A classification of isotropic affine hyperspheres

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A CLASSIFICATION OF ISOTROPIC AFFINE HYPERSPHERES

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Abstract. We study affine hypersurfaces M which have isotropic difference tensor. Note that any surface always has isotropic difference tensor. In case that the metric is positive definite such hypersurfaces have been previously studied in [2] and [**?**]. We first show that the dimension of an isotropic affine hypersurface is either 5, 8, 14 or 26. Next we assume that M is an affine hypersphere and we obtain in each of the possible dimensions a complete classification.

1. Introduction

The notion of a submanifold with isotropic second fundamental form was first introduced in [12] by O'Neill for immersions if Riemannian manifolds and recently extended by Cabrerizo et al. in [4] for pseudo-Riemannian manifolds. We say that M has isotropic second fundamental form h if and only if for any tangent vector X at a point p we have that

$$< h(X(p), X(p)), h(X(p), X(p)) >= \lambda(p) < X(p), X(p) >^2 .$$

If λ is independent of the point p, the submanifold is called constant isotropic. Given the similarities between the basic equations that charactherise the manifolds and the important role played by the difference tensor it is natural to introduce the equivalent notion of isotropy in affine geometry. That is, a hypersphere M has isotropic difference tensor K if and only if for any tangent vector X at a point p we have that

$$h(K(X(p), X(p)), K(X(p), X(p))) = \lambda(p)h(X(p), X(p))^2,$$

where h is the affine metric on the hypersurface. Note that a 2-dimensional affine surface is always isotropic. In case that the affine metric is positive definite such submanifolds have been previously studied in [2] and [**?**]. In [2], beside a restriction on the dimension, a complete classification was obtained in case that the affine hypersurface is an affine sphere. In [**?**] a complete classification was given of 5 dimensional positive definite affine hypersurfaces.

In this paper we deal with the case that the induced affine metric has arbitrary signature. We will first show that the restriction of the dimension remains valid in the indefinite case. Even though the proof remains based on the Hurwitz theorem it is essentially different from the proof in the definite case. This is because unlike in the definite case, the unit tangent bundle at a point p is no longer a compact manifold. Instead of this null vectors will play an important role in the proof of the restriction of the dimension.

In the second part of the paper we will then restrict ourselves to the case that M is an affine hypersphere and we will deduce that in that case the immersion also has parallel difference tensor (and is a pseudo-Riemannian symmetric space). We then look at each of the possible dimensions and determine in each case explicitly

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by elementary means the form of the difference tensor and the possible examples. Note that for this second part also a more involved Lie group approach would be possible. We show the following theorems.

Theorem 1. *Let M^5 ne a 5-dimensional affine hypersphere of $\mathbb{R}^6$. Assume that M is λ-isotropic with $\lambda \neq 0$. Then if we identify $\mathbb{R}^6$ with symmetric 3×3 -matrices, then M is congruent with the connected component of $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, of symmetric matrices with determinant 1.*

Note that in the positive definite case we had the connected component of the identity.

Theorem 2. *Let M^8 ne a 8-dimensional affine hypersphere of $\mathbb{R}^9$. Assume that M is λ-isotropic with $\lambda \neq 0$. Then if we identify*

Theorem 3. *Let M^8 ne a 14-dimensional affine hypersphere of $\mathbb{R}^9$. Assume that M is λ-isotropic with $\lambda \neq 0$. Then if we identify*

Theorem 4. *Let M^8 ne a 26-dimensional affine hypersphere of $\mathbb{R}^9$. Assume that M is λ-isotropic with $\lambda \neq 0$. Then if we identify*

2. Preliminaries

Let $f : M \longrightarrow \mathbb{R}^{n+1}$ be a nondegenerate affine hypersurface immersion. Let D be the covariant derivative on $\mathbb{R}^{n+1}$ and Ω the volume form given by $\Omega(u_1, \dots, u_{n+1}) = \det(u_1, \dots, u_{n+1})$, such that $\mathbb{R}^{n+1}$ is endowed with its standard equiaffine structure (D, Ω). In a general setting, an affine manifold (M^n, ∇) is said to be equiaffine if there exists a volume form ω, i.e. a non-vanishing n-form, on M which is parallel with respect to ∇:

$$(1)\quad (\nabla_X \omega)(X_1 \dots, X_n) = X(\omega(X_1, \dots, X_n)) - \omega(\nabla_X X_1, \dots, X_n) - \dots - \omega(X_1, \dots, \nabla_X X_n).$$

