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Braided algebras and their applications to Noncommutative Geometry

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Abstract

We introduce the notion of a braided algebra and study some examples of these. In particular, R -symmetric and R -skew-symmetric algebras of a linear space V equipped with a skew-invertible Hecke symmetry R are braided algebras. We prove the “mountain property” for the numerators and denominators of their Poincaré-Hilbert series (which are always rational functions).

Also, we further develop a differential calculus on modified Reflection Equation algebras. Thus, we exhibit a new form of the Leibniz rule for partial derivatives on such algebras related to involutive symmetries R . In particular, we present this rule for the algebra $U(\mathfrak{gl}(m))$. The case of the algebra $U(\mathfrak{gl}(2))$ and its compact form $U(\mathfrak{u}(2))$ (which can be treated as a deformation of the Minkowski space algebra) is considered in detail. On the algebra $U(\mathfrak{u}(2))$ we introduce the notion of the quantum radius, which is a deformation of the usual radius, and compute the action of rotationally invariant operators and in particular of the Laplace operator. This enables us to define analogs of the Laplace-Beltrami operators corresponding to certain Schwarzschild-type metrics and to compute their actions on the algebra $U(\mathfrak{u}(2))$ and its central extension. Some “physical” consequences of our considerations are presented.

AMS Mathematics Subject Classification, 2010: 17B37, 81R60

Key words: braiding, Hecke symmetry, Reflection Equation algebra, quantum radius, Schwarzschild metric, Laplace-Beltrami operator

1 Introduction

The aim of this paper is twofold. On the one hand, we introduce the notion of a braided algebra and further develop analysis on the Reflection Equation (RE) algebra, which is an example of such an algebra. On the other hand, we exhibit certain applications of the RE algebras to $U(\mathfrak{u}(2))$ -covariant Noncommutative (NC) Geometry.

By braided algebras we mean a subclass of unital algebras related to braidings, whereas the term braiding stands for an invertible operator $R : V^{\otimes 2} \rightarrow V^{\otimes 2}$ (V is a given finite dimensional linear space) satisfying the so-called Quantum Yang-Baxter Equation

$$R_{12} R_{23} R_{12} = R_{23} R_{12} R_{23}, \quad R_{12} = R \otimes I, \quad R_{23} = I \otimes R.$$

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Hereafter, $I \in \text{End}(V)$ is the identity operator and the lower indices indicate the factors of the tensor product on which a given operator acts.

Below, we are dealing with braidings satisfying an additional condition

$$(R - qI)(R + q^{-1}I) = 0, \quad q \in \mathbb{K},$$

where \mathbb{K} is the ground field (\mathbb{C} or \mathbb{R}) and the parameter q is assumed to be generic. Such braidings are called *Hecke symmetries*. For example, Hecke symmetries come from the quantum groups $U_q(\mathfrak{sl}(m))$. These Hecke symmetries (and other objects related to $U_q(\mathfrak{sl}(m))$) will be referred to as *standard* ones. Note, that quantum groups of the B , C and D series provide braidings of the Birman-Murakami-Wenzl type.

Our definition of braided algebras is given in terms of objects and morphisms of the so-called Schur-Weyl category $SW(V)$ generated by a given linear space V equipped with a skew-invertible Hecke symmetry $R : V^{\otimes 2} \rightarrow V^{\otimes 2}$ (see section 2). Typical examples of braided algebras are the R -symmetric, and the R -skew-symmetric algebras of the space V and the Reflection Equation (RE) algebra associated with a given Hecke symmetry R . By contrast, the well-known RTT algebra (which in the standard case is the *restricted dual* Hopf algebra to $U_q(\mathfrak{sl}(m))$) is not a braided algebra.

All aforementioned braided algebras have a good deformation property. For quadratic algebras this means that for a generic q the dimensions of their homogeneous components equal those for $q = 1$. As for the RE algebra, it has another important property: it admits a change of generators converting it into a quadratic-linear algebra (*modified* RE algebra) similar to an enveloping algebra. More precisely, in the standard case the modified RE algebra tends to the enveloping algebra $U(\mathfrak{gl}(m))$ as $q \rightarrow 1$. Also, if a Hecke symmetry R is a deformation of the super-flip on the super-space $V_{m,n}$, the corresponding modified RE algebra tends to the super-algebra $U(\mathfrak{gl}(m|n))$ as $q \rightarrow 1$. Besides, the RE algebra corresponding to a proper Hecke symmetry has a representation category looking like that of the algebra $U(\mathfrak{gl}(m))$ (or $U(\mathfrak{gl}(m|n))$) and turns into the latter one as $q \rightarrow 1$ (see [GPS1]). Due to this fact, generators of the RE algebra are good candidates for the role of vector fields. Such vector fields were introduced in the framework of quantum differential calculus on a pseudogroup (in fact, on the RTT algebra or on its compact version) in the series of papers [W, IP, FP].

In [GPS5] we suggested a more general calculus in which the role of the function algebra $\text{Fun}(GL(m))$ is played by a quantum matrix algebra while the role of vector fields is always played by a (modified) RE algebra. The most interesting particular case arises when the role of the algebra $\text{Fun}(GL(m))$ is taken by another copy of the RE algebra. In this case we get a braided analog of the Heisenberg double¹ constructed from two copies of the RE algebra. In the paper [GPS6] we used such a "braided Heisenberg double" in order to define braided analogs of partial derivatives on the RE algebra playing the role of $\text{Fun}(GL(m))$. Besides, by passing to the limit $q \rightarrow 1$ in the standard case, we constructed the partial derivatives on the NC algebras $U(\mathfrak{gl}(m))$.

These derivatives differ from the usual ones, defined on the commutative algebra $\text{Sym}(\mathfrak{gl}(m))$, by a modification of the Leibniz rule. Below, we exhibit a form of this rule well adapted to algebras $U(\mathfrak{gl}(m))$ and their generalized analogs associated with any skew-invertible involutive symmetry² R . Our construction includes the differential calculus (with the standard Leibniz rule) on the super-algebras $U(\mathfrak{gl}(m|n))$. Thus, by making use of braided algebras we get some elements of NC Geometry on algebras having no "braided ingredient" in their construction, the latter algebras are covariant with respect to the usual groups (or their super-analogs).

In the particular case $m = 2$, $n = 0$ we get a version of the differential calculus on the algebra $U(u(2))$ ³. An amazing fact is that partial derivatives defined on the algebra $U(u(2)_{\hbar})$ commute

¹The conventional Heisenberg double consists of two Hopf algebras dual to each other.

²We say that a braiding R is an involutive symmetry if $R^2 = I$.

³In fact, we deal with the algebra $U(u(2)_{\hbar})$ where the notation \mathfrak{g}_{\hbar} means that a deformation multiplier \hbar is introduced in the Lie bracket of a given Lie algebra \mathfrak{g} .

with each other. Consequently, any classical differential operator with constant coefficients can be easily re-defined on this algebra. Thus, analogs of the Klein-Gordon, Maxwell and Dirac operators on the algebra $U(u(2)_\hbar)$ ⁴ have forms just the same as the classical ones but with a new meaning of the partial derivatives.

The situation is more complicated for operators with coefficients from $\text{Sym}(u(2))$. In order to transfer such an operator to the algebra $U(u(2)_\hbar)$ we have to use a "quantizing map" $\text{Sym}(u(2)) \rightarrow U(u(2)_\hbar)$. Such a map enables us to quantize a differential operator with coefficients from $\text{Sym}(u(2))$ to an operator with coefficients from $U(u(2)_\hbar)$ and acting on this algebra.

Nevertheless, many interesting (and physically meaningful) operators have coefficients belonging to a larger algebra. As an example of such an algebra \mathcal{A} we consider the product $\mathbb{K}(t, r) \otimes \text{Sym}(u(2))$ where $\mathbb{K}(t, r)$ is the algebra of rational functions in the time t and the radius $r = \sqrt{x^2 + y^2 + z^2}$ (hereafter x, y, z are spatial variables). In particular, the Laplace-Beltrami operator corresponding to the Schwarzschild metric is of this form. We introduce a quantum counterpart \mathcal{A}_\hbar of the algebra \mathcal{A} by defining a quantum analog \hat{r} of the radius r . It should be emphasized that the "quantum radius" \hat{r} is naturally extracted from the Cayley-Hamilton identity valid for the generating matrix of the algebra $U(u(2)_\hbar)$. (In a more general context, such an identity is valid for the generating matrices of the RE algebras, see [GPS2, GPS3, GPS4].) Then, we extend the quantizing map to a map $\alpha : \mathcal{A} \rightarrow \mathcal{A}_\hbar$.

Let \mathcal{D} be a differential operator with coefficients from \mathcal{A} . By applying the quantizing map α to its coefficients we get the differential operator denoted $\alpha(\mathcal{D})$. Its coefficients belong to the algebra \mathcal{A}_\hbar and the derivatives $\partial_t, \dots, \partial_z$ appearing in this operator are assumed to act on the algebra $U(u(2)_\hbar)$. We show how to extend some operators of this form to the algebra \mathcal{A}_\hbar : in particular we do this for the Laplace-Beltrami operator, corresponding to the Schwarzschild metric.

Note that the quantized operators we are dealing with, being expressed via the commutative variables t and \hat{r} , become difference operators. Thus, in contrast with usual differential operators, we can consider them on a lattice, i.e. on a discrete space-time. In a sense, its discreteness is a consequence of its noncommutativity. Another "physical" observation is that the space and time on the algebra $U(u(2)_\hbar)$ do not exist independently of one another. Indeed, the partial derivative in t involves an action on the spatial variables and vice versa. It seems to be plausible that a similar NC calculus does not exist on the algebra $U(su(2)_\hbar)$, i.e. on the NC space without the time.

The paper is organized as follows. In the next section we introduce the notion of a braided algebra and consider some basic examples. In particular, we go back to the question of the possible form of a Hecke symmetry and prove the *mountain property* for the numerator and denominator of the Poincaré-Hilbert series corresponding to the (skew-)symmetric algebra defined via a skew-invertible Hecke symmetry. In section 3 we introduce Weyl algebras defined on RE algebras and consider their $q \rightarrow 1$ limits. Here, our main objective is to give a version of the Leibniz rule useful for further applications. In section 4 we introduce Laplace and other rotationally invariant operators acting on the RE algebras in question. We treat the case of the algebra $U(u(2)_\hbar)$ (which is a limit of a standard modified RE algebra) in detail. On this basis, in section 5 we construct a NC analog of a model describing the dynamics of a scalar massless field in a space equipped with a Schwarzschild-type metric. We consider this example as the starting point for a study of higher dimensional (matrix) models. Also note, that our approach is valid for the RE algebras (modified or not) but computations become much more complicated.

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⁴This algebra can be seen as a NC deformation of the Minkowski space algebra, though it does not admit any reasonable analog of the Lorentz subgroup.

2 Braided quadratic algebras: basic examples

Let $R = R(q) : V^{\otimes 2} \rightarrow V^{\otimes 2}$ be a Hecke symmetry analytically depending on the parameter q in a neighborhood \mathcal{V} (assumed to be connected) of 1. Note that the symmetry $R(1)$ is involutive. Let us associate with this Hecke symmetry two quotient algebras of the free tensor algebra $T(V)$ of the space V : R -symmetric $\text{Sym}_R(V)$ and R -skew-symmetric algebra $\bigwedge_R(V)$ defined as follows

$$\text{Sym}_R(V) = T(V)/\langle \text{Im}(qI - R) \rangle, \quad \bigwedge_R(V) = T(V)/\langle \text{Im}(q^{-1}I + R) \rangle. \quad (2.1)$$

Hereafter, the symbol $\langle X \rangle$ stands for the two-sided ideal generated by a subset X .

