nabla}_{\partial_{\bar{z}}} h(\partial_z, \partial_z), JP\partial_z) \\ &= -ig(\tilde{\nabla}_{\partial_{\bar{z}}} h(\partial_z, \partial_z), P\partial_z) + ig(\tilde{\nabla}_{\partial_{\bar{z}}} h(\partial_z, \partial_z), P\partial_z) \\ &= 0. \end{aligned}$$

Note that by equations (27) and (31) the complex valued function μ has to satisfy the second equation of (32). We complete the proof of Proposition 5.1. $\square$

As we mentioned before, the differential $\Lambda\, dz^2$ is holomorphic (Theorem 3.1). This fact follows from equations (24) and (25), but it can also be shown by an easy calculation.

Lemma 5.2. *The function $\Lambda = g(P\partial_z, \partial_z)$ is holomorphic.*

Proof. By the first equation of (30), we have $\Lambda_{\bar z} = 2g(\nabla_{\partial_z}\partial_{\bar z}, P\partial_z) = 2g(h(\partial_z, \partial_{\bar z}), P\partial_z) = 0$. □

We now distinguish two cases. The function Λ is holomorphic, so it is zero at isolated points or identically zero. In the latter case $|\Lambda|^2 = 0$ and since $e^{2\omega}$ is always positive, the frame $\mathcal{F}$ always exists on the surface. In the former case we can take a local coordinate system, on a possibly smaller neighbourhood, such that $\Lambda = 1$. The frame $\mathcal{F}$ then only exists at points where $\omega > 0$. As we have pointed out in Section 4 we can assume that the set of points where $\omega > 0$ is an open dense subset of M.
Case 1. $\Lambda \equiv 1$. From now on we assume that $\Lambda = 1$. In this case (32) simplifies to

$$\lambda = \frac{-\omega_z}{e^{\omega} - e^{-\omega}}, \qquad \frac{2}{3}(e^{2\omega} - 1)\mu = g(h(\partial_z, \partial_z), G(\partial_z, P\partial_{\bar z})).$$

The next step is to calculate $\omega_{z\bar z}$ and $\mu_{\bar z}$. For this we need the following lemma.

Lemma 5.3. *The normal covariant derivatives of the normal vectors are*

$$\begin{aligned}
\nabla^{\perp}_{\partial_{\bar z}} N &= -e^{-\omega}\bar\lambda N - e^{-\omega}\bar\mu G(\partial_z, P\partial_{\bar z}) + \frac{i}{2}G(\partial_{\bar z}, P\partial_z),\\
\nabla^{\perp}_{\partial_{\bar z}} \bar N &= \frac{1}{2}\omega_{\bar z} e^{\omega} \operatorname{csch}\omega \bar N + \bar\mu G(\partial_{\bar z}, P\partial_z),\\
\nabla^{\perp}_{\partial_{\bar z}} G(\partial_z, P\partial_{\bar z}) &= -\frac{2}{3}e^{\omega}\bar\mu N - \frac{i}{3}e^{\omega}\bar N + \frac{1}{2}\omega_{\bar z} e^{\omega}\operatorname{csch}\omega G(\partial_z, P\partial_{\bar z}),\\
\nabla^{\perp}_{\partial_{\bar z}} G(\partial_{\bar z}, P\partial_z) &= \frac{2}{3}\bar\mu \bar N + \frac{1}{2}\omega_{\bar z} e^{\omega}\operatorname{csch}\omega G(\partial_{\bar z}, P\partial_z).
\end{aligned}$$

Proof. The first two expressions can be calculated directly. Firstly we have

$$\begin{aligned}
\nabla^{\perp}_{\partial_{\bar z}} N &= \tilde\nabla_{\partial_{\bar z}}(P\partial_z - e^{-\omega}\partial_{\bar z}) + A_N \partial_{\bar z}\\
&= (\tilde\nabla_{\partial_{\bar z}} P)\partial_z + e^{-\omega}\omega_{\bar z}\partial_{\bar z} - e^{-\omega}\tilde\nabla_{\partial_{\bar z}}\partial_{\bar z}\\
&= \frac{1}{2}JG(\partial_{\bar z}, P\partial_z) - e^{-\omega}h(\partial_{\bar z}, \partial_{\bar z})\\
&= -e^{-\omega}\bar\lambda N - e^{-\omega}\bar\mu G(\partial_z, P\partial_{\bar z}) + \frac{i}{2}G(\partial_{\bar z}, P\partial_z).
\end{aligned}$$

From Proposition 5.1 we get

$$Ph(\partial_z, \partial_z) = \lambda((1 - e^{-2\omega})\partial_{\bar z} - e^{-\omega}N) + \mu G(\partial_z, P\partial_{\bar z}).$$

Therefore the second covariant derivative is

$$\begin{aligned}
\nabla^{\perp}_{\partial_{\bar z}} \bar N &= \tilde\nabla_{\partial_{\bar z}}(P\partial_{\bar z} - e^{-\omega}\partial_z) + A_{\bar N}\partial_{\bar z}\\
&= P\tilde\nabla_{\partial_{\bar z}}\partial_{\bar z} + \omega_{\bar z} e^{-\omega}\partial_z - e^{-\omega}\tilde\nabla_{\partial_{\bar z}}\partial_z + A_{\bar N}\partial_{\bar z}\\
&= Ph(\partial_{\bar z}, \partial_{\bar z})^{\perp} + \omega_{\bar z}(P\partial_{\bar z})^{\perp}\\
&= \frac{1}{2}\omega_{\bar z} e^{\omega}\operatorname{csch}\omega \bar N + \bar\mu G(\partial_{\bar z}, P\partial_z).
\end{aligned}$$

To obtain the third and fourth expression, we calculate the inner products of $\nabla^\perp_{\partial_{\bar z}} G(\partial_z, P\partial_{\bar z})$ and $\nabla^\perp_{\partial_{\bar z}} G(\partial_{\bar z}, P\partial_z)$ with the normal vectors. We already know that

$$\begin{aligned} g(\nabla^\perp_{\partial_{\bar z}} G(\partial_z, P\partial_{\bar z}), N) &= -g(G(\partial_z, P\partial_{\bar z}), \nabla^\perp_{\partial_{\bar z}} N) \\ &= -\frac{i}{3}(e^{2\omega} - 1), \\ g(\nabla^\perp_{\partial_{\bar z}} G(\partial_z, P\partial_{\bar z}), \bar N) &= -g(G(\partial_z, P\partial_{\bar z}), \nabla^\perp_{\partial_{\bar z}} \bar N) \\ &= -\frac{2}{3}\bar\mu(e^{2\omega} - 1). \end{aligned}$$

From (28) it follows that $g(\nabla^\perp_{\partial_{\bar z}} G(\partial_z, P\partial_{\bar z}), G(\partial_z, P\partial_{\bar z})) = 0$. By equation (30) we have

$$\begin{aligned} &g(\nabla^\perp_{\partial_{\bar z}} G(\partial_z, P\partial_{\bar z}), G(\partial_{\bar z}, P\partial_z)) \\ &\quad = g((\tilde\nabla_{\partial_{\bar z}} G)(\partial_z, P\partial_{\bar z}), G(\partial_{\bar z}, P\partial_z)) + g(G(\tilde\nabla_{\partial_{\bar z}}\partial_z, P\partial_{\bar z}), G(\partial_{\bar z}, P\partial_z)) \\ &\quad\quad + g(G(\partial_z, \tilde\nabla_{\partial_{\bar z}} P\partial_{\bar z}), G(\partial_{\bar z}, P\partial_z)). \end{aligned}$$