In this case we may also say that (∇, ω) is an equiaffine structure on M^n. In what follows, we briefly recall the construction of an equiaffine structure on an affine hypersurface M^n in $\mathbb{R}^{n+1}$. For more details we refer to [13].
First, let $p \in M$ and $X, Y \in T_pM$. If we choose an arbitrary transversal vector field η we can decompose

$$D_X Y = \nabla^\eta_X Y + h^\eta(X, Y)\eta.$$

It is easy to see that ∇^η is a connection on M and h^η is a symmetric bilinear form. Note that the fact whether this bilinear form is degenerate or not is independent of the choice of transversal vector field η. As such M is called nondegenerate if and only if this bilinear form is nondegenerate. Hence, locally there exists a volume form on M associated to h^η, given by

$$\omega_{h^\eta}(X_1, \dots, X_n) = \sqrt{|\det h^\eta(X_i, X_j)|}.$$

Next, we want to introduce a canonical transversal vector field ξ. In order to make a good choice, we define $\omega_\eta(X_1, \dots, X_n) := \Omega(X_1, \dots, X_n, \eta)$, for $X_1, \dots, X_n$ vector fields on M^n and we ask that the volume forms ω_ξ and ω_{h^ξ} coincide and that (∇^ξ, ω_ξ) is an equiaffine structure on M^n. Notice that these conditions guarantee the existence of a unique (up to sign) transversal vector field ξ, see [13]. It is called the affine normal vector field, or the Blaschke normal vector field. For convenience, we will denote from now on $\nabla := \nabla^\xi$.

Finally, in terms of this transversal vector field we get for M the formulas of Gauss and Weingarten, respectively, as follows:

$$D_X Y = \nabla_X Y + h(X,Y)\xi,$$
$$D_X \xi = -SX,$$

where we call ∇ the *induced affine connection*, h the *affine metric*, ξ the *affine normal field* or *Blaschke normal field* and S the affine *shape operator*. An affine hypersurface is called a (proper) affine sphere if S is a (non zero) multiple of the identity.
Moreover, let R denote the curvature tensor of M^n. Then, the following fundamental equations hold with respect to the induced affine connection:

Gauss equation:	$R(X,Y)Z = h(Y,Z)SX - h(X,Z)SY;$
Codazzi equation for h :	$(\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z);$
Codazzi equation for S :	$(\nabla_X S)Y = (\nabla_Y S)X;$
Ricci equation:	$h(SX,Y) = h(X,SY);$

The Codazzi equation implies that for a proper affine sphere, the multiple of the identity is constant, in which case by applying a homothety of the ambient space, we may assume that $S = \varepsilon I$, where $\varepsilon = \pm 1$. Moreover we have that $\xi + \varepsilon f$, where f denotes the position vector, is a constant vector which is called the center of the proper affine hypersphere. By applying a translation in the ambient space we may of course always assume that the center is the origin.
As ∇ is not necessarily compatible with the affine metric h, we can consider the *difference tensor* K, a $(1,2)$-type vector field defined as:

$$K(X,Y) = \nabla_X Y - \hat{\nabla}_X Y,$$

where $\hat{\nabla}$ is the Levi-Civita connection on M. By convention, one may also write $K_X Y$ instead of $K(X,Y)$. The classical Berwald theorem states that K vanishes identically if and only if M is congruent to a nondegenerate quadric.

Proposition 5. *We have the following properties for K:*

(1) $K(X,Y) = K(Y,X)$;
(2) for any X we have $Y \mapsto K_X Y$ is a symmetric linear map and $\mathrm{trace} K_X = 0$ (the apolarity condition);
(3) $h(K(X,Y),Z) = h(K(X,Z),Y)$.

It is easy to prove that ∇h is related to K by:

$$\nabla h(X,Y,Z) = -2h(Z,K(X,Y)).$$

Moreover, the equations of Gauss, Ricci and Codazzi, respectively, may also be written out with respect to the Levi-Civita connection as follows:

(2)
$$\begin{cases} \hat{R}(X,Y)Z = \frac{1}{2}\{h(Y,Z)SX - h(X,Z)SY + h(SY,Z)X - h(SX,Z)Y\} - [K_X,K_Y]Z, \\ \hat{\nabla}K(X,Y,Z) - \hat{\nabla}K(Y,X,Z) = \frac{1}{2}\{h(Y,Z)SX - h(X,Z)SY - h(SY,Z)X + h(SX,Z)Y,\} \end{cases}$$
$$\begin{cases} (\hat{\nabla}_X S)Y - (\hat{\nabla}_Y S)X = K(Y,SX) - K(X,SY), \\ (\nabla h)(X,Y,Z) = (\nabla h)(Y,X,Z), \end{cases}$$
$$h(X,SY) = h(SX,Y),$$

where

$$[K_X,K_Y]Z = K_X K_Y Z - K_Y K_X Z$$

and

$$\hat{\nabla K}(X,Y,Z) = \hat{\nabla}_X K(Y,Z) - K(\hat{\nabla}_X Y,Z) - K(Y,\hat{\nabla}_X Z).$$