Though the complete classification of Hecke symmetries is an open problem, some information concerning their possible forms can be drawn from the Poincaré-Hilbert (PH) series of the algebras introduced above

$$\mathcal{P}_+(t) = \sum_k \dim \text{Sym}_R^{(k)}(V) t^k, \quad \mathcal{P}_-(t) = \sum_k \dim \bigwedge_R^{(k)}(V) t^k,$$

where $A^{(k)}$ stands for the degree k homogenous component of a graded algebra A .

Constructed in the paper [G2], were two series of projectors $P_{\pm}^{(k)}$ which are q -analogs of the usual symmetrization and skew-symmetrization operators:

$$P_+^{(k)} : V^{\otimes k} \rightarrow \text{Sym}_R^{(k)}(V), \quad P_-^{(k)} : V^{\otimes k} \rightarrow \bigwedge_R^{(k)}(V), \quad k \geq 2.$$

Namely, if the parameter q is subject to the condition

$$n_q = \frac{q^n - q^{-n}}{q - q^{-1}} \neq 0, \quad n \in \mathbb{N} \quad (2.2)$$

then there are isomorphisms of vector spaces

$$\text{Sym}_R^{(k)}(V) \cong \text{Im } P_+^{(k)}, \quad \bigwedge_R^{(k)}(V) \cong \text{Im } P_-^{(k)}.$$

Since $\dim \text{Sym}_R^{(k)}(V)$ and $\dim \bigwedge_R^{(k)}(V)$ are lower semi-continuous functions in q while the functions $\dim \text{Im } P_{\pm}^{(k)}$ are upper semi-continuous, we conclude that these functions are constant on the connected neighborhood \mathcal{V} with the possible exception of the values of q violating the condition (2.2). Moreover, as was shown in [G2] for a generic q (i.e. for all q excepting a countable set not including 1) the relation

$$\mathcal{P}_+(t) \mathcal{P}_-(-t) = 1$$

is valid. Hereafter, we consider the PH series $\mathcal{P}_{\pm}(t)$ at generic values of the parameter q .

Let us recall, that a braiding R is called skew-invertible if there exists an endomorphism $\Psi : V^{\otimes 2} \rightarrow V^{\otimes 2}$ such that

$$\text{Tr}_{(2)} R_{12} \Psi_{23} = P_{13} = \text{Tr}_{(2)} \Psi_{12} R_{23}. \quad (2.3)$$

Hereafter, P stands for the usual flip. Note that if a Hecke symmetry $R(q)$ is skew-invertible for a value of the parameter q , it is skew-invertible for all generic q .

A skew-invertible braiding $R : V^{\otimes 2} \rightarrow V^{\otimes 2}$ can be extended to a braiding

$$R : (V \oplus V^*)^{\otimes 2} \rightarrow (V \oplus V^*)^{\otimes 2}, \quad (2.4)$$

where we keep for the extended braiding the same notation R . In the above formula the space V^* is dual to V , i.e. the spaces V and V^* are equipped with a nondegenerated R -invariant pairing $\langle, \rangle : V \otimes V^* \rightarrow \mathbb{K}$. The pairing is R -invariant if the following property holds true (below the space W stands for either V or V^*)

$$\begin{aligned} R\langle, \rangle_{23} &= \langle, \rangle_{12} R_{23} R_{12} & \text{on} & \quad W \otimes V \otimes V^* \\ R\langle, \rangle_{12} &= \langle, \rangle_{23} R_{12} R_{23} & \text{on} & \quad V \otimes V^* \otimes W, \end{aligned} \quad (2.5)$$

where we assume that the ground field \mathbb{K} commutes with V and V^* in the usual way:

$$R(w \otimes k) = k \otimes w, \quad R(k \otimes w) = w \otimes k, \quad \forall k \in \mathbb{K}, w \in W.$$

The prolongation (2.4) exists and is unique. In order to describe it explicitly, we fix dual basis $\{x_i\}$ and $\{x^j\}$ in the spaces V and V^* respectively:

$$\langle x_i, x^j \rangle = \delta_i^j.$$

Remark 1 Note that in the space V^* one can define right and left dual basis to a given basis of V . We are dealing with the right dual basis.

Let $\{x_i \otimes x_j\}$ be the corresponding basis of the space $V^{\otimes 2}$. In this basis the skew-invertible Hecke symmetry $R: V^{\otimes 2} \rightarrow V^{\otimes 2}$ is represented by a $(\dim V)^2 \times (\dim V)^2$ matrix $\|R_{ij}^{kl}\|$

$$R(x_i \otimes x_j) = x_k \otimes x_l R_{ij}^{kl}.$$

Hereafter, summation over repeated indices is always assumed.

Then the extension (2.4) can be represented by the following matrices on the components $V^{\otimes 2}$, $(V^*)^{\otimes 2}$, $V \otimes V^*$ and $V^* \otimes V$ of the tensor square $(V \oplus V^*)^{\otimes 2}$

$$\begin{aligned} R(x_i \otimes x_j) &= x_k \otimes x_l R_{ij}^{kl}, & R(x^i \otimes x^j) &= x^k \otimes x^l R_{lk}^{ji}, \\ R(x_i \otimes x^j) &= x^k \otimes x_l (R^{-1})_{ki}^{lj}, & R(x^j \otimes x_i) &= x_k \otimes x^l \Psi_{li}^{kj}. \end{aligned} \quad (2.6)$$

It is easy to show that the embedding $\mathbb{K} \rightarrow V^* \otimes V$ generated by the correspondence

$$1 \mapsto \sum_i x^i \otimes x_i \quad (2.7)$$

is R -invariant.

Let us introduce two endomorphisms B and C of the space V via the morphism Ψ (2.3) as $B := \text{Tr}_{(1)} \Psi$, $C := \text{Tr}_{(2)} \Psi$ or, in coordinate form

$$B_i^j = \Psi_{ki}^{kj}, \quad C_i^j = \Psi_{ik}^{jk}. \quad (2.8)$$

Note that if R is a super-flip, the operators B and C are equal to each other and coincide with the parity operator.

Proposition 2 ([H, Da, DH]). *The HP series $\mathcal{P}_-(t)$ (and hence $\mathcal{P}_+(t)$) is a rational function:*

$$\mathcal{P}_-(t) = \frac{N(t)}{D(t)} = \frac{1 + a_1 t + \dots + a_m t^m}{1 - b_1 t + \dots + (-1)^n b_n t^n} = \frac{\prod_{i=1}^m (1 + x_i t)}{\prod_{j=1}^n (1 - y_j t)}, \quad (2.9)$$

where a_i and b_i are positive integers, the polynomials $N(t)$ and $D(t)$ are coprime, and all the numbers x_i and y_i are real positive.

If, in addition, the Hecke symmetry is skew-invertible, then the polynomials $N(t)$ and $D(-t)$ are reciprocal.

Recall, that a polynomial $p(t) = c_0 + c_1 t + \dots + c_n t^n$ is called *reciprocal* if $p(t) = t^n p(t^{-1})$ or, equivalently, $c_i = c_{n-i}$, $0 \leq i \leq n$. Note, that if a real number $z \neq 1$ is a root of a reciprocal polynomial, the number z^{-1} is a root as well with the same multiplicity. This entails that the polynomials $N(t)$ and $D(-t)$ factorize into a product of terms of the form

$$(1 + t) \quad \text{and} \quad (1 + zt)(1 + z^{-1}t) = 1 + ct + t^2 \quad \text{where} \quad c \geq 2. \quad (2.10)$$

For such a polynomial the following claim is valid.

Proposition 3 *Any polynomial which is a product of terms listed in (2.10) has the "mountain property": its coefficients strictly increase up to the middle (and, consequently, strictly decrease after it).*

The proof of the above proposition can be easily obtained by induction on the number of the factors in such a polynomial.

Proposition 3 provides new information on the possible form of a Hecke symmetry. In particular, if a skew-invertible Hecke symmetry is even (see definition 4 below) then its HP series $\mathcal{P}_-(t)$ is a polynomial possessing the mountain property.

However, we do not know whether the coefficients c in the multipliers (2.10) entering the factorized form of $N(t)$ and $D(-t)$ must be integer: we do not know, for example, whether there exists a Hecke symmetry such that its HP series $\mathcal{P}_-(t)$ is equal to

$$1 + 10t + 12t^2 + 10t^3 + t^4 = (1 + (5 + \sqrt{15})t + t^2)(1 + (5 - \sqrt{15})t + t^2).$$

Nevertheless, if the monic polynomials $N(t)$ and $D(-t)$ are products of factors (2.10) with integer middle coefficients c , then there exists a Hecke symmetry $R(q)$ for which $\mathcal{P}_-(t) = \frac{N(t)}{D(t)}$. It can be constructed by methods of [G2].

Definition 4 Let $R : V^{\otimes 2} \rightarrow V^{\otimes 2}$ be a skew-invertible Hecke symmetry. Let m (respectively n) be the degree of the numerator (respectively denominator) of the corresponding PH series $\mathcal{P}_-(t)$. The couple $(m|n)$ is called the *bi-rank* of the space V or of the Hecke symmetry R . If the bi-rank is $(m|0)$ (respectively $(0|n)$) the corresponding Hecke symmetry is called *even* (respectively *odd*).

This notion is a generalization of the super-dimension for super-spaces and equals the latter provided R is a super-flip.

Now, consider the monoidal quasitensor rigid category introduced in the paper [GPS1]. This category is generated by the spaces V and V^* . Following [GPS1] we call it the *Schur-Weyl* category and denote by the symbol $SW(V)$. We briefly describe the category $SW(V)$ with minor modifications comparing with [GPS1].

Let

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k), \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k, \quad \sum \lambda_i = k$$

be a partition of an integer $k \geq 2$. There is known a Schur functor $V \rightarrow V_\lambda$ which assigns a space $V_\lambda \subset V^{\otimes k}$ to the *basic* space V . In [GPS1] "braided analogs" of the spaces V_λ were introduced as images of some projectors $P_\lambda : V^{\otimes k} \rightarrow V_\lambda$ constructed via a given Hecke symmetry R . In general, there is a family of equivalent projectors (and, consequently, isomorphic spaces V_λ) which differ from each other by embeddings into the space $V^{\otimes k}$. By V_λ we mean any of these spaces. In a similar manner the subspaces $V_\mu^* \subset (V^*)^{\otimes k}$ can be introduced.

By definition, the class $\text{Ob}(SW(V))$ of objects of the category $SW(V)$ includes all spaces V_λ , V_μ^* for all possible partitions λ and μ as well as all their tensor products and direct sums of these products⁵.

Now, we want to describe the family $\text{Mor}(SW(V))$ of morphisms of this category. For this purpose, we note that any object of the category can be given a comodule structure over the so-called RTT algebra. This is an associative unital algebra generated by the elements t_i^j , $1 \leq i, j \leq \dim V$, which are subject to the quadratic relations

$$RT_1 T_2 = T_1 T_2 R, \quad T_1 = T \otimes I, \quad T_2 = I \otimes T, \quad T = \|\!| t_i^j \|\!|.$$

⁵Note that for any given Hecke symmetry R with a bi-rank $(m|n)$ some of the projectors P_λ are zero operators (and, consequently, spaces V_λ and V_λ^* vanish). If the Hecke symmetry R is even or odd, the category $SW(V)$ can be introduced with the use of the only space V (see [G2] for detail).

For arbitrary R this is a bialgebra, but for the standard R the RTT algebra is a Hopf algebra⁶ restricted dual to the quantum universal enveloping algebra $U_q(\mathfrak{gl}(m))$ $m = \dim V$ (see [FRT] for detail).