Equation (11) says that $(\tilde\nabla_{\partial_{\bar z}} G)(\partial_z, P\partial_{\bar z})$ is a linear combination of $\partial_z, \partial_{\bar z}, P\partial_z, P\partial_{\bar z}$ and therefore the first term vanishes. The second term vanishes as well since $\nabla_{\partial_{\bar z}}\partial_z = 0$. Equations (27), (31) and (12) then give

$$\begin{aligned} g(\nabla^\perp_{\partial_{\bar z}} G(\partial_z, P\partial_{\bar z}), G(\partial_{\bar z}, P\partial_z)) &= g(G(\partial_z, P(\nabla_{\partial_{\bar z}}\partial_{\bar z} + h(\partial_{\bar z}, \partial_{\bar z})), G(\partial_{\bar z}, P\partial_z)) \\ &= \frac{2}{3}\omega_{\bar z} e^{2\omega}. \end{aligned}$$

Combining these equations with (26) and (27), one obtains the third expression. Finally we have

$$\begin{aligned} g(\nabla^\perp_{\partial_{\bar z}} G(\partial_{\bar z}, P\partial_z), N) &= -g(G(\partial_{\bar z}, P\partial_z), \nabla^\perp_{\partial_{\bar z}} N) = \frac{2}{3}\bar\mu(e^{\omega} - e^{-\omega}), \\ g(\nabla^\perp_{\partial_{\bar z}} G(\partial_{\bar z}, P\partial_z), \bar N) &= -g(G(\partial_{\bar z}, P\partial_z), \nabla^\perp_{\partial_{\bar z}} \bar N) = 0, \\ g(\nabla^\perp_{\partial_{\bar z}} G(\partial_{\bar z}, P\partial_z), G(\partial_{\bar z}, P\partial_z)) &= 0, \\ g(\nabla^\perp_{\partial_{\bar z}} G(\partial_{\bar z}, P\partial_z), G(\partial_z, P\partial_{\bar z})) &= \frac{2}{3}\partial_{\bar z}(e^{2\omega} - 1) - g(\nabla^\perp_{\partial_{\bar z}} G(\partial_z, P\partial_{\bar z}), G(\partial_{\bar z}, P\partial_z)) \\ &= \frac{2}{3}\omega_{\bar z} e^{2\omega}. \end{aligned}$$

These equations together with (26) and (27) give the last covariant derivative. □

The expression for the curvature tensor $\tilde R$ yields $\big(\tilde R(\partial_{\bar z}, \partial_z)\partial_z\big)^\perp = \frac{2}{3}\bar N$, so the Codazzi equation becomes $\nabla^\perp_{\partial_{\bar z}} h(\partial_z, \partial_z) = \frac{2}{3}\bar N$. It follows from Lemma 5.3 that

$$\begin{aligned} \omega_{z\bar z}\sinh\omega - \frac{e^{-\omega}}{2}|\omega_z|^2 - \frac{4}{3}\sinh^2\omega(1 - |\mu|^2) &= 0, \\ \mu_{\bar z} + \frac{e^{\omega}\mu\omega_{\bar z} - \omega_z\bar\mu}{2\sinh\omega} &= 0. \end{aligned} \tag{34}$$

Case 2. Λ is identically zero. The calculations and expressions in this case are simpler. In this case $h(\partial_z, \partial_z) = \mu G(\partial_{\bar z}, P\partial_z)$, where

$$\frac{2}{3}e^{2\omega}\mu = g(h(\partial_z, \partial_z), G(\partial_z, P\partial_{\bar z})).$$

Using the same approach as in Lemma 5.3, we get

$$\begin{aligned}\nabla^\perp_{\partial_{\bar z}} G(\partial_{\bar z}, P\partial_z) &= (\tilde\nabla G)(\partial_{\bar z}, \partial_{\bar z}, P\partial_z) + G(\tilde\nabla_{\partial_{\bar z}}\partial_{\bar z}, P\partial_z)\\ &\quad + G(\partial_{\bar z}, \tilde\nabla_{\partial_{\bar z}} P\partial_z) + A_{G(\partial_{\bar z}, P\partial_z)}\partial_{\bar z}\\ &= \omega_{\bar z} G(\partial_{\bar z}, P\partial_z) + \bar\mu G(G(\partial_z, P\partial_{\bar z}), P\partial_z)\\ &\quad + \frac{1}{2} G(\partial_{\bar z}, JG(\partial_{\bar z}, P\partial_z)) + A_{G(\partial_{\bar z}, P\partial_z)}\partial_{\bar z}\\ &= \omega_{\bar z} G(\partial_{\bar z}, P\partial_z).\end{aligned}$$

From the Codazzi equation it follows that

$$0 = \nabla^\perp_{\partial_{\bar z}} h(\partial_z, \partial_z) = (\mu_{\bar z} + \omega_{\bar z}\mu) G(\partial_{\bar z}, P\partial_z).$$

Together with the Gauss equation $R(\partial_z, \partial_{\bar z}, \partial_{\bar z}, \partial_z) = \tilde R(\partial_z, \partial_{\bar z}, \partial_{\bar z}, \partial_z) + \|h(\partial_z, \partial_z)\|^2$ we obtain

$$\begin{aligned}\omega_{z\bar z} + \frac{2}{3} e^\omega (1 - |\mu|^2) &= 0,\\ \mu_{\bar z} + \omega_{\bar z}\mu &= 0.\end{aligned} \tag{35}$$

In order to finish the proof, we will use the correspondence between almost complex surfaces and H-systems (Theorem 3.1). We look at the equation

$$X_{z\bar z} = \frac{2i}{\sqrt{3}} X_z \times X_{\bar z}.$$

The definition of g, the equations (22) and the fact that α, β and $\tilde\alpha, \tilde\beta$ are equal up to a rotation give

$$\begin{aligned}e^\omega &= \frac{1}{4}\big(\langle \partial_u, \partial_u\rangle + \langle \partial_v, \partial_v\rangle\big)\\ &= \frac{1}{4}\big(\langle\tilde\alpha, \tilde\alpha\rangle + \langle\tilde\beta, \tilde\beta\rangle + \langle\tilde\gamma, \tilde\gamma\rangle + \langle\tilde\delta, \tilde\delta\rangle\big) = \frac{1}{2}\big(\langle\tilde\alpha, \tilde\alpha\rangle + \langle\tilde\beta, \tilde\beta\rangle\big)\\ &= \frac{1}{2}\big(\langle\alpha, \alpha\rangle + \langle\beta, \beta\rangle\big) = \frac{1}{2}\big(\langle X_u, X_u\rangle + \langle X_v, X_v\rangle\big) = 2\langle X_z, X_{\bar z}\rangle.\end{aligned}$$

Moreover we know from Theorem 3.1

$$\Lambda = g(P\partial_z, \partial_z) = 2e^{i\frac{\pi}{3}}\langle X_z, X_z\rangle.$$

So we have the relations

$$\langle X_z, X_{\bar z}\rangle = \frac{1}{2} e^\omega, \qquad \langle X_z, X_z\rangle = \frac{1}{2} e^{-i\frac{\pi}{3}}\Lambda. \tag{36}$$