We have the following *Ricci identity*:

$$\hat\nabla^2 K(X,Y,Z,W) - \hat\nabla^2 K(Y,X,Z,W) = \hat R(X,Y)K(Z,W) - K(\hat R(X,Y)Z,W) - h(Z,\hat R(X,Y)W). \tag{3}$$

A nondegenerate hypersurface M of the equiaffine space $\mathbb{R}^{n+1}$ is called locally homogeneous if for all points p and q of M, there exists a neighborhood U_p of p in M, and an equiaffine transformation A of $\mathbb{R}^{n+1}$, i.e. $A \in \mathrm{SL}(n+1,\mathbb{R}) \ltimes \mathbb{R}^{n+1}$, such that $A(p) = q$ and $A(U_p) \subset M$. If $U_p = M$ for all p, then M is called homogeneous. Let G be the pseudogroup defined by

$$G = \{A \in \mathrm{SL}(n+1,\mathbb{R}) \ltimes \mathbb{R}^{n+1} \mid \exists U, \text{ open in } M : A(U) \subset M\},$$

then M is locally homogeneous if and only if G "acts" transitively on M. If M is homogeneous, then G is a group and every element of G maps the whole of M into M. The following proposition is probably well known, however, as we did not find an explicit reference, we include a small proof.

Proposition 6. *Let M^n be a nondegenerate homogeneous affine hypersurface. Assume that $G \subset SL(n+1,\mathbb{R})$. Then M is an affine sphere centered at the origin.*

Proof. We denote the immersion by f. Let p and q be in M and let g be the affine transformation which maps p to q. We have that

$$\xi(g(p)) = dg(\xi(p)),$$

and

$$dg(f(p)) = g(f(p)) = f(q)$$

Moreover as M is homogeneous we know that the position vector can not be a tangent vector at one point (and therefore at every point). Indeed if that were the case, we would habe a tangent vector field X such that $X(p) = f(p)$. This would imply that $D_Y X = Y$, and therefore $h(X,Y) = 0$ for any vector field Y. This implies that the immersion f would be degenerate.
Therefore we may write $\xi = \rho f + Z$, where Z is a tangent vector field and ρ a function. As M is locally homogeneous and g belongs to $\mathrm{SL}(n+1,\mathbb{R})$ it follows that ρ is constant. The construction of the affine normal of [13] then implies that M is an affine sphere centered at the origin. ◻

The equivalent notion in affine geometry for isotropic submanifolds from Riemannian geometry, which was introduced by O'Neill in [12], corresponds to submanifolds for which the difference tensor K is isotropic, that is, it satisfies:

$$h(K(X,X),K(X,X)) = \lambda(p)h(X,X)^2,$$

for some tangent vector field X. Here we will always deal with the case that $\lambda \neq 0$. Therefore, if necessary, by replacing ξ with $-\xi$, we may assume that λ is positive and therefore there exists a positive function μ such that $\lambda = \mu^2$. We will also use the following lemma from [13]:

Lemma 7. *Let $F : M \to \mathbb{R}^{n+1}$ be an equiaffine immersion. If the metric on $\mathbb{R}^{n+1}$ is indefinite, then the immersion is isotropic if and only if for any tangent vectors $X_1, X_2, X_3, X_4 \in T_pM$, we have that*

$$h(K(X_1,X_2),K(X_3,X_4)) + h(K(X_1,X_3),K(X_2,X_4)) + h(K(X_1,X_4),K(X_2,X_3)) = \lambda(p)\{h(X_1,X_2)h(X_3,X_4) + h(X_1,X_3)h(X_2,X_4) + h(X_1,X_4)h(X_2,X_3)\}. \tag{4}$$

By using lemma (7) and property (3) in Proposition 5 we get that an affine submanifold M^n in $\mathbb{R}^{n+1}$ is isotropic if and only if for any tangent vectors $X_1, X_2, X_3, X_4 \in T_pM$ we have that

$$(5)\quad K_{X_1}K_{X_2}X_3 + K_{X_2}K_{X_1}X_3 + K_{X_3}K_{X_1}X_2 = \lambda(p)\left(h(X_2,X_3)X_1 + h(X_1,X_3)X_2 + h(X_1,X_2)X_3\right).$$