In general, if R is an even Hecke symmetry, this bialgebra structure can be also extended up to the Hopf one (see [GPS1] for detail). Then the comodule structure on the basis space V , on its dual V^* and on $V \otimes V^* \cong \text{End}(V)$ reads

$$x_i \mapsto t_i^k \otimes x_k, \quad x^j \mapsto s_k^j \otimes x^k, \quad x_i x^j \mapsto t_i^k s_p^j \otimes x_k x^p, \quad (2.11)$$

where s_k^j are entries of the "inverse matrix" $S(T)$. Here S is the antipode. The comodule structure on an arbitrary $U \in \text{Ob}(SW(V))$ is now easily defined since the above coaction can be naturally extended up to any tensor products of V and V^* and it commutes with the projectors P_λ .

By definition, a linear map $\beta : U_1 \rightarrow U_2$, $U_1, U_2 \in \text{Ob}(SW(V))$ belongs to $\text{Mor}(SW(V))$ iff it commutes with the coaction of the RTT algebra on the spaces U_1 and U_2 . In the standard case the RTT coaction can be changed for the action of the quantum group $U_q(\mathfrak{gl}(m))$. So, in this case a morphism is, by definition, a linear map commuting with the action of the quantum group.

For instance, it turns out that the initial skew-invertible braiding R can be naturally extended up to a family of linear maps

$$R_{U_1, U_2} : U_1 \otimes U_2 \rightarrow U_2 \otimes U_1, \quad \forall U_1, U_2 \in \text{Ob}(SW(V)).$$

All these maps commute with the RTT coaction and form the family of *permutation morphisms* of the Schur-Weyl category.

There exists another way of introducing the set $\text{Mor}(SW(V))$ (see [GPS1]) but that given above is technically more useful.

Definition 5 Algebras of the form

$$T(U)/\langle \beta(W) \rangle, \quad U, W \in \text{Ob}(SW(V)),$$

where W is a subspace of the free tensor algebra $T(U)$ and $\beta : W \rightarrow W$ is an element of $\text{Mor}(SW(V))$ are called braided. The relations of the form $\beta(W) = 0$ are called *admissible*.

It is clear that the algebra $\text{Sym}_R(V)$ (respectively $\Lambda_R(V)$) is braided. To show this we set $U = V$, $W = V^{\otimes 2}$ and $\beta = qI - R$ (respectively $\beta = q^{-1}I + R$).

Remark 6 If $U \in \text{Ob}(SW(V))$ is an object, then the direct sum $U \oplus U$ is an object as well. In order to distinguish these two copies of the object U we denote one of them by U' . Let $W \subset T(U \oplus U')$ be a subspace of the form $(U \otimes U) \oplus (U' \otimes U') \oplus (U \otimes U' \oplus U' \otimes U)$. Let β be a map such that its restrictions to each of the three above components are morphisms. Then it is a morphism and the corresponding braided algebra is defined by three sets of relations:

$$\beta(U \otimes U) = 0, \quad \beta(U' \otimes U') = 0, \quad \beta(U \otimes U' \oplus U' \otimes U) = 0.$$

Below, we consider such algebras constructed from two copies of an object and a morphism β of the indicated form. The third set of relations plays the role of permutation relations between the algebras $T(U)/\langle \beta(U \otimes U) \rangle$ and $T(U')/\langle \beta(U' \otimes U') \rangle$.

Now, we consider the object $\text{End}(V) \cong V \otimes V^* \in \text{Ob}(SW(V))$ and exhibit some admissible relations on the object $\text{End}(V)^{\otimes 2}$. Below, $L = \|l_i^j\|$ is a matrix and each of the entries l_i^j is identified with the element $x_i \otimes x^j \in V \otimes V^*$.

⁶To introduce the Hopf structure one needs an antipodal map in the bialgebra. For this purpose, the RTT bialgebra should be extended by the element $(\det_R T)^{-1}$, where $\det_R T$ is the so-called quantum determinant.

Proposition 7 *The relations*

$$R^{\varepsilon_1} L_1 R^{\varepsilon_2} L_1 - L_1 R^{\varepsilon_3} L_1 R^{\varepsilon_4} = 0, \quad (2.12)$$

where R is a Hecke symmetry and $\varepsilon_i \in \{1, -1\}$, $i = 1, 2, 3, 4$ are admissible.

Proof. In virtue of the identification $l_i^j = x_i \otimes x^j \in V \otimes V^*$, the coaction (2.11) leads to the following transformation for elements of the matrix L : $l_i^j \mapsto t_i^k s_p^j \otimes l_k^p$. This coaction can be symbolically written in the matrix form

$$L \mapsto TLS, \quad (2.13)$$

where we assume that the entries of the matrix L commute with those of S and T .

According to Definition 5 we have to verify, that the linear span, generated by the left hand side⁷ of (2.12) is an invariant subcomodule in $\text{End}(V)^{\otimes 2}$ with respect to the coaction (2.13). This can easily be done with the use of the following direct consequence of the RTT relations

$$S_1 R^\varepsilon T_1 = T_2 R^\varepsilon S_2, \quad \varepsilon \in \{1, -1\}.$$

Note that the RTT-type relations on the elements l_i^j

$$R_{12} L_1 L_2 - L_1 L_2 R_{12} = 0$$

are not admissible since the corresponding linear span is not invariant comodule with respect to the coaction (2.13). So, the RTT algebra is not braided (according to the definition above). ■

Definition 8 The algebra $\mathcal{L}(R)$ generated by matrix element of the matrix $L = \|l_i^j\|$ subject to a particular case of the relations (2.12)

$$R L_1 R L_1 - L_1 R L_1 R = 0 \quad (2.14)$$

is called the Reflection Equation (RE) algebra. The matrix L is called the generating matrix of the algebra $\mathcal{L}(R)$.

Remark 9 The relation (2.12) with $\varepsilon_1 = \varepsilon_4$ and $\varepsilon_2 = \varepsilon_3$ is equivalent to (2.14) up to a possible change of R for R^{-1} . Note that R^{-1} is a braiding. Besides, R^{-1} is a Hecke symmetry if and only if R is also.

It was discovered by Sh. Majid that any RE algebra has a braided bialgebra structure (see [M]). This structure is defined via a braiding $R_{\text{End}(V)} : \text{End}(V)^{\otimes 2} \rightarrow \text{End}(V)^{\otimes 2}$ extended to the whole RE algebra. Note that in the standard case this braiding is the image of the $U_q(\mathfrak{sl}(m))$ universal R-matrix in the space $\text{End}(V)^{\otimes 2}$.

The relation $R_{\text{End}(V)}(L \otimes L) - L \otimes L = 0$ is admissible since it can be rewritten as a particular case of (2.12)

$$R L_1 R^{-1} L_1 = L_1 R L_1 R^{-1}.$$

However, in contrast with the RE algebra, the algebra defined by this system of relations does not in general possess the good deformation property in the sense of [GPS2].

In this connection we would like to mention the paper [BZ] where another way was suggested of defining analogs of symmetric and skew-symmetric algebras of irreducible modules over the quantum groups. Note that except for the case $m = 2$ the ‘‘R-symmetric algebra’’ of the adjoint $U_q(\mathfrak{sl}(m))$ -module as defined in [BZ] does not possess the good deformation property.

⁷With the use of endomorphism Ψ (see definition 2.3) or its analog $\Psi_{R^{-1}}$ for the inverse matrix R^{-1} (see (4.1) below) we can write this linear span as the image of the map $\beta = I - \gamma$, where $\gamma : \text{End}(V)^{\otimes 2} \rightarrow \text{End}(V)^{\otimes 2}$ reads

$$\gamma(L_1 L_2) = \text{Tr}_{(0)} \left(R_{10}^{-\varepsilon_1} L_0 R_{10}^{\varepsilon_3} L_0 R_{10}^{\varepsilon_4} (\Psi_{R^{\varepsilon_2}})_{02} \right) P_{12}.$$

3 Braided Weyl algebras and their $q \rightarrow 1$ limits

Let $R = R(q)$ be a skew-invertible Hecke symmetry and $\mathcal{L}(R)$ the corresponding RE algebra defined by relations (2.14) on their generators. We perform the following linear change of the generators

$$L = \hbar I - (q - q^{-1}) \tilde{L}, \quad \tilde{L} = \|\tilde{l}_i^j\|. \quad (3.1)$$

Then the system (2.14) being rewritten via the matrix \tilde{L} becomes

$$R \tilde{L}_1 R \tilde{L}_1 - \tilde{L}_1 R \tilde{L}_1 R = \hbar (R \tilde{L}_1 - \tilde{L}_1 R). \quad (3.2)$$

We call the algebra defined by generators \hat{l}_i^j subject to (3.2) the *modified* Reflection Equation algebra. In fact, it is nothing but another form of the RE algebra.

Now, passing to the limit $q \rightarrow 1$ we get just the system (3.2) but with an involutive braiding $R(1)$. Assuming $R(1)$ to be the *usual* flip P we get the defining relations of the algebra $U(\mathfrak{gl}(m|n)_{\hbar})$. If a Hecke symmetry R is a deformation of the *super-flip* acting on the space $V_{m,n}^{\otimes 2}$ where $V_{m,n} = V_0 \oplus V_1$, $\dim V_0 = m$, $V_1 = n$, we get the enveloping algebra $U(\mathfrak{gl}(m|n)_{\hbar})$ as $q \rightarrow 1$. Thus, though the algebras $\text{Sym}(\mathfrak{gl}(m|n))$ and $U(\mathfrak{gl}(m|n)_{\hbar})$ are not isomorphic to each other, their q -deformations (2.14) and (3.2) are.

In general, as $q \rightarrow 1$ we get an algebra which can be considered as the enveloping algebra of a generalized Lie algebra. Such algebras have been introduced by one of the authors in [G1] (also see [G2]).

Now, we consider an associative unital algebra $\mathcal{W}(\mathcal{N})$ generated by entries n_i^j and d_i^j of two $\dim V \times \dim V$ matrices $N = \|n_i^j\|$ and $D = \|d_i^j\|$ satisfying the following relations

$$\begin{aligned} R N_1 R N_1 - N_1 R N_1 R &= \hbar (R N_1 - N_1 R), \\ R^{-1} D_1 R^{-1} D_1 &= D_1 R^{-1} D_1 R^{-1}, \\ D_1 R N_1 R - R N_1 R^{-1} D_1 &= R + \hbar D_1 R. \end{aligned} \quad (3.3)$$

We call the algebra $\mathcal{W}(\mathcal{N})$ the *braided Weyl algebra*. Our terminology is motivated by the fact that in the case $q \rightarrow 1$ and $\hbar \rightarrow 0$ (provided R is the standard Hecke symmetry) the algebra $\mathcal{W}(\mathcal{N})$ is isomorphic to the usual Weyl algebra generated by commutative indeterminates n_i^j and the partial derivatives $\partial/\partial n_i^j$ (see (3.12–3.14)). Note that the algebra $\mathcal{W}(\mathcal{N})$ was introduced in [GPS6]. It is just the braided Heisenberg double mentioned in the Introduction.

The algebra $\mathcal{W}(\mathcal{N})$ possesses two subalgebras. The modified RE subalgebra \mathcal{N} , generated by the matrix $N = \|n_i^j\|$, plays the role of a function algebra. The other subalgebra \mathcal{D} , generated by the matrix $D = \|d_i^j\|$, is an RE algebra corresponding to the Hecke symmetry R^{-1} . The matrix elements d_i^j are braided analogs of the partial derivatives on the algebra \mathcal{N} . The third set of relations in the system (3.3) (the permutation relations) plays the role of the Leibniz rule.

In order to show that the algebra defined by (3.3) is a braided algebra indeed, we identify entries of the matrices N and D with elements of $\text{End}(V)$ as above. Consequently, the coaction of the RTT algebra is given by the same formula (2.13) on the both matrices. We leave to the reader checking that the relations (3.3) are admissible (see Remark 6).