Case 1. Λ is not identically zero. By changing the coordinate system we may assume that Λ and thus $\langle X_z, X_z\rangle$ are constant. Then $\langle X_{zz}, X_{\bar z}\rangle = \frac{1}{2}\omega_z e^\omega$ and $\langle X_{zz}, X_z\rangle = 0$. One obtains the system

$$\begin{aligned}X_{z\bar z} &= \frac{2i}{\sqrt{3}} X_z \times X_{\bar z},\\ X_{zz} &= \frac{\omega_z}{1 - |\Lambda|^2 e^{-2\omega}} X_z - \frac{e^{-\omega}\omega_z}{1 - |\Lambda|^2 e^{-2\omega}}\Lambda e^{-i\frac{\pi}{3}} X_{\bar z} + \Omega X_z \times X_{\bar z}.\end{aligned}$$

for some local function Ω. We have

$$\begin{aligned}(X_{z\bar z})_z &= -\Big(\frac{ie^{i\frac{\pi}{3}}}{\sqrt{3}}\bar\Lambda\Omega + \frac{2}{3}e^\omega\Big) X_z + \Big(\frac{ie^\omega}{\sqrt{3}}\Omega + \frac{2}{3}e^{-i\frac{\pi}{3}}\Lambda\Big) X_{\bar z}\\ &\quad + \frac{2i}{\sqrt{3}}\frac{\omega_z}{1 - |\Lambda|^2 e^{-2\omega}} X_z \times X_{\bar z}.\end{aligned}$$

Comparing this with $(X_{zz})_{\bar z}$ we then get

$$\Omega_{\bar z} = \frac{\omega_{\bar z}}{|\Lambda|^2 e^{-2\omega}-1}\Omega + \frac{e^{-\omega}e^{-i\frac{\pi}{3}}\omega_z}{|\Lambda|^2 e^{-2\omega}-1}\Lambda\bar\Omega,$$

$$0 = \omega_{z\bar z}(|\Lambda|^2 e^{-2\omega}-1) + |\omega_z|^2|\Lambda|^2 e^{-2\omega} + e^{\omega}(|\Lambda|^2 e^{-2\omega}-1)^2\Big(\frac{1}{2}|\Omega|^2-\frac{2}{3}\Big).$$

If we set

$$\Omega = -\frac{2i}{\sqrt{3}}e^{-i\frac{\pi}{6}}\mu, \qquad \Lambda = 1,$$

we see that the above compatibility conditions are the same as equations (34). Therefore such an H-system indeed exists. Consequently we get a corresponding almost complex surface in $S^3\times S^3$.

Case 2. Λ is identically zero. If Λ vanishes, we obtain the system

$$X_{zz} = \omega_z X_z + \Omega X_z \times X_{\bar z}.$$

The derivatives $(X_{z\bar z})_z$ and $(X_{zz})_{\bar z}$ are

$$(X_{z\bar z})_z = -\frac{2}{3}e^{\omega}X_z + \frac{ie^{\omega}}{\sqrt{3}}\Omega X_{\bar z} + \frac{2i}{\sqrt{3}}\omega_z X_z\times X_{\bar z}$$

$$(X_{zz})_{\bar z} = \Big(\omega_{z\bar z} - \frac{e^{\omega}}{2}|\Omega|^2\Big)X_z + \frac{ie^{\omega}}{\sqrt{3}}\Omega X_{\bar z} + \Big(\Omega_{\bar z} + \omega_{\bar z}\Omega + \frac{2i}{\sqrt{3}}\omega_z\Big)X_z\times X_{\bar z}.$$

The compatibility condition then gives

$$\omega_{z\bar z} = \frac{e^{\omega}}{2}\big(|\Omega|^2 - \frac{4}{3}\big), \qquad \Omega_{\bar z} + \omega_{\bar z}\Omega = 0.$$

So for $\Omega = \frac{2}{\sqrt{3}}\mu$ we get the same equations as (35). This proves the existence and uniqueness theorems.

6. Flat almost complex surfaces

In the last section we give a method to obtain all flat almost complex surfaces in $S^3\times S^3$. There is a 2-parameter family of flat almost complex surfaces and one isolated example that was already described in [?]. Our proof consists of three steps. First we choose a suitable adapted frame for the almost complex surfaces and determine the second fundamental form. For this, we calculate several compatibility conditions. In the second step, we show that the components of the flat surface are Clifford tori. This fact will be used in the final step to obtain a parametrization of the flat almost complex surfaces.

6.1. **Finding the second fundamental form.** Let M be a flat almost complex surface. If $PTM = TM$ then the surface is locally congruent to the torus $f(u,v) = (\cos u + i\sin u, \cos v + i\sin v)$ (see Theorem 4.1 in [?]). So we will assume that $PTM \neq TM$. Choose a local frame e_1, e_2 such that e_1 is a maximum of $g(Pu,u)$ for all unit vectors u and $e_2 = Je_1$. From a standard argument it follows that $g(Pe_1, e_2) = 0$. Consider the adapted frame $e_1,\dots,e_6$ along M where

$$e_3 = Pe_1, \qquad e_5 = G(e_1, Pe_1),$$
$$e_4 = Pe_2 = -Je_3, \qquad e_6 = G(e_1, Pe_2) = Je_5.$$

By the assumption $PTM \neq TM$, these vectors are indeed linearly independent. Let a and b be local functions such that the connection is given by

$$\nabla_{e_1}e_1 = ae_2, \qquad \nabla_{e_2}e_1 = -be_2,$$
$$\nabla_{e_1}e_2 = -ae_1, \qquad \nabla_{e_2}e_2 = be_1.$$

As ∇ is torsion free, one has $[e_1, e_2] = -ae_1 + be_2$. The curvature is zero, so

$$e_1(b) + e_2(a) = a^2 + b^2. \tag{37}$$

Furthermore if we write $\cos\theta = g(Pe_1, e_1)$, then

$$Pe_1 - \cos\theta e_1 = e_3 - \cos\theta e_1, \qquad -J(Pe_1 - \cos\theta e_1) = e_4 + \cos\theta e_2$$

are normal vectors. By (5) the vectors e_5 and e_6 are also normal ones. We choose c_1, c_2, d_1, d_2 such that

$$h(e_1, e_1) = c_1(e_3 - \cos\theta e_1) + c_2(e_4 + \cos\theta e_2) + d_1 e_5 + d_2 e_6.$$

Since M is almost complex, $Jh(e_1, e_1) = h(e_1, e_2)$ and $h(e_1, e_1) = -h(e_2, e_2)$. Therefore

$$\begin{aligned}
\tilde\nabla_{e_1} e_1 &= -c_1\cos\theta e_1 + (a + c_2\cos\theta)e_2 + c_1 e_3 + c_2 e_4 + d_1 e_5 + d_2 e_6,\\
\tilde\nabla_{e_1} e_2 &= -(a + c_2\cos\theta)e_1 - c_1\cos\theta e_2 + c_2 e_3 - c_1 e_4 - d_2 e_5 + d_1 e_6,\\
\tilde\nabla_{e_2} e_1 &= -c_2\cos\theta e_1 - (b + c_1\cos\theta)e_2 + c_2 e_3 - c_1 e_4 - d_2 e_5 + d_1 e_6,\\
\tilde\nabla_{e_2} e_2 &= (b + c_1\cos\theta)e_1 - c_2\cos\theta e_2 - c_1 e_3 - c_2 e_4 - d_1 e_5 - d_2 e_6.
\end{aligned}$$