Theorem 8. *([5]) Let (M^n_k, h) be an $n-$dimensional simply connected pseudo-Riemannian manifold with index k. Let $\hat{\nabla}$ denote the Levi Civita connection, $\hat{R}$ its curvature tensor and let TM denote the tangent bundle of M^n_k. If K is a TM-valued symmetric bilinear form on M^n_k satisfying that*
i) $h(K(X,Y),Z)$ is totally symmetric
ii) $(\hat{\nabla}K)(X,Y,Z) = \hat{\nabla}_X K(Y,Z) - K(\hat{\nabla}_X Y, Z) - K(Y, \hat{\nabla}_X Z)$ is totally symmetric,
iii) $\hat{R}(X,Y)Z = c(h(Y,Z)X - h(X,Z)Y) + K(K(Y,Z),X) - K(K(X,Z),Y)$,
then there exists an affine immersion $\phi : M^n_k \to \mathbb{R}^{n+1}$ as an affine sphere with induced difference tensor K and induced affine metric h.

Theorem 9. *([5]) Let $\phi^1, \phi^2 : M^n_k \longrightarrow \mathbb{R}^{n+1}$ be two affine immersions of an pseudo-Riemannian n-manifold (M^n_k, h) with difference tensors K^1, K^2, respectively. If*

$$h(K^1(X,Y), \phi^1_* Z) = h(K^2(X,Y), \phi^2_* Z)$$

for all tangent vectors fields $X, Y, Z \in T_pM^n_k$, then there exists an isometry ϕ of $\mathbb{R}^{n+1}$ such that $\phi^1 = \phi \circ \phi^2$.

3. Possible dimensions and choice of frame

From now on we will always assume that M^n_k is an affine isotropic hypersurface in $\mathbb{R}^{n+1}$. Here n denotes the dimension and k the index of the affine metric. In case that the metric is definite a classification was obtained already in [2], In view of this we will also assume thay M is neither positive nor negative definite, i.e. $1 \le k < n$. Also recall that because of the properties of K any surface is isotropic. Therefore we will also assume that $n > 2$. First, we have the following lemma:

Lemma 10. *Let M^n_k be an n-dimensional isotropic affine hypersurface and let $p \in M^n_k$. If for any null vector $v \in T_pM$ we have that $K(v,v)$ is a null vector such that $h(K(v,v),v) = 0$, then the difference tensor K vanishes.*

As its proof is very similar to the proof of Lemma 3.1 in [9], we omit it here. From now on , we will assume that $\lambda \neq 0$. By Lemma 10, there exists a null vector v_0 such that v_0 and $K(v_0,v_0)$ are linearly independent and $h(v_0, K(v_0,v_0)) \neq 0$. Using Lemma 7, we have that for any null vector u

$$(6)\qquad h(K(v_0,v_0), K(v_0,u)) = \lambda h(v_0,v_0)h(v_0,u) = 0.$$

As K_{v_0} is a symmetric operator with respect to the metric h, we get that $K_{v_0}K_{v_0}v_0 = 0$. Moreover, taking in particular $u = v_0$ in (6), we get that $K(v_0,v_0)$ is a null vector.
We can now take a null frame such that

$$e_1 = v_0, \quad e_2 = K_{v_0}v_0.$$

By rescaling v_0 if necessary, we may assume that $h(K(v_0,v_0),v_0) = -4\lambda^2$. Then we get

$$(7)\qquad \begin{array}{l} h(e_1,e_1) = h(e_2,e_2) = 0 \quad h(e_1,e_2) = -4\lambda^2, \\ K(e_1,e_1) = e_2, \quad K(e_1,e_2) = K_{v_0}K_{v_0}v_0 = 0. \end{array}$$

Using the isotropy condition in (5) for $X_1 = X_2 = e_1, X_3 = e_3$ we get that

$$K_{e_2}e_2 = -8\lambda^3 e_1.$$

From relation (7) we can see that the space $span\{e_1, e_2\}$ is invariant under the operator K_{e_1}. As the operator K_{e_1} is symmetric with respect to the metric, it follows that also the space $span\{e_1, e_2\}^\perp$ is invariant under K_{e_1}.