We define the action of the generators d_i^j on the unit $1_{\mathcal{N}}$ of the subalgebra \mathcal{N} by setting⁸ $d_i^j \triangleright 1_{\mathcal{N}} = 0$. Now, the result of action $d_i^j \triangleright n \equiv d_i^j \triangleright (n \cdot 1_{\mathcal{N}})$ on an arbitrary element $n \in \mathcal{N}$ can be found by permuting d_i^j with n and then evaluating it on the unit element $1_{\mathcal{N}}$ according to the above rule.

Equivalently, the latter operation can be presented in terms of the counit map $\varepsilon : \mathcal{D} \rightarrow \mathbb{K}$ defined by the following rule

$$\varepsilon(1_{\mathcal{D}}) = 1, \quad \varepsilon(d_i^j) = 0, \quad \varepsilon(d_1 d_2) = \varepsilon(d_1) \varepsilon(d_2), \quad d_1, d_2 \in \mathcal{D}. \quad (3.4)$$

⁸Hereafter, we denote the action of an operator A on an element x in two ways: either $A \triangleright x$ or $A(x)$ depending on our convenience.

Checking that the operation \triangleright is well defined on the algebra \mathcal{N} (i.e. the set of defining relations of the algebra \mathcal{N} is invariant with respect to this action) follows from the results of [GPS5].

Proposition 10 *The action of generators d_i^j on the subalgebra \mathcal{N} completed with the natural action of the unit $1_{\mathcal{D}} \triangleright n = n$ can be extended up to the action of the whole subalgebra \mathcal{D} on \mathcal{N} : $\mathcal{D} \otimes \mathcal{N} \xrightarrow{\triangleright} \mathcal{N}$. This action gives a representation of the algebra \mathcal{D} in \mathcal{N} .*

This proposition also follows from [GPS5].

Let now R be an involutive symmetry. Using the permutation relations among D and N we derive a form of the Leibniz rule convenient for our subsequent considerations.

We introduce the following chains of R -matrices:

$$\mathcal{R}_{kp} = R_{p-1}R_{p-2} \dots R_{k+1}R_kR_{k+1} \dots R_{p-2}R_{p-1}, \quad 1 \leq k < p.$$

The Yang-Baxter equation for R allows us to rewrite the above definition in an equivalent form

$$\mathcal{R}_{kp} = R_kR_{k+1} \dots R_{p-2}R_{p-1}R_{p-2} \dots R_{k+1}R_k, \quad 1 \leq k < p.$$

Besides, we assume that $\mathcal{R}_{kp} = \mathcal{R}_{pk}$ by definition.

For any i, j, k we have the following "index exchange rules", well known for the usual flip operators:

$$\mathcal{R}_{ij}\mathcal{R}_{ik} = \mathcal{R}_{jk}\mathcal{R}_{ij} = \mathcal{R}_{ik}\mathcal{R}_{kj}.$$

For any braiding $R : V^{\otimes 2} \rightarrow V^{\otimes 2}$ and any $\dim V \times \dim V$ matrix N we introduce a convenient notation:

$$N_{\bar{k}} = R_{k-1} \dots R_1 N_1 R_1^{-1} \dots R_{k-1}^{-1}. \quad (3.5)$$

Then, the following relations are valid

$$\begin{aligned} \mathcal{R}_{pk} N_{\bar{k}} &= N_{\bar{p}} \mathcal{R}_{pk}, & k > p, \\ \mathcal{R}_{kp} N_{\bar{k}} &= N_{\bar{p}} \mathcal{R}_{kp}, & k < p, \\ \mathcal{R}_{pk} N_{\bar{s}} &= N_{\bar{s}} \mathcal{R}_{pk}, & \forall s \notin \{p, k\}. \end{aligned} \quad (3.6)$$

For an involutive symmetry R the relation (3.5) can be presented as

$$N_{\bar{k}} = R_{k-1} \dots R_1 N_1 R_1 \dots R_{k-1}.$$

Now, we rewrite the permutation relations of D and N (the third line in (3.3)) in the form

$$D_1 N_{\bar{2}} = N_{\bar{2}} D_1 + \hbar D_1 R_1 + R_1. \quad (3.7)$$

It is convenient to modify the generating matrix D as follows

$$\tilde{D} = \hbar^{-1} \text{Id} + D. \quad (3.8)$$

Then, the permutation rules (3.7) take the form

$$\tilde{D}_1 N_{\bar{2}} = N_{\bar{2}} \tilde{D}_1 + \hbar \tilde{D}_1 \mathcal{R}_{12}.$$

The above relation can be easily generalized to

$$\tilde{D}_1 N_{\bar{k}} = N_{\bar{k}} \tilde{D}_1 + \hbar \tilde{D}_1 \mathcal{R}_{1k}, \quad \forall k \geq 2. \quad (3.9)$$

The action of \tilde{D} on any element of \mathcal{N} can be obtained from the same scheme as above with the only modification: $\varepsilon(\tilde{D}) = \hbar^{-1}$.

Now, with the use of (3.9) we get the commutation of \tilde{D} with some low degree polynomials in N

$$\begin{aligned}\tilde{D}_1 N_{\bar{2}} N_{\bar{3}} &= N_{\bar{2}} N_{\bar{3}} \tilde{D}_1 + \hbar \left(N_{\bar{2}} \tilde{D}_1 \mathcal{R}_{13} + N_{\bar{3}} \tilde{D}_1 \mathcal{R}_{12} \right) + \hbar^2 \tilde{D}_1 \mathcal{R}_{12} \mathcal{R}_{23}, \\ \tilde{D}_1 N_{\bar{2}} N_{\bar{3}} N_{\bar{4}} &= N_{\bar{2}} N_{\bar{3}} N_{\bar{4}} \tilde{D}_1 + \hbar \left(N_{\bar{2}} N_{\bar{3}} \tilde{D}_1 \mathcal{R}_{14} + N_{\bar{2}} N_{\bar{4}} \tilde{D}_1 \mathcal{R}_{13} + N_{\bar{3}} N_{\bar{4}} \tilde{D}_1 \mathcal{R}_{12} \right) \\ &\quad + \hbar^2 \left(N_{\bar{2}} \tilde{D}_1 \mathcal{R}_{13} \mathcal{R}_{34} + N_{\bar{3}} \tilde{D}_1 \mathcal{R}_{12} \mathcal{R}_{24} + N_{\bar{4}} \tilde{D}_1 \mathcal{R}_{12} \mathcal{R}_{23} \right) + \hbar^3 \tilde{D}_1 \mathcal{R}_{12} \mathcal{R}_{23} \mathcal{R}_{34}.\end{aligned}$$

These relations immediately give us the action of the matrix D on the same polynomials:

$$\begin{aligned}D_1 \triangleright N_{\bar{2}} &= \mathcal{R}_{12}, \\ D_1 \triangleright N_{\bar{2}} N_{\bar{3}} &= N_{\bar{2}} \mathcal{R}_{13} + N_{\bar{3}} \mathcal{R}_{12} + \hbar \mathcal{R}_{12} \mathcal{R}_{23}, \\ D_1 \triangleright N_{\bar{2}} N_{\bar{3}} N_{\bar{4}} &= N_{\bar{2}} N_{\bar{3}} \mathcal{R}_{14} + N_{\bar{2}} N_{\bar{4}} \mathcal{R}_{13} + N_{\bar{3}} N_{\bar{4}} \mathcal{R}_{12} \\ &\quad + \hbar \left(N_{\bar{2}} \mathcal{R}_{13} \mathcal{R}_{34} + N_{\bar{3}} \mathcal{R}_{12} \mathcal{R}_{24} + N_{\bar{4}} \mathcal{R}_{12} \mathcal{R}_{23} \right) + \hbar^2 \mathcal{R}_{12} \mathcal{R}_{23} \mathcal{R}_{34}.\end{aligned}$$

To write down the general result we introduce some more notation. First, for any set of integers $2 \leq k_1 < k_2 < \dots < k_s \leq p$ we denote

$$\mathcal{R}_{(1k_1 \dots k_s)} = \mathcal{R}_{1k_1} \mathcal{R}_{k_1 k_2} \dots \mathcal{R}_{k_{s-1} k_s},$$

an analog of the cycle permutation $(1k_1 k_2 \dots k_s)$ in the permutation group S_p . Second, for the same set of integers we denote

$$(N_{\bar{2}} \dots N_{\bar{p}})^{(k_1 \dots k_s)} = N_{\bar{2}} \dots N_{\overline{k_1-1}} N_{\overline{k_1+1}} \dots N_{\overline{k_s-1}} N_{\overline{k_s+1}} \dots N_{\bar{p}},$$

that is the product $N_{\bar{2}} \dots N_{\bar{p}}$ where the multipliers $N_{\overline{k_1}}, N_{\overline{k_2}}, \dots, N_{\overline{k_s}}$ are omitted.

Then, the following result can be proved by induction in $p \geq 2$

$$\tilde{D}_1 N_{\bar{2}} \dots N_{\bar{p}} = N_{\bar{2}} \dots N_{\bar{p}} \tilde{D}_1 + \sum_{s=1}^{p-1} \hbar^s \sum_{2 \leq k_1 < \dots < k_s \leq p} (N_{\bar{2}} \dots N_{\bar{p}})^{(k_1 \dots k_s)} \tilde{D}_1 \mathcal{R}_{(1k_1 \dots k_s)}. \quad (3.10)$$

Finally, we get the following formula for the action of the matrix D :

$$D_1 \triangleright N_{\bar{2}} \dots N_{\bar{p}} = \sum_{s=1}^{p-1} \hbar^{s-1} \sum_{2 \leq k_1 < \dots < k_s \leq p} (N_{\bar{2}} \dots N_{\bar{p}})^{(k_1 \dots k_s)} \mathcal{R}_{(1k_1 \dots k_s)}. \quad (3.11)$$

Now, we consider the particular case $R = P$, where P is the usual flip. Then, for the first order monomials formula (3.11) gives

$$D_1 \triangleright N_2 = P_{12} \quad \text{or} \quad d_i^j \triangleright n_k^l = P_{ik}^{jl} = \delta_i^l \delta_k^j. \quad (3.12)$$

First, note that if $\hbar = 0$ (and, therefore, $\mathcal{N} \cong \text{Sym}(gl(m))$) formula (3.11) assumes that the action (3.12) is extended to an arbitrary order monomial via the classical Leibniz rule. This allows us to identify $d_i^j = \partial / \partial n_k^j$.

Second, for $\hbar \neq 0$ we consider the $GL(m)$ -covariant product

$$n_i^j \circ n_k^l = \hbar \delta_k^j n_i^l, \quad (3.13)$$

well defined in the algebra $gl(m)_\hbar$. The Lie bracket in $gl(m)_\hbar$ is related to this product in the usual way

$$[n_i^j, n_k^l] = n_i^j \circ n_k^l - n_k^l \circ n_i^j. \quad (3.14)$$

Now, in the case $R = P$ formula (3.11) can be obtained by a sequence of several steps. At the first step we apply an element d_i^j to a given monomial $n_{i_1}^{j_1} n_{i_2}^{j_2} \dots n_{i_p}^{j_p}$ in accordance with the rule (3.12) and the classical Leibniz rule. This gives us the sum of $p - 1$ terms corresponding to the value $s = 1$ of the summation index in (3.11).

At the second step we choose an arbitrary pair of elements $n_{i_s}^{j_s}$ and $n_{i_t}^{j_t}$, $1 \leq s < t \leq p$, in the initial monomial and calculate their product (3.13) preserving the order of the elements. After that we apply d_i^j to the result of this product:

$$d_i^j \triangleright (n_{i_s}^{j_s} \circ n_{i_t}^{j_t}) = \hbar \delta_i^{j_t} \delta_{i_t}^{j_s} \delta_{i_s}^j \quad \text{or} \quad D_1 \triangleright (N_s \circ N_t) = \hbar P_{1s} P_{st}.$$

This gives us $(p - 1)(p - 2)/2$ terms proportional to the first order of the parameter \hbar .