Using equation (8) we obtain the covariant derivatives of e_3 and e_4. For example, for $\tilde\nabla_{e_1} e_3 = \tilde\nabla_{e_1} Pe_1$, we have

$$\tilde\nabla_{e_1} Pe_1 = P\tilde\nabla_{e_1} e_1 + \frac{1}{2}G(e_1, Pe_2).$$

One obtains

$$\begin{aligned}
\tilde\nabla_{e_1} e_3 &= c_1 e_1 + c_2 e_2 - c_1\cos\theta e_3 + (a + c_2\cos\theta)e_4 + d_1 e_5 + (\tfrac{1}{2} - d_2)e_6,\\
\tilde\nabla_{e_1} e_4 &= c_2 e_1 - c_1 e_2 - (a + c_2\cos\theta)e_3 - c_1\cos\theta e_4 - (\tfrac{1}{2} + d_2)e_5 - d_1 e_6,\\
\tilde\nabla_{e_2} e_3 &= c_2 e_1 - c_1 e_2 - c_2\cos\theta e_3 - (b + c_1\cos\theta)e_4 + (\tfrac{1}{2} - d_2)e_5 - d_1 e_6,\\
\tilde\nabla_{e_2} e_4 &= -c_1 e_1 - c_2 e_2 + (b + c_1\cos\theta)e_3 - c_2\cos\theta e_4 - d_1 e_5 + (\tfrac{1}{2} + d_2)e_6.
\end{aligned}$$

The compatibility of the Levi-Civita connection $\tilde\nabla$ with the metric gives

$$e_i(g(e_1, e_k)) = g(\tilde\nabla_{e_i} e_1, e_k) + g(e_1, \tilde\nabla_{e_i} e_k)$$

for $i = 1, 2$ and $k = 3, 4$. These four equations give

$$\begin{aligned}
e_1(\theta) &= -2c_1\sin\theta, &\qquad c_1 &= b\cot\theta\csc\theta,\\
e_2(\theta) &= -2c_2\sin\theta, &\qquad c_2 &= a\cot\theta\csc\theta.
\end{aligned}$$

Using equation (11), we can compute the covariant derivatives of e_5 and e_6. For example, for $\tilde\nabla_{e_1} e_5 = \tilde\nabla_{e_1} G(e_1, Pe_1)$ the calculation goes as follows:

$$\begin{aligned}
\tilde\nabla_{e_1} G(e_1, Pe_1) &= (\tilde\nabla_{e_1} G)(e_1, Pe_1) + G(\tilde\nabla_{e_1} e_1, Pe_1) + G(e_1, \tilde\nabla_{e_1} Pe_1)\\
&= \tfrac{1}{3}(g(e_1, Pe_1)e_2 - JPe_1) + G(\tilde\nabla_{e_1} e_1, Pe_1) + G(e_1, \tilde\nabla_{e_1} Pe_1).
\end{aligned}$$

We obtain

$$\begin{aligned}
\tilde{\nabla}_{e_1} e_5 &= \tfrac{1}{3} d_1(\cos\theta - 1)e_1 + \tfrac{1}{6}(2d_2(1+\cos\theta) + \cos\theta)e_2 \\
&\quad + \tfrac{1}{3} d_1(\cos\theta - 1)e_3 + \tfrac{1}{6}(1 + 2d_2(1+\cos\theta))e_4 - 2b\cot^2\theta e_5, \\
\tilde{\nabla}_{e_1} e_6 &= \tfrac{1}{6}(\cos\theta - 2d_2(1+\cos\theta))e_1 + \tfrac{1}{3} d_1(\cos\theta - 1)e_2 \\
&\quad + \tfrac{1}{6}(2d_2(1+\cos\theta) - 1)e_3 + \tfrac{1}{3} d_1(1-\cos\theta)e_4 - 2b\cot^2\theta e_6, \\
\tilde{\nabla}_{e_2} e_5 &= \tfrac{1}{6}(2d_2(1-\cos\theta) + \cos\theta)e_1 + \tfrac{1}{3} d_1(1+\cos\theta)e_2 \\
&\quad + \tfrac{1}{6}(2d_2(1-\cos\theta) - 1)e_3 + \tfrac{1}{3} d_1(1+\cos\theta)e_4 - 2a\cot^2\theta e_5, \\
\tilde{\nabla}_{e_2} e_6 &= -\tfrac{1}{3} d_1(1+\cos\theta)e_1 + \tfrac{1}{6}(2d_2(1-\cos\theta) - \cos\theta)e_2 \\
&\quad + \tfrac{1}{3} d_1(1+\cos\theta)e_3 + \tfrac{1}{6}(2d_2(\cos\theta - 1) - 1)e_4 - 2a\cot^2\theta e_6.
\end{aligned}$$

We have $\tilde{R}(e_1, e_2)e_1 = \tilde{\nabla}_{e_1}\tilde{\nabla}_{e_2} e_1 - \tilde{\nabla}_{e_2}\tilde{\nabla}_{e_1} e_1 - \tilde{\nabla}_{[e_1,e_2]} e_1$. The left hand side is given by the expression (10) for $\tilde{R}$ and the right hand side can be written using all covariant derivatives above. The e_1-, e_2-, e_5- and e_6-components of this equation then give

$$0 = \cot^2\theta\big(e_1(a) - e_2(b)\big), \tag{38a}$$

$$0 = \frac{2}{3}(d_1^2 + d_2^2 - 1) + \csc^2\theta\big((a^2+b^2)(2\csc^2\theta - 3) + e_2(a) + e_1(b)\big), \tag{38b}$$

$$0 = 2\frac{ad_1 + bd_2}{\cos\theta - 1} + e_2(d_1) + e_1(d_2), \tag{38c}$$

$$0 = e_1(d_1) - e_2(d_2) - (bd_1 - ad_2)\sec^2(\frac{\theta}{2}). \tag{38d}$$

Note that the second equation (38b) together with (37) becomes

$$\frac{1}{3} = \frac{1}{3}(d_1^2 + d_2^2) + \csc^2\theta\cot^2\theta(a^2+b^2).$$

In view of this equation we write from now on

$$\begin{aligned}
a &= \frac{1}{\sqrt{3}}\sin\theta\tan\theta\cos\phi_1\cos\alpha, & d_1 &= \cos\phi_2\sin\alpha, \\
b &= \frac{1}{\sqrt{3}}\sin\theta\tan\theta\sin\phi_1\cos\alpha, & d_2 &= \sin\phi_2\sin\alpha
\end{aligned}$$

and furthermore we introduce the variables

$$\begin{aligned}
U &= e_1(\alpha) + \frac{1}{\sqrt{3}}\sin\alpha\cos(\phi_1 - 2\phi_2), \\
V &= e_2(\alpha) - \frac{1}{\sqrt{3}}\sin\alpha\sin(\phi_1 - 2\phi_2).
\end{aligned}$$

Substitute U and V in (37), (38a), (38c) and (38d) and solve this system of equations to obtain the derivatives of ϕ_1 and ϕ_2:

$$\begin{aligned}
e_1(\phi_1) &= \tfrac{1}{6}\big(\sqrt{3}\cos\alpha\sec\theta\cos\phi_1(7+\cos2\theta) + 2\tan\alpha(\sqrt{3}\sin\alpha\cos(\phi_1 - 2\phi_2) + 3V)\big), \\
e_1(\phi_2) &= \tfrac{1}{6}\big(-\sqrt{3}\cos\alpha\sec\theta\sin\phi_1(7+\cos2\theta) + 2\tan\alpha(\sqrt{3}\sin\alpha\sin(\phi_1 - 2\phi_2) - 3U)\big), \\
e_2(\phi_1) &= -V\cot\alpha + \tfrac{1}{\sqrt{3}}(\cos\alpha\cos(\phi_1 - 2\phi_2) + 2\cos\phi_1\sec\theta), \\
e_2(\phi_2) &= U\cot\alpha + \tfrac{1}{\sqrt{3}}\cos\alpha(\sin(\phi_1 - 2\phi_2) - 2\sec\theta\sin\phi_1).
\end{aligned}$$