In order to get the \hbar^2 -order terms we apply the same procedure to all triples of the factors in the given monomial. Thus, any triple of chosen elements $n_{i_s}^{j_s}$, $n_{i_t}^{j_t}$, $n_{i_r}^{j_r}$ is replaced by the numerical factor $\hbar^2 \delta_i^{j_r} \delta_{i_r}^{j_t} \delta_{i_t}^{j_s} \delta_{i_s}^j$ or $\hbar^2 P_{1s} P_{st} P_{tr}$. And so on: at the k -th step we obtain the terms proportional to \hbar^{k-1} .

Note, that in [GPS6] the above procedure of calculating the action of an element d_i^j on a monomial in n (the “ \hbar -Leibniz rule”) was defined in a different but equivalent way via a coproduct $\Delta(d_i^j) = d_i^j \otimes 1 + 1 \otimes d_i^j + \hbar d_k^j \otimes d_i^k$ which was found by S. Meljanac and Z. Škoda.

Remark 11 Note, that the method of computing the action of an element d_i^j on monomials can be readily generalized to super-algebras $U(\mathfrak{gl}(m|n)_\hbar)$. The only difference is that all transpositions of factors in monomials (we have to put them aside in order to apply the product \circ) and their transpositions with elements d_i^j must be done with taking into account the parity of elements.

A similar version of the Leibniz rule is valid for any ”generalized Lie algebra” corresponding to a skew-invertible involutive symmetry R (see [GPS1]). In this case all aforementioned transpositions must be performed in terms of the involutive symmetry coming in the definition of such a generalized Lie algebra. Unfortunately, if the initial symmetry R is not involutive (i.e. $q \neq 1$), such a simple form of the Leibniz rule is not found.

Our scheme of defining partial derivatives can be also generalized to the enveloping algebras of current Lie algebra $\widehat{\mathfrak{gl}(m)}$ (and their super-analogs) since the Lie brackets in these Lie algebras can be realized via a product \circ analogous to (3.13). We do not know any way of defining similar partial derivatives on the enveloping algebras of Lie algebras without such a product, for instance on $U(\mathfrak{sl}(m))$.

4 Invariant differential operators, an example

Let R be a skew-invertible Hecke symmetry. It is well known that the elements $\text{Tr}_R L^k = \text{Tr}(CL^k)$ (the matrix C is defined in (2.8)), are central in the algebra $\mathcal{L}(R)$ for any k . Note, that a similar statement is valid for a modified RE algebra, in particular, for the algebra \mathcal{N} coming in the definition (3.3) of the Weyl algebra $\mathcal{W}(\mathcal{N})$. Namely, the elements $\text{Tr}_R N^k$ are central in this algebra. We call the map $L^k \mapsto \text{Tr}_R L^k$ *braided trace*.

The subalgebra $\mathcal{D} \subset \mathcal{W}(\mathcal{N})$ is also an RE algebra with R replaced by R^{-1} . For any Hecke symmetry R with the bi-rank $(m|n)$ we have $R^{-1} = R - (q - q^{-1})I$ and (see [GPS1])

$$\text{Tr } B = \text{Tr } C = \frac{(m - n)_q}{q^{m-n}}.$$

Emphasize that the braiding R^{-1} is also skew-invertible since the corresponding endomorphism $\Psi_{R^{-1}}$ exists and can be expressed via Ψ (see (2.3)) by the relation

$$\Psi_{R^{-1}} = \Psi_{12} + (q - q^{-1})q^{2(m-n)} C_1 B_2. \quad (4.1)$$

This formula enables us to compute the endomorphisms $B_{R^{-1}}$ and $C_{R^{-1}}$. Namely, we have

$$B_{R^{-1}} = \text{Tr}_{(1)} \Psi_{R^{-1}} = q^{2(m-n)} B, \quad C_{R^{-1}} = \text{Tr}_{(2)} \Psi_{R^{-1}} = q^{2(m-n)} C.$$

Thus, the matrices $B_{R^{-1}}$ and $C_{R^{-1}}$ are proportional to these B and C respectively and the elements $\text{Tr}_R D^k$ differ from central elements $\text{Tr}_{R^{-1}} D^k = \text{Tr}(C_{R^{-1}} D^k)$ of the subalgebra \mathcal{D} by the factor $q^{-2(m-n)} \neq 0$. So, we associate "braided differential operators" with these elements by treating matrix elements d_i^j of the matrix D to be derivations on the algebra \mathcal{N} as was described in the previous section. We call $\text{Tr}_R D^k$ *Laplace operators*.

In the same way, we define *sl-type Laplace operators* associated with the elements $\text{Tr}_R \tilde{D}^k$ where

$$\tilde{D} = D - \frac{\text{Tr}_R D}{\text{Tr}_R I} I \quad (4.2)$$

is the R-traceless part of the matrix D . Since

$$\text{Tr}_R I = \text{Tr} C = q^{n-m}(m-n)_q,$$

the operator \tilde{D} is well defined provided $m \neq n$ and q is subject to the condition (2.2). Note, that the families of the operators D^k and \tilde{D}^k can be expressed via each other.

Also, we need the operators $\text{Tr}_R(N^l D^k)$, $\text{Tr}_R(N^l \tilde{D}^k)$ and $\text{Tr}_R(\tilde{N}^l \tilde{D}^k)$ which are called *invariant*. In the standard case they are invariant with respect to the action of the quantum group $U_q(\mathfrak{sl}(m))$. In general, they are invariant with respect to the coaction of the corresponding RTT algebra.

We are interested in the action of the Laplace operators on the algebra \mathcal{N} , in particular, on its center $Z(\mathcal{N})$.

Conjecture 12 *All the Laplace operators $\text{Tr}_R D^k$ map the center $Z(\mathcal{N})$ into itself.*

Below, we consider invariant differential operators in a particular case $R = P$, $m = 2$, $n = 0$. Also we assume $\mathbb{K} = \mathbb{R}$. The case $\mathbb{K} = \mathbb{C}$ can be treated as the complexification of the objects and morphisms over the real field.

Let us recall some results and notations from [GPS6]. Consider the algebra $U(u(2)_{\hbar})$ with the following defining relations between generators

$$[x, y] = \hbar z, \quad [y, z] = \hbar x, \quad [z, x] = \hbar y, \quad [t, x] = [t, y] = [t, z] = 0.$$

These generators are connected with the generators a , b , c and d of the algebra $U(\mathfrak{gl}(2)_{\hbar})$ in the standard way

$$a = t - iz, \quad b = -ix - y, \quad c = -ix + y, \quad d = t + iz. \quad (4.3)$$

The center $Z(U(u(2)_{\hbar}))$ is generated by t and the Casimir element

$$\text{Cas} = x^2 + y^2 + z^2.$$

First, we consider the algebra $U(\mathfrak{su}(2)_{\hbar})$ which is known to be flat over its center (see [D]). More precisely, as an $\mathfrak{su}(2)$ -module it is isomorphic to the following direct sum of $\mathfrak{su}(2)$ -modules

$$U(\mathfrak{su}(2)_{\hbar}) \cong \bigoplus_{k=0}^{\infty} (Z(U(\mathfrak{su}(2)_{\hbar})) \otimes V^k), \quad (4.4)$$

where V^k is the $\mathfrak{su}(2)$ -module of the highest weight k . We fix its highest weight vector in the complexification of the algebra $U(\mathfrak{su}(2)_{\hbar})$ to be $b^k = (-ix - y)^k$.

Remark 13 In fact, the module V^k contains the elements $\text{Re}(b^k)$ and $\text{Im}(b^k)$.

Definition 14 The component $Z(U(\mathfrak{su}(2)_{\hbar})) \otimes V^k$ is called isotypic.

The algebra $U(u(2)_{\hbar})$ is the first ingredient of the Weyl algebra we are going to construct. The second ingredient is the algebra \mathcal{D} generated by the partial derivatives $\partial_x, \partial_y, \partial_z$ and $\tilde{\partial}_t = \partial_t + \frac{2}{\hbar} I$. This shift of the t -derivative is convenient for technical reasons.

The properties of the partial derivatives follow from the defining relations (3.3). Namely, one can show (see [GPS6] for detail) that the partial derivatives commute with each other, while their permutation relations with the generators x, y, z and t read

$$\begin{aligned}
\tilde{\partial}_t t - t \tilde{\partial}_t &= \frac{\hbar}{2} \tilde{\partial}_t & \tilde{\partial}_t x - x \tilde{\partial}_t &= -\frac{\hbar}{2} \partial_x & \tilde{\partial}_t y - y \tilde{\partial}_t &= -\frac{\hbar}{2} \partial_y & \tilde{\partial}_t z - z \tilde{\partial}_t &= -\frac{\hbar}{2} \partial_z \\
\partial_x t - t \partial_x &= \frac{\hbar}{2} \partial_x & \partial_x x - x \partial_x &= \frac{\hbar}{2} \tilde{\partial}_t & \partial_x y - y \partial_x &= \frac{\hbar}{2} \partial_z & \partial_x z - z \partial_x &= -\frac{\hbar}{2} \partial_y \\
\partial_y t - t \partial_y &= \frac{\hbar}{2} \partial_y & \partial_y x - x \partial_y &= -\frac{\hbar}{2} \partial_z & \partial_y y - y \partial_y &= \frac{\hbar}{2} \tilde{\partial}_t & \partial_y z - z \partial_y &= \frac{\hbar}{2} \partial_x \\
\partial_z t - t \partial_z &= \frac{\hbar}{2} \partial_z & \partial_z x - x \partial_z &= \frac{\hbar}{2} \partial_y & \partial_z y - y \partial_z &= -\frac{\hbar}{2} \partial_x & \partial_z z - z \partial_z &= \frac{\hbar}{2} \tilde{\partial}_t.
\end{aligned} \tag{4.5}$$

Our next aim is to compute the action of the invariant operators $\tilde{\partial}_t$ and $\text{Tr}_R \tilde{D}^2$ on elements t^k and $fb^k, \forall f \in Z(U(su(2)_{\hbar}))$ from the complexification of the algebra $U(u(2)_{\hbar})$. Since the action of these operators is $SU(2)$ -invariant (or $SL(2)$ -invariant if we consider the complexification), we easily extend this action up to the isotypic component $Z(U(su(2)_{\hbar})) \otimes V^k$ and, consequently, on the whole algebra $U(u(2)_{\hbar})$ due to the isomorphism (4.4). Note, that on the algebra $U(su(2)_{\hbar})$ the Laplace operator $\text{Tr}_R \tilde{D}^2$ coincides (up to a numerical factor) with the invariant differential operator

$$\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2.$$

Below we deal with Δ instead of $\text{Tr}_R \tilde{D}^2$.

In contrast with the classical case $\hbar = 0$, when the function algebra is $\text{Sym}(u(2))$, the computation of the action of an invariant operator on $U(u(2)_{\hbar})$ involves other invariant operators. Bellow, we will deal with two invariant first order differential operators

$$\tilde{\partial}_t \quad \text{and} \quad Q = x \partial_x + y \partial_y + z \partial_z \tag{4.6}$$

and with four invariant second order differential operators

$$\Delta_0 = \tilde{\partial}_t^2, \quad \Delta_1 = \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2, \quad \Delta_2 = Q \tilde{\partial}_t, \quad \Delta_3 = Q^2. \tag{4.7}$$

It is evident, that any polynomial in the above operators with coefficients from $Z(U(u(2)_{\hbar}))$ is also an invariant operator.