Now we will calculate several compatibility conditions to obtain Lemma 6.1. The compatibility conditions $e_1(e_2(\phi_i)) - e_2(e_1(\phi_i)) = [e_1, e_2](\phi_i)$ give

$$\begin{aligned}
&\frac{1}{24}\big(24(U^2+V^2)\sec^2\alpha + \cos^2\alpha\sec^2\theta(103+28\cos 2\theta - 3\cos 4\theta)\\
&\qquad + 8\sec\alpha\tan\alpha(3\cos\alpha(e_1(U)+e_2(V)) + \sin 3\alpha + \sqrt{3}(V\cos(\phi_1-2\phi_2) - U\sin(\phi_1-2\phi_2)))\\
&\qquad + 8\tan^2\alpha + 4\tan\theta(-2\sqrt{3}\sin\alpha\sin\theta(V\cos\phi_1+U\sin\phi_1) - 3\cos^2\alpha(7+\cos 2\theta)\tan\theta)\big) = 0,\\
&\frac{1}{3}\big(2\cos 2\alpha - 3\cot\alpha(e_1(U)+e_2(V)) + \cos^2\alpha\sec^2\theta(7+\cos 2\theta)\\
&\qquad + \csc\alpha(3\csc\alpha(U^2+V^2) + \sqrt{3}(-V\cos(\phi_1-2\phi_2) + U\sin(\phi_1-2\phi_2))\\
&\qquad + \sqrt{3}\cos\alpha\cot\alpha\sin\theta\tan\theta(V\cos\phi_1+U\sin\phi_1) - 6\cos^2\alpha\tan^2\theta\big) = 0.
\end{aligned}$$

Multiplying the second expression by $\tan^2\alpha$ and adding it to the first one, gives us

$$6 + 2\sec^2\alpha(U^2+V^2-\frac{1}{3}) = 0,$$

or $U^2+V^2 = \frac{1}{3}\cos^2\alpha(\sec^2\alpha - 9)$. Since U^2+V^2 is positive, $\alpha \in [\arccos(\frac{1}{3}), \arccos(-\frac{1}{3})]$.

We denote

$$r = \Big(\frac{1}{3}\cos^2\alpha(\sec^2\alpha-9)\Big)^{1/2}$$

and $U = r\cos\delta$ and $V = r\sin\delta$. The compatibility conditions for ϕ_2 and α yield two equations linear in $e_1(\delta)$ and $e_2(\delta)$. Solving these two equations gives

$$\begin{aligned}
e_1(\delta) &= \frac{\sec\alpha}{8\sqrt{3}(7+9\cos 2\alpha)^2}\big(A\cos\delta - B\sin\delta\big),\\
e_2(\delta) &= \frac{\sec\alpha}{8\sqrt{3}(7+9\cos 2\alpha)^2}\big(A\sin\delta + B\cos\delta\big)
\end{aligned}$$

with

$$\begin{aligned}
A &= (590+927\cos 2\alpha+450\cos 4\alpha+81\cos 6\alpha)\cos(\delta-\phi_1+2\phi_2)\\
&\quad - 2\sin\theta\tan\theta\cos(\delta+\phi_1)(23\cos\alpha+9\cos 3\alpha)^2,\\
B &= (7+9\cos 2\alpha)(\sqrt{2}\csc\alpha\sqrt{-7-9\cos 2\alpha}(33+28\cos 2\alpha+3\cos 4\alpha)\\
&\quad - 2(19+36\cos 2\alpha+9\cos 4\alpha)\sin(\delta-\phi_1+2\phi_2)\\
&\qquad + 8\cos^2\alpha\sin\theta\tan\theta\sin(\delta+\phi_1)(7+9\cos 2\alpha)).
\end{aligned}$$

The compatibility condition $e_1(e_2(\delta)) - e_2(e_1(\delta)) - [e_1,e_2](\delta) = 0$ then gives

$$\sin(\delta-\phi_1+2\phi_2) = \frac{(25+36\cos 2\alpha+3\cos 4\alpha)\csc^3\alpha}{4\sqrt{2}\sqrt{-7-9\cos 2\alpha}}. \tag{39}$$

Lemma 6.1. *The numbers θ and $\phi := \phi_2$ are constants. Moreover a, b, c_1, c_2 are all zero and $d_1 = \cos\phi$, $d_2 = \sin\phi$.*

Proof. The expression on the right hand side of (39) is smaller than or equal to -1 for every $\alpha \in [\arccos(\frac{1}{3}), \arccos(-\frac{1}{3})]$ and equality holds if $\alpha = \pi/2$. Therefore $\alpha = \pi/2$ and thus a, b, c_1 and c_2 vanish and $d_1 = \cos\phi_2$ and $d_2 = \sin\phi_2$. The equations for $e_i(\theta)$ and $e_i(\phi_2)$ imply that θ and ϕ_2 are constant. □

The second fundamental form thus is given by

$$h(e_1,e_1) = \cos\phi\, e_5 + \sin\phi\, e_6, \qquad h(e_1,e_2) = -\sin\phi\, e_5 + \cos\phi\, e_6,$$

where ϕ is a constant.

6.2. **Both components are homogeneous tori.** In order to obtain a parametrization f for flat almost complex surfaces in $S^3 \times S^3$, we will rewrite the structure equations for the almost complex immersion f as equations for the immersions p, q in S^3 where $f = (p, q)$. The calculations are not that difficult, but the expressions get rather lengthy.

Consider $S^3 \times S^3$ in $\mathbb{R}^8 = \mathbb{R}^4 \oplus \mathbb{R}^4$. Since $[e_1, e_2] = 0$, there are coordinates (u, v) such that $\partial_u = e_1$ and $\partial_v = e_2$. Write $f = (p, q)$, $Qf = (-p, q)$ and consider the frame

$$\mathcal{G} = \{f, Qf, f_u, f_v, Pf_u, Pf_v, G(f_u, Pf_u), JG(f_u, Pf_u)\};$$

this is just the frame $e_1, \ldots, e_6$ with f and Qf added to it. Let D denote the Euclidean connection on $\mathbb{R}^8$ and A and B the matrices such that

$$D_{\partial_u}\mathcal{G} = A\mathcal{G} \quad \text{and} \quad D_{\partial_v}\mathcal{G} = B\mathcal{G}.$$

Now we explain how to calculate all the Euclidean covariant derivatives. We have $D_{\partial_u} f = f_u$. By $DQ = 0$ and (14)

$$D_{\partial_u} Qf = Qf_u = -\frac{1}{\sqrt{3}} f_v + \frac{2}{\sqrt{3}} Pf_v.$$

By Lemma 2.2, equations (16), (17) and the expression $\tilde{\nabla}_{e_1} e_1$ we have

$$\begin{aligned} D_{\partial_u} f_u &= \nabla^E_{\partial_u} f_u + \tfrac{1}{2}\langle D_{\partial_u} f_u, f\rangle f + \tfrac{1}{2}\langle D_{\partial_u} f_u, Qf\rangle Qf \\ &= \tilde{\nabla}_{\partial_u} f_u + JG(f_u, Pf_u) - \tfrac{1}{2}(\langle f_u, f_u\rangle f + \langle f_u, Qf_u\rangle Qf) \\ &= \cos\phi G(f_u, Pf_u) + (\sin\phi + 1) JG(f_u, Pf_u) - (\tfrac{1}{2} + \tfrac{1}{4}\cos\theta) f. \end{aligned}$$

Note that f and Qf have Euclidean length 2, hence the factor $\frac{1}{2}$ in the last two terms. The derivatives of f_v, Pf_u and Pf_v are calculated in the same way. To obtain $D_{\partial_u} G(f_u, Pf_u)$ and $D_{\partial_u} JG(f_u, Pf_u)$, we use (13). For instance,

$$\begin{aligned} D_{\partial_u} G(f_u, Pf_u) &= \nabla^E_{\partial_u} G(f_u, Pf_u) + \tfrac{1}{2}\langle D_{\partial_u} G(f_u, Pf_u), f\rangle f + \tfrac{1}{2}\langle D_{\partial_u} G(f_u, Pf_u), Qf\rangle Qf \\ &= \tilde{\nabla}_{\partial_u} G(f_u, Pf_u) + \tfrac{1}{2}\big(JG(f_u, PG(f_u, Pf_u)) + JG(G(f_u, Pf_u), Pf_u) \\ &\quad - \langle G(f_u, Pf_u), f_u\rangle f - \langle G(f_u, Pf_u), Qf_u\rangle Qf\big) \\ &= \tfrac{1}{3}\cos\phi(\cos\theta - 1) f_u + \tfrac{1}{6}(2(1 + \sin\phi)\cos\theta + 2\sin\phi - 1) f_v \\ &\quad + \tfrac{1}{3}\cos\phi(\cos\theta - 1) Pf_u + \tfrac{1}{6}(2\sin\phi(\cos\theta + 1) - \cos\theta + 2) Pf_v. \end{aligned}$$

Proceeding in this way, one obtains the matrices A and B. The matrix A is

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\phi & 1 + \sin\phi \\ 0 & 0 & 0 & 0 & -\sin\phi & \cos\phi \\ 0 & 0 & 0 & 0 & \cos\phi & \frac{1}{2} - \sin\phi \\ 0 & 0 & 0 & 0 & -(\frac{1}{2} + \sin\phi) & -\cos\phi \\ \frac{1}{3}\cos\phi(\cos\theta - 1) & \frac{1}{6}(2\cos\theta(\sin\phi + 1) + 2\sin\phi - 1) & \frac{1}{3}\cos\phi(\cos\theta - 1) & \frac{1}{6}(2\sin\phi(\cos\theta + 1) - \cos\theta + 2) & 0 & 0 \\ -\frac{1}{6}(2\sin\phi(\cos\theta + 1) + 1) & \frac{1}{3}\cos\phi(\cos\theta - 1) & \frac{1}{6}(2\sin\phi(\cos\theta + 1) + \cos\theta) & \frac{1}{3}\cos\phi(1 - \cos\theta) & 0 & 0 \end{matrix} \Bigg]$$

and B equals

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sin\phi & \cos\phi \\ 0 & 0 & 0 & 0 & -\cos\phi & 1 - \sin\phi \\ 0 & 0 & 0 & 0 & \frac{1}{2} - \sin\phi & -\cos\phi \\ 0 & 0 & 0 & 0 & -\cos\phi & \frac{1}{2} + \sin\phi \\ \frac{1}{6}(2\sin\phi(1 - \cos\theta) + 2\cos\theta + 1) & \frac{1}{3}\cos\phi(\cos\theta + 1) & \frac{1}{6}(2\sin\phi(1 - \cos\theta) - \cos\theta - 2) & \frac{1}{3}\cos\phi(1 + \cos\theta) & 0 & 0 \\ -\frac{1}{3}\cos\phi(\cos\theta + 1) & \frac{1}{6}(2\sin\phi(1 - \cos\theta) - 1) & \frac{1}{3}\cos\phi(\cos\theta + 1) & \frac{1}{6}(2\sin\phi(\cos\theta - 1) - \cos\theta) & 0 & 0 \end{matrix} \Bigg]$$

Note that $f(u, v) = \exp(Au + Bv) = \exp(Au)\exp(Bv)$ gives the parametrization of flat almost complex surfaces in $S^3 \times S^3$ as a solution of the system of differential equations of first order. At this point we have equations for the immersion $f\colon M \to (S^3 \times S^3, \langle \cdot, \cdot \rangle)$, that is, f is regarded as an immersion in $S^3 \times S^3$ with the usual product metric.

In order to find the equations for the components p and q, we rewrite these first order equations as a system of second order equations. Note that $p = \frac{1}{2}(f - Qf)$ and $q = \frac{1}{2}(f + Qf)$ and consider the vectors

$$\xi_1 = \frac{1}{\sqrt{3}} G(f_u, Pf_u) + JG(f_u, Pf_u),$$
$$\xi_2 = -\frac{1}{\sqrt{3}} G(f_u, Pf_u) + JG(f_u, Pf_u).$$

The derivatives of p and q can readily be found using the calculations above. If we write

$$\mathcal{G}_1 = (p, p_u, p_v, \xi_1), \qquad \mathcal{G}_2 = (q, q_u, q_v, \xi_2),$$

then we obtain the following system of equations

$$D_{\partial_u} \mathcal{G}_1 = A_1 \mathcal{G}_1, \qquad D_{\partial_v} \mathcal{G}_1 = B_1 \mathcal{G}_1, \tag{40}$$
$$D_{\partial_u} \mathcal{G}_2 = A_2 \mathcal{G}_2, \qquad D_{\partial_v} \mathcal{G}_2 = B_2 \mathcal{G}_2, \tag{41}$$

where the matrices A_1, B_1, A_2 and B_2 are

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{4}(2+\cos\theta) & 0 & 0 & \frac{1}{2}(1+\sqrt{3}\cos\phi+\sin\phi) \\ -\frac{\sqrt{3}}{4}\cos\theta & 0 & 0 & \frac{1}{2}(\cos\phi-\sqrt{3}\sin\phi) \\ 0 & \begin{gathered}\frac{1}{6}(\cos\theta(1+2\sqrt{3}\cos\phi \\ -2\sin\phi) - 2(1+\sqrt{3}\cos\phi+\sin\phi))\end{gathered} & \begin{gathered}\frac{1}{6\sqrt{3}}(3\cos\theta + 2\sqrt{3}\cos\phi(\cos\theta-1) \\ +6\sin\phi(1+\cos\theta))\end{gathered} & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ -\frac{\sqrt{3}}{4}\cos\theta & 0 & 0 & \frac{1}{2}(\cos\phi-\sqrt{3}\sin\phi) \\ -\frac{1}{4}(2-\cos\theta) & 0 & 0 & \frac{1}{2}(1-\sqrt{3}\cos\phi-\sin\phi) \\ 0 & \begin{gathered}-\frac{1}{3}\cos\phi(1+\cos\theta) \\ +\frac{1}{2\sqrt{3}}(\cos\theta-2\sin\phi(\cos\theta-1))\end{gathered} & \begin{gathered}\frac{1}{\sqrt{3}}\cos\phi(1+\cos\theta) \\ -\frac{1}{6}(2+\cos\theta+2\sin\phi(\cos\theta-1))\end{gathered} & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{1}{4}(2+\cos\theta) & 0 & 0 & \frac{1}{2}(\sin\phi-\sqrt{3}\cos\phi+1) \\ \frac{\sqrt{3}}{4}\cos\theta & 0 & 0 & \frac{1}{2}(\cos\phi+\sqrt{3}\sin\phi) \\ 0 & \begin{gathered}\frac{1}{\sqrt{3}}\cos\phi(1-\cos\theta) \\ -\frac{1}{6}(2\sin\phi(1+\cos\theta)-\cos\theta+2)\end{gathered} & \begin{gathered}\frac{1}{3}\cos\phi(\cos\theta-1) \\ -\frac{1}{2\sqrt{3}}(2\sin\phi(1+\cos\theta)+\cos\theta)\end{gathered} & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ \frac{\sqrt{3}}{4}\cos\theta & 0 & 0 & \frac{1}{2}(\cos\phi+\sqrt{3}\sin\phi) \\ \frac{1}{4}(\cos\theta-2) & 0 & 0 & \frac{1}{2}(\sqrt{3}\cos\phi-\sin\phi+1) \\ 0 & \begin{gathered}\frac{1}{2\sqrt{3}}(\cos\theta(2\sin\phi-1)-2\sin\phi) \\ -\frac{1}{3}\cos\phi(1+\cos\theta)\end{gathered} & \begin{gathered}\frac{1}{6}(2\sin\phi(1-\cos\theta)-\cos\theta-2) \\ -\frac{1}{\sqrt{3}}\cos\phi(1+\cos\theta)\end{gathered} & 0 \end{bmatrix}.$$