First, we find the permutation relations of the operators (4.6) and (4.7) with the central elements $t^k, k \in \mathbb{N}$:

$$\tilde{\partial}_t t^k = (t + \frac{\hbar}{2})^k \tilde{\partial}_t, \quad Q t^k = (t + \frac{\hbar}{2})^k Q, \quad \Delta_i t^k = (t + \hbar)^k \Delta_i, \quad \forall i = 0, 1, 2, 3. \tag{4.8}$$

Second, the permutation relations of the first order operators (4.6) with the Casimir element Cas are as follows

$$\begin{aligned}
\tilde{\partial}_t \text{Cas} &= \left(\text{Cas} - \frac{3}{4} \hbar^2 \right) \tilde{\partial}_t - \hbar Q, \\
Q \text{Cas} &= \hbar \text{Cas} \tilde{\partial}_t + \left(\text{Cas} + \frac{\hbar^2}{4} \right) Q,
\end{aligned}$$

or in the matrix form

$$\begin{pmatrix} \tilde{\partial}_t \\ Q \end{pmatrix} \text{Cas} = \begin{pmatrix} \text{Cas} - \frac{3\hbar^2}{4} & -\hbar \\ \hbar \text{Cas} & \text{Cas} + \frac{\hbar^2}{4} \end{pmatrix} \begin{pmatrix} \tilde{\partial}_t \\ Q \end{pmatrix}. \tag{4.9}$$

We denote $\Phi = \Phi(\text{Cas})$ the 2×2 matrix coming in this formula. Then the permutation relations of the operators (4.6) with the central element Cas^p , $p \in \mathbb{N}$ can be expressed via the p -th power of the matrix Φ .

Now, recall that the generating matrix

$$N = \begin{pmatrix} t - iz & -ix - y \\ -ix + y & t + iz \end{pmatrix} \quad (4.10)$$

of the algebra $U(u(2)_\hbar)$ satisfies the Cayley-Hamilton (CH) identity of the form

$$N^2 - (2t + \hbar)N + (t^2 + x^2 + y^2 + z^2 + \hbar t)I = 0.$$

Let μ_1 and μ_2 obey the relations

$$\mu_1 + \mu_2 = 2t + \hbar, \quad \mu_1 \mu_2 = t^2 + x^2 + y^2 + z^2 + \hbar t.$$

This means that the variables μ_1 and μ_2 belong to an algebraic extension of the center $Z(U(u(2)_\hbar))$. We call them the *eigenvalues* of the matrix N . Note, that our ordering of the eigenvalues is arbitrary.

These eigenvalues are useful for parameterizing all central elements. In particular, for the Casimir element Cas we get

$$\text{Cas} = \frac{\hbar^2 - \mu^2}{4}, \quad \mu = \mu_1 - \mu_2. \quad (4.11)$$

Also, we need the eigenvalues λ_1 and λ_2 of the matrix $\Phi(\text{Cas})$

$$\lambda_1 = \frac{\mu}{4}(2\hbar - \mu), \quad \lambda_2 = -\frac{\mu}{4}(2\hbar + \mu).$$

The matrix $\Phi(\text{Cas})$ has the following spectral decomposition

$$\Phi(\text{Cas}) = \lambda_1 \cdot \frac{\Phi - \lambda_2}{\lambda_1 - \lambda_2} + \lambda_2 \cdot \frac{\Phi - \lambda_1}{\lambda_2 - \lambda_1} = \lambda_1 P_1(\Phi) + \lambda_2 P_2(\Phi),$$

where the explicit form of the matrices $P_i(\Phi)$ is as follows

$$P_1(\Phi) = \frac{1}{\mu} \begin{pmatrix} \frac{\mu - \hbar}{2} & -1 \\ \frac{\hbar^2 - \mu^2}{4} & \frac{\mu + \hbar}{2} \end{pmatrix}, \quad P_2(\Phi) = \frac{1}{\mu} \begin{pmatrix} \frac{\mu + \hbar}{2} & 1 \\ \frac{\mu^2 - \hbar^2}{4} & \frac{\mu - \hbar}{2} \end{pmatrix}.$$

The matrices $P_i(\Phi)$, $i = 1, 2$, are complementary projectors, i.e. they satisfy the relations

$$P_i(\Phi)P_j(\Phi) = \delta_{ij}P_i(\Phi), \quad P_1(\Phi) + P_2(\Phi) = I.$$

The spectral decomposition enables us to compute the matrix $\Phi^p(\text{Cas})$ explicitly:

$$\begin{aligned} \Phi^p &= \lambda_1^p P_1(\Phi) + \lambda_2^p P_2(\Phi) = \\ & \frac{\lambda_1^p}{\mu} \begin{pmatrix} \frac{\mu - \hbar}{2} & -1 \\ \frac{\hbar^2 - \mu^2}{4} & \frac{\mu + \hbar}{2} \end{pmatrix} + \frac{\lambda_2^p}{\mu} \begin{pmatrix} \frac{\mu + \hbar}{2} & 1 \\ \frac{\mu^2 - \hbar^2}{4} & \frac{\mu - \hbar}{2} \end{pmatrix}. \end{aligned} \quad (4.12)$$

Now, we are able to compute the action of the operators (4.6) on the elements $\text{Cas}^p b^k$, $p, k \in \mathbb{N}$. To this end we first apply these operators to the elements b^k .

Proposition 15 *The following relations hold true*

$$\tilde{\partial}_t(b^k) = \frac{2}{\hbar}b^k, \quad Q(b^k) = kb^k \quad (4.13)$$

and, consequently, $\partial_t(b^k) = 0$.

Proof. To prove formulae (4.13) it is sufficient to use the \hbar -Leibniz rule, described at the end of the previous section. Namely, since $b \circ b = 0$ (see (3.13)), the action of derivatives on b is, actually, classical (the terms proportional to \hbar are absent). This immediately involves the result (4.13). ■

Thus, in order to compute the action of $\tilde{\partial}_t$ and Q on $\text{Cas}^p b^k$ we transpose these operators with the element Cas^p with the help of (4.9) and then apply them to the element b^k in accordance with (4.13). Finally, we get

$$\begin{pmatrix} \tilde{\partial}_t \\ Q \end{pmatrix} (\text{Cas}^p b^k) = \Phi^p \begin{pmatrix} \frac{2}{\hbar} \\ k \end{pmatrix} b^k = b^k \begin{pmatrix} \frac{1}{\hbar}(\lambda_1^p + \lambda_2^p) - \frac{(\lambda_1^p - \lambda_2^p)}{\mu}(k+1) \\ (\lambda_1^p + \lambda_2^p)\frac{k}{2} + \frac{(\lambda_1^p - \lambda_2^p)}{\mu} \left(\frac{\hbar(k+1)}{2} - \frac{\mu^2}{2\hbar} \right) \end{pmatrix}$$

or, in more detail,

$$\begin{aligned} \partial_t(\text{Cas}^p b^k) &= b^k \left(\frac{1}{\hbar}(\lambda_1^p + \lambda_2^p - 2) - \frac{(\lambda_1^p - \lambda_2^p)}{\mu}(k+1) \right), \\ Q(\text{Cas}^p b^k) &= b^k \left((\lambda_1^p + \lambda_2^p)\frac{k}{2} + \frac{(\lambda_1^p - \lambda_2^p)}{\mu} \left(\frac{\hbar(k+1)}{2} - \frac{\mu^2}{2\hbar} \right) \right). \end{aligned} \quad (4.14)$$

These formulae enable us to compute the action of the operators (4.6) on an element $f(\text{Cas}) b^k$ where $f(u)$ is a polynomial or a formal series in one variable. This action is defined by formula (4.14) where λ_1^p and λ_2^p are respectively replaced by $f(\lambda_1)$ and $f(\lambda_2)$.

However, we are interested in the action of the operators (4.6) on μ and its integer powers. To find this action, we take into account that $\mu = \sqrt{\hbar^2 - 4\text{Cas}}$ and choose $f(u) = (\sqrt{\hbar^2 - 4u})^p$, where $p \in \mathbb{Z}$ is an integer (may be negative). Then we get

$$f(\lambda_1) = \left(\sqrt{(\mu - \hbar)^2} \right)^p = (\mu - \hbar)^p, \quad f(\lambda_2) = \left(\sqrt{(\mu + \hbar)^2} \right)^p = (\mu + \hbar)^p.$$

Our choice of the roots in these formulae is motivated by the classical limit $\hbar \rightarrow 0$. Finally, we get the following proposition.

Proposition 16 *The following relations take place (here $k \in \mathbb{N}$ and $p \in \mathbb{Z}$)*

$$\begin{aligned} \partial_t(\mu^p b^k) &= b^k \left(\frac{1}{\hbar} \left((\mu - \hbar)^p + (\mu + \hbar)^p - 2 \right) - \frac{(\mu - \hbar)^p - (\mu + \hbar)^p}{\mu} (k+1) \right), \\ Q(\mu^p b^k) &= b^k \left(\left((\mu - \hbar)^p + (\mu + \hbar)^p \right) \frac{k}{2} + \frac{(\mu - \hbar)^p - (\mu + \hbar)^p}{\mu} \left(\frac{\hbar(k+1)}{2} - \frac{\mu^2}{2\hbar} \right) \right). \end{aligned}$$

For any rational function $f(u)$ the action of the operators ∂_t and Q on elements $f(\mu) b^k$ can be generalized in the obvious way: we replace $(\mu - \hbar)^p$ and $(\mu + \hbar)^p$ in the right hand side of the above formulae by $f(\mu - \hbar)$ and $f(\mu + \hbar)$ respectively. Consequently, by assuming $f(t, \mu)$ to be a rational function in two variables we define the action of the operators ∂_t and Q on elements $f(t, \mu) b^k$ by similar formulae but with t in the right hand side replaced by $t + \frac{\hbar}{2}$.

Remark 17 As follows from the proposition 16, the operators $b^{-k}\partial_t b^k$ and $b^{-k}Qb^k$ (gauge equivalent to ∂_t and Q respectively) are well defined on the space of polynomials (series) in μ . Also, observe that the factor b^k in all formulae above can be replaced by any element from the module V^k .

Now, we turn to computing the action of the second order operators (4.7) on the elements $\text{Cas}^p b^k$. To this end we reproduce the permutation relations of these operators with the Casimir element Cas .

Proposition 18 [GPS6] *The following permutation relations hold true:*

$$\Delta_i \text{Cas} = \sum_{j=0}^3 \Pi_{ij} \Delta_j, \quad 0 \leq i \leq 3, \quad (4.15)$$

where the matrix $\Pi = \Pi(\text{Cas})$ reads

$$\Pi = \begin{pmatrix} \text{Cas} - \frac{3}{2}\hbar^2 & \frac{\hbar^2}{2} & -2\hbar & 0 \\ \frac{3}{2}\hbar^2 & \text{Cas} - \frac{\hbar^2}{2} & 2\hbar & 0 \\ \hbar \text{Cas} & 0 & \text{Cas} - \frac{\hbar^2}{2} & -\hbar \\ \hbar^2 \text{Cas} & -\frac{\hbar^2}{2} \text{Cas} & \hbar \left(2\text{Cas} + \frac{\hbar^2}{4} \right) & \text{Cas} + \frac{\hbar^2}{2} \end{pmatrix}.$$

The matrix Π plays the same role for the operators (4.7) as Φ plays for the operators (4.6). Thus, we get

$$\Delta_i \text{Cas}^p = \sum_{j=0}^3 (\Pi^p)_{ij} \Delta_j, \quad p \in \mathbb{N}.$$

By a direct calculation one can verify that the matrix Π is semisimple:

$$\Pi \sim \text{diag}(\lambda_0, \lambda_0, \lambda_+, \lambda_-),$$

where (recall that $\mu = \mu_1 - \mu_2$)

$$\lambda_0 = \frac{1}{4}(\hbar^2 - \mu^2), \quad \lambda_{\pm} = \frac{1}{4}(\hbar^2 - (\mu \pm 2\hbar)^2).$$

Similarly to (4.12) we get

$$\Pi^p = \sum_{a=0,\pm} \lambda_a^p \prod_{\substack{b=0,\pm \\ b \neq a}} \frac{(\Pi - \lambda_b I)}{(\lambda_a - \lambda_b)} = \lambda_0^p P_0(\Pi) + \lambda_+^p P_+(\Pi) + \lambda_-^p P_-(\Pi). \quad (4.16)$$

Here $P_a(\Pi)$, $a \in \{0, +, -\}$, are complementary projectors:

$$P_a(\Pi)P_b(\Pi) = \delta_{ab}P_a(\Pi), \quad P_0(\Pi) + P_+(\Pi) + P_-(\Pi) = I.$$

Their explicit form reads:

$$P_0(\Pi) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{\hbar}{\hbar^2 - \mu^2} & \frac{2}{\hbar^2 - \mu^2} \\ \frac{1}{2} & 1 & \frac{-\hbar}{\hbar^2 - \mu^2} & \frac{-2}{\hbar^2 - \mu^2} \\ \frac{-\hbar}{4} & \frac{\hbar}{4} & \frac{-\hbar^2}{\hbar^2 - \mu^2} & \frac{-2\hbar}{\hbar^2 - \mu^2} \\ \frac{2\hbar^2 - \mu^2}{8} & \frac{-\hbar^2}{8} & \frac{\hbar(3\hbar^2 - \mu^2)}{4(\hbar^2 - \mu^2)} & \frac{3\hbar^2 - \mu^2}{2(\hbar^2 - \mu^2)} \end{pmatrix}$$

$$P_+(\Pi) = \begin{pmatrix} \frac{2\hbar + \mu}{4\mu} & \frac{-\hbar}{4\mu} & \frac{\frac{3}{2}\hbar + \mu}{\mu(\hbar + \mu)} & \frac{1}{\mu(\hbar + \mu)} \\ -\frac{2\hbar + \mu}{4\mu} & \frac{\hbar}{4\mu} & -\frac{\frac{3}{2}\hbar + \mu}{\mu(\hbar + \mu)} & \frac{-1}{\mu(\hbar + \mu)} \\ \frac{-2\hbar^2 + \hbar\mu + \mu^2}{8\mu} & \frac{\hbar(\hbar - \mu)}{8\mu} & \frac{-3\hbar^2 + \hbar\mu + 2\mu^2}{4\mu(\hbar + \mu)} & \frac{-\hbar + \mu}{2\mu(\hbar + \mu)} \\ \frac{-2\hbar^2 + \hbar\mu + \mu^2}{16} & \frac{\hbar(\hbar - \mu)}{16} & \frac{-3\hbar^2 + \hbar\mu + 2\mu^2}{8(\hbar + \mu)} & \frac{-\hbar + \mu}{4(\hbar + \mu)} \end{pmatrix}$$

$$P_-(\Pi) = \begin{pmatrix} \frac{-2\hbar + \mu}{4\mu} & \frac{\hbar}{4\mu} & \frac{\frac{3}{2}\hbar - \mu}{\mu(\mu - \hbar)} & \frac{1}{\mu(\mu - \hbar)} \\ \frac{2\hbar - \mu}{4\mu} & \frac{-\hbar}{4\mu} & \frac{-\frac{3}{2}\hbar + \mu}{\mu(\mu - \hbar)} & \frac{-1}{\mu(\mu - \hbar)} \\ \frac{2\hbar^2 + \hbar\mu - \mu^2}{8\mu} & \frac{-\hbar(\hbar + \mu)}{8\mu} & \frac{3\hbar^2 + \hbar\mu - 2\mu^2}{4\mu(\hbar - \mu)} & \frac{\hbar + \mu}{2\mu(\hbar - \mu)} \\ \frac{-2\hbar^2 - \hbar\mu + \mu^2}{16} & \frac{\hbar(\hbar + \mu)}{16} & \frac{-3\hbar^2 - \hbar\mu + 2\mu^2}{8(\hbar - \mu)} & \frac{-\hbar + \mu}{4(\hbar - \mu)} \end{pmatrix}$$

Now, compute the action of the operators Δ_i on the elements b^k . Applying the same reasoning as in the proposition 15, we get that the \hbar -Leibniz rule reduces to the classical one. Thus, we come to the proposition below.

Proposition 19 *The following action holds true*

$$\Delta_0(b^k) = \frac{4}{\hbar^2} b^k, \quad \Delta_1(b^k) = 0, \quad \Delta_2(b^k) = \frac{2k}{\hbar} b^k, \quad \Delta_3(b^k) = k^2 b^k.$$

Thus, the action of the operators ∂_t , Q , Δ on the elements b^k and, consequently, on the whole modules V^k are given by the same formulae as in the classical case. Therefore, the elements from V^k are *harmonic* in the usual sense of the words: they are killed by the operators $\tilde{\partial}_t$ and Δ .

Using the same method as above, we can compute the action of the operators (4.7) on elements $\text{Cas}^p b^k$. Namely, we have

$$\begin{pmatrix} \Delta_0 \\ \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix} (\text{Cas}^p b^k) = b^k \left(\lambda_0^p P_0(\Pi) + \lambda_+^p P_+(\Pi) + \lambda_-^p P_-(\Pi) \right) \begin{pmatrix} \frac{4}{\hbar^2} \\ 0 \\ \frac{2k}{\hbar} \\ k^2 \end{pmatrix}.$$

Now, we want to extend this formula to elements $\mu^p b^k$, $p \in \mathbb{Z}, k \in \mathbb{N}$. Since

$$\sqrt{\hbar^2 - 4\lambda_0} = \mu, \quad \sqrt{\hbar^2 - 4\lambda_{\pm}} = \mu \pm 2\hbar$$

(as above the signs are motivated by the classical limit) we have

$$\begin{pmatrix} \Delta_0 \\ \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{pmatrix} (\mu^p b^k) = b^k \left(\mu^p P_0(\Pi) + (\mu + 2\hbar)^p P_+(\Pi) + (\mu - 2\hbar)^p P_-(\Pi) \right) \begin{pmatrix} \frac{4}{\hbar^2} \\ 0 \\ \frac{2k}{\hbar} \\ k^2 \end{pmatrix}. \quad (4.17)$$

This enables us to compute the action of the operators in question on elements $f(\mu)b^k$ where $f(u)$ is a rational function. For example, the action of the operator $\Delta_1 = \Delta$ in the case $k = 0$ is as follows:

$$\Delta(f(\mu)) = \frac{1}{\hbar^2} \left(2f(\mu) - f(\mu - 2\hbar) - f(\mu + 2\hbar) \right) + \frac{2}{\mu\hbar} \left(f(\mu - 2\hbar) - f(\mu + 2\hbar) \right). \quad (4.18)$$

This formula together with (4.8) entails that if f is a rational function in two variables then

$$\begin{aligned} \Delta(f(t, \mu)) &= \frac{1}{\hbar^2} \left(2f(t + \hbar, \mu) - f(t + \hbar, \mu - 2\hbar) - f(t + \hbar, \mu + 2\hbar) \right) \\ &\quad + \frac{2}{\mu\hbar} \left(f(t + \hbar, \mu - 2\hbar) - f(t + \hbar, \mu + 2\hbar) \right). \end{aligned} \quad (4.19)$$

Remark 20 Formula (4.18) (expressed via $(\mu_1 - \mu_2)^2$) has been found in [GPS6]. The currently used element $\mu = \mu_1 - \mu_2$ is much more convenient since the expression $\sqrt{\hbar^2 - 4u}$, $u \in \{\lambda_1, \lambda_2, \lambda_0, \lambda_{\pm}\}$ is a first order polynomial in μ . This property considerably simplifies all calculations.

Other formulae expressing the action of the operators (4.7) on elements $f(\mu)b^k$ (and consequently, on elements $f(\mu)u$ with any $u \in V^k$) can be deduced from relation (4.17) in the same way. However, they are too cumbersome and we do not exhibit their explicit form. Finally, the following proposition takes place.

Proposition 21 *Each of the operators (4.6) and (4.7), maps elements of the form $f(\mu)u$, where $f(\mu)$ is a rational function and $u \in V^k$, into elements of the same form, namely, $g(\mu)u$ with a rational g . The same claim is valid if $f(t, \mu)$ is a ration function in two variables.*

5 Quantization of differential operators and dynamical models

In this section we use different notation for classical and quantum objects. Namely, for elements of the algebra $U(u(2)_{\hbar})$, as well as for derivatives on this algebra, we use the Latin letters with hats (for instance, \hat{x} , $\hat{\partial}_x$ etc.), while for elements of the commutative algebra $\text{Sym}(u(2))$ and the corresponding derivatives we keep the previous notation (x , ∂_x etc.).

Given a differential operator \mathcal{D} on the algebra $\text{Sym}(u(2))$, we shall associate with it an operator acting on the algebra $U(u(2)_{\hbar})$. (We are mainly interested in operators describing dynamical models.)

If such an operator has constant (i.e., numerical) coefficients, we define its analog on $U(u(2)_{\hbar})$ by the same formula but with a new meaning of the derivatives (i.e. all partial derivatives are assumed to act on the algebra $U(u(2)_{\hbar})$ and to be subject to (4.5)). Thus, the d'Alembert operator

$$\hat{\square} = \hat{\partial}_t^2 - \hat{\partial}_x^2 - \hat{\partial}_y^2 - \hat{\partial}_z^2$$

is well defined on the algebra $U(u(2)_{\hbar})$.

In a similar manner we can define quantum analogs of the Dirac and Maxwell operators. We introduce the quantum analog of the Dirac operator by the classical formula employing the usual Dirac matrices:

$$\hat{D} = \gamma^0 \hat{\partial}_t - \gamma^1 \hat{\partial}_x - \gamma^2 \hat{\partial}_y - \gamma^3 \hat{\partial}_z.$$

As usual, the Dirac matrices γ^μ realize a representation of the Clifford algebra

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} I, \quad \mu, \nu \in \{0, 1, 2, 3\}$$

with the standard Minkowski metric $g^{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Since the *quantum partial derivatives* $\hat{\partial}_t, \dots, \hat{\partial}_x$ commute with each other, the relation between the quantum d'Alembert and Dirac operators does not change compared with the classical case

$$\hat{D}^2 = \hat{\square} I.$$

For defining the quantum analog of the Maxwell operator we first identify a differential 1-form⁹

$$\omega = \alpha dt + \beta dx + \gamma dy + \delta dz, \quad \alpha, \beta, \gamma, \delta \in \text{Sym}(u(2))$$

with the vector-function $(\alpha, \beta, \gamma, \delta)^t$ (here, the upper script t stands for the transposing) and realize the classical Maxwell operator Mw as follows

$$\text{Mw} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \square(\alpha) \\ \square(\beta) \\ \square(\gamma) \\ \square(\delta) \end{pmatrix} - \begin{pmatrix} \partial_t \\ \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} (\partial_t - \partial_x, -\partial_y, -\partial_z) \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}.$$

Its quantum analog acts on a vector-function $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta})^t$, where $\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ are some elements of $U(u(2)_\hbar)$, by the same formula with the change of the classical partial derivatives for the quantum ones: $\partial \rightarrow \hat{\partial}, \square \rightarrow \hat{\square}$.

If the coefficients of a given differential operator \mathcal{D} are some non-numerical elements of $\text{Sym}(u(2))$ the problem of its quantization becomes more subtle since it involves a quantization of the function algebra $\text{Sym}(u(2))$. Namely, we need an $SU(2)$ -isomorphism of linear spaces $\alpha : \text{Sym}(u(2)) \rightarrow U(u(2)_\hbar)$ satisfying the following properties.