In particular p and q are parametrizations of surfaces in S^3. An elementary calculation shows that these surfaces have zero Gaussian curvature and constant mean curvatures

$$\frac{\csc\theta}{\sqrt{3}}\bigl(2\cos\theta\sin\bigl(\tfrac{\pi}{3}-\phi\bigr)-1\bigr), \qquad -\frac{\csc\theta}{\sqrt{3}}\bigl(2\cos\theta\sin\bigl(\tfrac{\pi}{3}+\phi\bigr)+1\bigr) \tag{42}$$

respectively. It is well-known that the only surfaces with constant Gaussian and mean curvature in S^3 are locally totally geodesic spheres or homogenous tori, hence

Proposition 6.2. *The immersions p and q are congruent to (parts of) the homogeneous tori.*

Recall that for $-1 < a < 1$, if

$$x_a(u,v) = \Bigl(\sqrt{\frac{1+a}{2}}e^{iu}, \sqrt{\frac{1-a}{2}}e^{iv}\Bigr),$$

is the homogeneous torus with normal given by

$$\xi(u,v) = (\sqrt{\frac{1-a}{2}}e^{iu}, -\sqrt{\frac{1+a}{2}}e^{iv}).$$

Then we have that

$$\begin{aligned} \xi_u &= \sqrt{\tfrac{1-a}{1+a}}x_u, \\ \xi_v &= -\sqrt{\tfrac{1+a}{1-a}}x_v, \end{aligned}$$

from which we deduce that the mean curvature $H_a = -\frac{1}{2}(\sqrt{\frac{1-a}{1+a}} - \sqrt{\frac{1+a}{1-a}}) = \frac{a}{\sqrt{1-a^2}}$.

The immersions p and q are $SO(4)$-congruent to a standard flat torus in S^3. Also $S^3 \times S^3 = SU(2) \times SU(2)$ is the double cover of $SO(4)$, so any rotation $R \in SO(4)$ can be written as $R(x) = \alpha x \beta$, where $\alpha, \beta \in S^3$. Therefore we can write

$$\begin{aligned} p(u,v) &= a_1(r_1 \cos u, r_1 \sin u, r_2 \cos v, r_2 \sin v)\, a_2, \\ q(\tilde{u},\tilde{v}) &= a_3(s_1 \cos \tilde{u}, s_1 \sin \tilde{u}, s_2 \cos \tilde{v}, s_2 \sin \tilde{v})\, a_4, \end{aligned}$$

where r_1, r_2, s_1, s_2 are positive real numbers with $r_1^2 + r_2^2 = 1$, $s_1^2 + s_2^2 = 1$ and $a_1, a_2, a_3, a_4 \in S^3$ are unit quaternions. Recall that the isometries of the nearly Kähler $S^3 \times S^3$ are $(p,q) \mapsto (apc^{-1}, bqc^{-1})$ for unit quaternions a, b and c. So by applying an appropriate isometry, p and q become

$$\begin{aligned} p(u,v) &= (r_1 \cos u''(u,v), r_1 \sin u''(u,v), r_2 \cos v''(u,v), r_2 \sin v''(u,v)), \\ q(u,v) &= (s_1 \cos u'(u,v), s_1 \sin u'(u,v), s_2 \cos v'(u,v), s_2 \sin v'(u,v))\, d, \end{aligned} \tag{43}$$

for some $d \in S^3$. We already know that for both maps p and q are congruent to homogenous tori and that moreover their Christoffel symbols with respect to the usual metric on S^3 vanish. Therefore the coordinates u', v' and u'', v'' are related by an affine transformation to the standard coordinates of the homogeneous torus. We thus have

$$\begin{aligned} u' &= b_1 u + b_2 v, & v' &= b_3 u + b_4 v, \\ u'' &= c_1 u + c_2 v, & v'' &= c_3 u + c_4 v. \end{aligned}$$

Without loss of generality, by applying a rotation and a homothety we may moreover assume that $c_1 = 1$, $c_2 = 0$.

6.3. **Proof of Theorem 1.3.** We first assume that neither $P(TM) \subset TM$ nor $P(TM) \perp TM$. The first step of the proof will be to apply the existence and uniqueness theorem given in Theorem 1.1. In this case the holomorphic differential does not vanish identically. We have seen that in this case $0 < \theta < \frac{\pi}{2}$ and that the connection coefficients of the basis e_1 and e_2 vanish. As such we can introduce a complex coordinate $z = x + iy$, by

$$\frac{\partial}{\partial z} = \partial z = \tfrac{1}{2}w(e_1 - ie_2),$$

where at the moment w is an arbitrary positive constant. As

$$g(\partial z, P\partial z) = 1/4w^2 g(e_1 - iJe_1, Pe_1 - iPJe_1) = \tfrac{1}{2}w^2 \cos\theta,$$

we see that taking $w = \frac{\sqrt{2}}{\sqrt{\cos\theta}}$ yields our preferred complex coordinate.

It then follows that

$$e^{\omega} = g(\partial z, \partial \bar{z}) = 1/2w^2 = \tfrac{1}{\cos\theta},$$

which implies that ω is a positive constant. Moreover, we have that

$$\begin{aligned} h(\partial z, \partial z) &= \tfrac{1}{4} w^2 h(e_1 - iJe_1, e_1 - iJe_1) \\ &= \tfrac{1}{\cos\theta}(h(e_1,e_1) - ih(e_1,e_2)) \\ &= \tfrac{1}{\cos\theta}(\cos\phi e_5 + \sin\phi e_6 - i\sin\phi e_5 + i\cos\phi e_6) \\ &= \tfrac{1}{\cos\theta} e^{i\phi}(e_5 - ie_6), \end{aligned}$$

and

$$\begin{aligned} G(\partial\bar{z}, P\partial z) &= \tfrac{1}{4} w^2 G(e_1 + iJe_1, Pe_1 - iPJe_1) \\ &= \tfrac{1}{2\cos\theta} G(e_1 + iJe_1, Pe_1 + iJPe_1) \\ &= \tfrac{1}{\cos\theta}(G(e_1, Pe_1) + iG(Je_1, Pe_1)) \\ &= \tfrac{1}{\cos\theta}(G(e_1, Pe_1) - iG(e_1, PJe_1)) \\ &= \tfrac{1}{\cos\theta}(e_5 - ie_6), \end{aligned}$$

which implies that $e^{i\phi} = \mu$. Note that these are indeed trivial solutions of the system of differential equations in Theorem 1.1 and that therefore for every value of $0 < \theta < \frac{\pi}{2}$ and $0 \leq \phi < 2\pi$ we get a unique solution. Therefore, in order to complete the proof in this case, by the uniqueness part of the theorem it is sufficient for value of $0 < \theta < \frac{\pi}{2}$ and $0 \leq \phi < 2\pi$ to give an example of a flat almost complex surface.

In order to do so we will look at the mean curvatures of the immersions p and q. We define the map

$$g_1 :]0, \tfrac{\pi}{2}[\times]0, 2\pi[\to \mathbb{R}^2 : (\theta,\phi) \mapsto \left(\tfrac{\csc\theta}{\sqrt{3}}\left(2\cos\theta\sin\left(\tfrac{\pi}{3} - \phi\right) - 1\right), -\tfrac{\csc\theta}{\sqrt{3}}\left(2\cos\theta\sin\left(\tfrac{\pi}{3} + \phi\right) + 1\right)\right).$$

Note that the image of the ϕ-lines under g_1 is the ellipse with equation

$$\tfrac{3}{4}(x+y)^2 + \tfrac{1}{4}(x-y)^2 + \sqrt{3}\csc\theta\,(x+y) + 1 = 0.$$

This implies first that the map is injective and that its image is given by

$$g_1(]0, \tfrac{\pi}{2}[\times]0, 2\pi[) = \{(x,y) \in \mathbb{R}^2 | x + y < 0\} \backslash \{(-\tfrac{1}{\sqrt{3}}, -\tfrac{1}{\sqrt{3}})\}.$$

On the other hand as we have seen before, the homogeneous torus

$$C_a = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| = \sqrt{\frac{1+a}{2}}, |z_2| = \sqrt{\frac{1-a}{2}}\},$$

where $a \in]-1,1[$ has mean curvature $\frac{a}{\sqrt{1-a^2}}$. Note that the function $a \mapsto \frac{a}{\sqrt{1-a^2}}$ is an odd function which is a bijection between $]-1,1[$ and $\mathbb{R}$. Moreover we have that $\frac{a}{\sqrt{1-a^2}} = -\frac{b}{\sqrt{1-b^2}}$ if and only $a + b = 0$. Also $\frac{a}{\sqrt{1-a^2}} = -\frac{1}{\sqrt{3}}$ if and only if $a = -\frac{1}{2}$. Therefore, in order to complete the proof of the theorem in this case, it is sufficient to give homogeneous tori p and q, with parameters a and b satisfying $a + b < 0$ and $(a,b) \neq (-\frac{1}{2}, -\frac{1}{2})$. A straightforward computation now implies that

$$p(u,v) = (\sqrt{\frac{1+a}{2}} e^{i(c_1 u + c_2 v)}, \sqrt{\frac{1-a}{2}} e^{i(c_3 u + c_4 v)}),$$

and

$$q(u,v) = (\sqrt{\frac{1+b}{2}} e^{i(b_1 u + b_2 v)}, \sqrt{\frac{1-a}{2}} e^{i(b_3 u + b_4 v)}) * i,$$

where $*$ denotes the quaternion multiplication and $c_1, c_2, c_3, c_4, b_1, b_2, b_3, b_4$ are constants given by

$$c_1 = 1, \tag{44}$$

$$c_2 = 0, \tag{45}$$

$$c_3 = \frac{a\left(2\sqrt{1-a^2}\sqrt{1-b^2}b - 2ab^2 + a\right) + 2b^2 - 1}{a^2\left(2b^2-1\right) - a\left(2\sqrt{1-a^2}\sqrt{1-b^2}b + b^2 - 1\right) + \sqrt{1-a^2}b\sqrt{1-b^2} - 1}, \tag{46}$$

$$c_4 = -\frac{\sqrt{3}\left(\sqrt{1-a^2}\sqrt{1-b^2}b - ab^2 + a\right)}{a^2\left(2b^2-1\right) - a\left(2\sqrt{1-a^2}\sqrt{1-b^2}b + b^2 - 1\right) + \sqrt{1-a^2}b\sqrt{1-b^2} - 1}, \tag{47}$$

$$b_1 = -\frac{\sqrt{1-a^2}\left(\sqrt{3}c_1 + 3c_2 + \sqrt{3}c_3 + 3c_4\right) + 2\sqrt{3-3b^2}(c_3 - c_1)}{4\sqrt{3-3b^2}}, \tag{48}$$

$$b_2 = \frac{\sqrt{3}\left(a^2 b\sqrt{1-b^2} + a\left(b^2-1\right)\left(\sqrt{1-a^2} + \sqrt{1-b^2}\right) + b\left(\sqrt{1-a^2}(b-1)b - \sqrt{1-a^2} - \sqrt{1-b^2}\right) + \sqrt{1-a^2}\right)}{2\sqrt{1-b^2}\left(a^2\left(1-2b^2\right) + a\left(2\sqrt{1-a^2}\sqrt{1-b^2}b + b^2 - 1\right) - \sqrt{1-a^2}b\sqrt{1-b^2} + 1\right)}, \tag{49}$$

$$b_3 = -\frac{\sqrt{1-a^2}\left(\sqrt{3}c_1 + 3c_2 + \sqrt{3}c_3 + 3c_4\right) + 2\sqrt{3-3b^2}(c_3 - c_1)}{4\sqrt{3-3b^2}} - c_1 + c_3, \tag{50}$$

$$b_4 = \frac{\sqrt{3}\left(a^2 b\sqrt{1-b^2} + a\left(b^2-1\right)\left(\sqrt{1-a^2} - \sqrt{1-b^2}\right) - b\left(\sqrt{1-a^2}b(b+1) - \sqrt{1-a^2} + \sqrt{1-b^2}\right) + \sqrt{1-a^2}\right)}{2\sqrt{1-b^2}\left(a^2\left(1-2b^2\right) + a\left(2\sqrt{1-a^2}\sqrt{1-b^2}b + b^2 - 1\right) - \sqrt{1-a^2}b\sqrt{1-b^2} + 1\right)}. \tag{51}$$

As

$$\begin{aligned} &-\frac{3}{4}\left(\sqrt{1-a^2}b + (a-1)\sqrt{1-b^2}\right)^2 - \frac{1}{4}\left(\sqrt{1-a^2}b + (a+1)\sqrt{1-b^2}\right)^2 \\ &\quad = a^2\left(2b^2-1\right) - a\left(2\sqrt{1-a^2}\sqrt{1-b^2}b + b^2 - 1\right) + \sqrt{1-a^2}b\sqrt{1-b^2} - 1 \end{aligned}$$

the above constants are actually well defined for any $(a,b) \in]-1,1[\times]-1,1[$. This completes the proof in the general case.

If $P(TM) \subset TM$, we know that M is congruent to the totally geodesic torus, which can be obtained from the previous formula by taking any allowable $(a,-a)$.

If $P(TM) \perp TM$ we know that the corresponding H-system is a flat constant mean curvature surface (with mean curvature $-\frac{2}{\sqrt{3}}$). Up to congruence in $\mathbb{R}^3$ such a surface is unique and therefore all such surfaces in $S^3 \times S^3$ must be mutually congruent. So again it is sufficient to give an example of such a surface which is obtained by taking $(a,b) = (-\frac{1}{2}, -\frac{1}{2})$ in the previous formulas.

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