The induced product

$$f \star_\hbar g = \alpha^{-1}(\alpha(f) * \alpha(g)), \quad f, g \in \text{Sym}(u(2))$$

in the algebra $\text{Sym}(u(2))$ should smoothly depend on the parameter \hbar (here $*$ stands for the product in the algebra $U(u(2)_\hbar)$) so that

$$\lim_{\hbar \rightarrow 0} f \star_\hbar g = f \cdot g \quad \text{and} \quad \lim_{\hbar \rightarrow 0} \frac{f \star_\hbar g - g \star_\hbar f}{\hbar} = \{f, g\}_{u(2)}.$$

Here $f \cdot g$ is the product in the algebra $\text{Sym}(u(2))$ and the notation $\{, \}_{u(2)}$ stands for the linear Poisson-Lie bracket on the algebra $\text{Sym}(u(2))$.

Besides, we want the restriction of α to the Poisson center¹⁰ $Z(\text{Sym}(u(2)))$ of the algebra $\text{Sym}(u(2))$, to be an algebraic isomorphism between $Z(\text{Sym}(u(2)))$ and $Z(U(u(2)_\hbar))$. Such a map α can be constructed via the Harish-Chandra isomorphism. We introduce it on a larger algebra and by using another method.

Namely, we consider the following algebra

$$\mathcal{A} = (\mathbb{K}(t, r) \otimes \text{Sym}(su(2))) / \langle x^2 + y^2 + z^2 - r^2 \rangle,$$

where r is a new central generator and $\mathbb{K}(t, r)$ is the algebra of rational functions in t and r . Thus, r has the meaning of the length of the radius vector of a point (x, y, z) .

⁹In Electrodynamics the components of this form are those of the four-potential $A_\mu, \mu \in \{0, 1, 2, 3\}$. The Maxwell operator comes in the left hand side of the well-known differential equation $\square A_\mu - \partial_\mu(\partial \cdot A) = 0$.

¹⁰By the Poisson center we mean the set of elements from $\text{Sym}(u(2))$ central with respect to the Poisson-Lie brackets

$$\{x, y\}_{u(2)} = z, \quad \{y, z\}_{u(2)} = x, \quad \{z, x\}_{u(2)} = y, \quad \{t, g\}_{u(2)} = 0, \quad g \in \{x, y, z\}.$$

This center is generated by the elements t and $\text{cas} = x^2 + y^2 + z^2$.

Now, introduce the "quantum radius" by setting

$$\hat{r} = \frac{\hat{\mu}}{2i} = \frac{\sqrt{\hbar^2 - 4\text{Cas}}}{2i}, \quad (5.1)$$

where $\hat{\mu}$ is a new notation for the element introduced in (4.11). Note that \hat{r} coincides with the usual radius provided $\hbar = 0$.

This motivates the following quantum analog of the algebra \mathcal{A}

$$\mathcal{A}_{\hbar} = (\mathbb{K}(\hat{t}, \hat{r}) \otimes U(\mathfrak{su}(2)_{\hbar})) / \langle \hat{x}^2 + \hat{y}^2 + \hat{z}^2 + \hbar^2 - \hat{r}^2 \rangle$$

where we put $\hbar = 2ih$. Besides, if we assume h to be real we get that \hbar is purely imaginary and the generators $\hat{x}, \hat{y}, \hat{z}$ can be represented as self-adjoint operators in a Hilbert space or a Verma module.

Now, define the quantizing map $\alpha : \mathcal{A} \rightarrow \mathcal{A}_{\hbar}$ by the following relation

$$\alpha(f(t, r)u) = f(\hat{t}, \hat{r})\hat{u}, \quad (5.2)$$

where \hat{u} is any element of the module V^k above (see (4.4)) and u is its classical counterpart. Considering the complexification of these modules and taking into account that α is $SU(2)$ -invariant, it suffices to put

$$u = b^k, \quad \hat{u} = \hat{b}^k, \quad \text{where } b = -(y + ix) \in \text{Sym}(\mathfrak{sl}(2)), \quad \hat{b} = -(\hat{y} + i\hat{x}) \in U(\mathfrak{sl}(2)_{\hbar}).$$

Now, extend the quantizing map on differential operators. With any given differential operators $\mathcal{D} = \sum_{\beta} a_{\beta} \partial_{\beta}$ where β is a multi-index, the sum is finite, and $a_{\beta} \in \mathcal{A}$, we associate the operator

$$\alpha(\mathcal{D}) := \sum_{\beta} \alpha(a_{\beta}) \hat{\partial}_{\beta} \quad (5.3)$$

with coefficients from \mathcal{A}_{\hbar} .

Definition 22 The map $\mathcal{D} \mapsto \alpha(\mathcal{D})$ is called the quantization of differential operators.

The operator $\alpha(\mathcal{D})$ is acting on the algebra $U(\mathfrak{u}(2)_{\hbar})$. However, we want it to be defined on the algebra \mathcal{A}_{\hbar} . The results of the previous section enable us to define such an action for the invariant operators (4.6) and (4.7) and consequently for any their combination with coefficients from \mathcal{A}_{\hbar} . (Note that defining an action of the partial derivatives on the whole algebra \mathcal{A}_{\hbar} is a more subtle deal and we leave it for subsequent publications.)

Similarly to formula (4.4) we present the algebra \mathcal{A}_{\hbar} as follows

$$\mathcal{A}_{\hbar} \cong \bigoplus_{k=0}^{\infty} (\mathbb{K}(\hat{t}, \hat{r}) \otimes V^k).$$

The components $\mathbb{K}(\hat{t}, \hat{r}) \otimes V^k$ are also called *isotypic*.

According to proposition 21 each of the operators (4.6) and (4.7) maps such an isotypic component to itself. Consequently, the same claim is valid for any linear combination of these operators with coefficients from \mathcal{A}_{\hbar} . Below, we consider an example of such an operator, namely, the Laplace-Beltrami operator corresponding to some rotationally symmetric metrics. However, first, we construct the quantization of some invariant operators.

Consider the quantization of the operators $Q = x\partial_x + y\partial_y + z\partial_z$ and Q^2 . First, we present the operator Q^2 in the ordered form, moving all the partial derivatives to the right hand side position:

$$Q^2 = Q + 2(xy \partial_x \partial_y + yz \partial_y \partial_z + zx \partial_z \partial_x) + x^2 \partial_x^2 + y^2 \partial_y^2 + z^2 \partial_z^2.$$

Then, we have to find the quantum images of the coefficients at the derivatives (see (5.3)). The definition (5.2) of the quantizing map α gives

$$\alpha(x) = \hat{x}, \quad \alpha(x^2) = \hat{x}^2 + \frac{\hbar^2}{12}, \quad \alpha(xy) = \frac{\hat{x}\hat{y} + \hat{y}\hat{x}}{2}.$$

The answer for other coefficients of Q^2 can be obtained from the above formulae by the cyclic substitution $x \rightarrow y \rightarrow z$. Finally, we get the answer

$$\hat{Q} =: \alpha(Q) = \hat{x}\hat{\partial}_x + \hat{y}\hat{\partial}_y + \hat{z}\hat{\partial}_z,$$

$$\alpha(Q^2) = \hat{Q}^2 + \frac{\hbar^2}{12}\hat{\Delta} - \frac{\hbar}{2}\hat{Q}\hat{\partial}_t.$$

where $\hat{\Delta} = \hat{\partial}_x^2 + \hat{\partial}_y^2 + \hat{\partial}_z^2$. Emphasize that $\alpha(Q^2) \neq \hat{Q}^2$.

Now, we consider the quantization of a free massless scalar field in the space equipped with a Schwarzschild-type metric

$$\varphi(r) dt^2 - \varphi(r)^{-1} dr^2 - r^2 d\Omega^2.$$

Here $d\Omega^2$ is the area form of the unit sphere and $\varphi(r)$ is a rational function. It is just Schwarzschild metric provided $\varphi(r) = 1 - \frac{r_g}{r}$.

The corresponding Laplace-Beltrami (LB) operator describing the dynamics reads

$$\square_{LB} = \varphi(r)^{-1}\partial_t^2 - \varphi(r)\partial_r^2 - \frac{1}{r^2}(X^2 + Y^2 + Z^2) - \frac{1}{r^2}\partial_r(\varphi(r)r^2)\partial_r,$$

where $X = y\partial_z - z\partial_y$, $Y = z\partial_x - x\partial_z$, $Z = x\partial_y - y\partial_x$. By using the relation

$$\frac{1}{r^2}(X^2 + Y^2 + Z^2) = \Delta - \partial_r^2 - \frac{2}{r}\partial_r,$$

we rewrite \square_{LB} in the form

$$\square_{LB} = \varphi(r)^{-1}\partial_t^2 - (\varphi(r) - 1)\partial_r^2 - \left(\frac{2}{r}(\varphi(r) - 1) + \partial_r\varphi(r)\right)\partial_r - \Delta. \quad (5.4)$$

In order to quantize this operator we have to find quantum analogs of the operators ∂_r and ∂_r^2 . Note, that in the classical setting

$$\partial_r = \frac{1}{r}(x\partial_x + y\partial_y + z\partial_z) = \frac{1}{r}Q.$$

As for the operator ∂_r^2 , we have

$$\partial_r^2 = \frac{1}{r^2}(x^2\partial_x^2 + y^2\partial_y^2 + z^2\partial_z^2 + 2xy\partial_x\partial_y + 2yz\partial_y\partial_z + 2zx\partial_z\partial_x) = \frac{1}{r^2}(Q^2 - Q).$$

Finally, formula (5.4) can be presented as

$$\square_{LB} = \varphi(r)^{-1}\partial_t^2 - \frac{\varphi(r) - 1}{r^2}Q^2 - \frac{1}{r}\left(\frac{\varphi(r) - 1}{r} + \partial_r\varphi(r)\right)Q - \Delta. \quad (5.5)$$

So, according to our scheme, the quantum counterpart $\hat{\square}_{LB} = \alpha(\square_{LB})$ of the operator \square_{LB} is

$$\hat{\square}_{LB} = \varphi(\hat{r})^{-1}\hat{\partial}_t^2 - \frac{\varphi(\hat{r}) - 1}{\hat{r}^2}\hat{Q}^2 - \left(\frac{\varphi(\hat{r}) - 1}{\hat{r}^2} + \frac{(\partial_r\varphi)(\hat{r})}{\hat{r}}\right)\hat{Q} + \frac{\hbar}{2}\frac{\varphi(\hat{r}) - 1}{\hat{r}^2}\hat{Q}\hat{\partial}_t - \left(1 + \frac{\hbar^2}{12}\frac{\varphi(\hat{r}) - 1}{\hat{r}^2}\right)\hat{\Delta}.$$

Thus, the operator $\hat{\square}_{LB}$ is a linear combination of the operators (4.6) and (4.7) with coefficients from $\mathbb{K}(t, \mu)$. Consequently, it is well defined on the whole algebra \mathcal{A}_\hbar and maps each isotypic component to itself. More precisely, the result of the action $\hat{\square}_{LB}(f(\hat{t}, \hat{r})\hat{u})$, where f is a rational

function in two variables and $\hat{u} \in V^k$, is an element of the form $g_k(\hat{t}, \hat{r})\hat{u}$, where $g_k(\hat{t}, \hat{r})$ is also a rational function. Thus, the action of the operator $\hat{\square}_{LB}$ can be described by a series of the maps

$$\mathbb{K}(\hat{t}, \hat{r}) \rightarrow \mathbb{K}(\hat{t}, \hat{r}) : \quad f(\hat{t}, \hat{r}) \mapsto g_k(\hat{t}, \hat{r}), \quad \forall k \in \mathbb{N}.$$

In conclusion, we want to emphasize that the variables \hat{t} and \hat{r} , coming in the final version of the above model, are commutative. So, this model can be treated by means of the commutative algebra and analysis. Thus, it would be interesting to find the spectrum of the operator $\hat{\square}_{LB}$. However, in contrast with the classical case our model is based on difference (not differential) operators. Also, note that our method is hopefully valid on the RE and modified RE algebras. The only problem in this case consists in a reasonable introducing a quantum radius and extending the action of invariant operators to rational functions. We plan to exhibit this construction in our subsequent publications.

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