Now we follow precisely the computations of [9]. We get a basis $\{e_1, e_2, u_1, \dots, u_r, \omega_1^1, \omega_2^1, \dots, \omega_1^r, \omega_2^r\}$, which satisfies that $\{u_1, \dots, u_r, \omega_1^1, \omega_2^1, \dots, \omega_1^r, \omega_2^r\}$ is an orthogonal basis of $\{e_1, e_2\}^\perp$ and

$$(8) \qquad \begin{cases} h(e_1,e_1) = h(e_2,e_2) = 0, \quad h(e_1,e_2) = -4\lambda^2, \\ h(u_i,u_j) = \varepsilon_i\delta_{ij}, \quad \varepsilon_i = \pm 1, \quad h(\omega_1^\alpha,\omega_1^\alpha) = 1, \quad h(\omega_2^\alpha,\omega_2^\alpha) = -1, \end{cases}$$

$$(9) \qquad \begin{cases} K_{e_1}e_1 = e_2, \quad K_{e_1}e_2 = 0, \quad K_{e_1}u_i = \lambda u_i, \\ K_{e_1}\omega_1^\alpha = -\frac{1}{2}\lambda\omega_1^\alpha - \frac{\sqrt{3}}{2}\lambda\omega_2^\alpha, \quad K_{e_1}\omega_2^\alpha = -\frac{1}{2}\lambda\omega_2^\alpha + \frac{\sqrt{3}}{2}\lambda\omega_1^\alpha, \\ K_{e_2}e_2 = -8\lambda^3 e_1, \quad K_{e_2}u_i = -2\lambda^2 u_i, \\ K_{e_2}\omega_1^\alpha = \lambda^2\omega_1^\alpha - \sqrt{3}\lambda^2\omega_2^\alpha, \quad K_{e_2}\omega_2^\alpha = \sqrt{3}\lambda^2\omega_1^\alpha + \lambda^2\omega_2^\alpha, \\ K_{e_1}K_{u_i}u_j = \frac{\delta_{ij}\varepsilon_i}{4\lambda}(2\lambda e_1 - e_2), \quad K_{\omega_1^\alpha}\omega_1^\alpha = L(\omega_1^\alpha,\omega_1^\alpha) - \frac{1}{8\lambda}(2\lambda e_1 - e_2), \\ K_{\omega_2^\alpha}\omega_2^\alpha = L(\omega_1^\alpha,\omega_1^\alpha) + \frac{1}{8\lambda}(2\lambda e_1 - e_2), \quad K_{\omega_1^\alpha}\omega_2^\alpha = \frac{\sqrt{3}}{8\lambda}(2\lambda e_1 + e_2), \\ K_{\omega_k^\alpha}\omega_l^\beta = L(\omega_k^\alpha,\omega_l^\beta), k,l \in 1,2, 1 \le \alpha \ne \beta \le r. \end{cases}$$

In the above formulas, U and W correspond to the invariant subspaces of K_{e_1} and the operator L is an operator on $W \times W$, defined by

$$(10) \qquad L(\omega,\tilde\omega) = K_\omega\tilde\omega + \frac{1}{4\lambda^2}h(K_\omega\tilde\omega, e_2)e_1 + \frac{1}{4\lambda^2}h(K_\omega\tilde\omega, e_1)e_2, \quad \omega,\tilde\omega \in W.$$

which is a symmetric operator, satisfies $ImL \subset U = span\{u_1, \dots, u_k\}$ and

$$(11) \qquad \begin{array}{l} L(\omega_1^\alpha,\omega_1^\alpha) = L(\omega_2^\alpha,\omega_2^\alpha), \quad L(\omega_1^\alpha,\omega_2^\alpha) = 0, \quad K_{\omega_1^\alpha}\omega_2^\alpha = \frac{\sqrt{3}}{4}\left(e_1 + \frac{1}{2\lambda}e_2\right), \\ L(\omega_1^\alpha,\omega_1^\beta) = L(\omega_2^\alpha,\omega_2^\beta), \quad L(\omega_1^\alpha,\omega_2^\beta) = -L(\omega_2^\alpha,\omega_1^\beta). \end{array}$$

As in [9] changing the frame by taking

$$(12) \qquad f_1 = (2\lambda e_1 - e_2)/(4\mu^3), \quad f_2 = (2\lambda e_1 + e_2)/(4\mu^3),$$

we get that

$$(13) \qquad h(f_1,f_1) = -h(f_2,f_2) = \varepsilon_0, \quad h(f_1,f_2) = 0$$

and

$$(14) \qquad \begin{cases} K_{f_1}f_1 = -\mu f_1, \quad K_{f_1}f_2 = \mu f_2, \quad K_{f_2}f_2 = -\mu f_1, \quad K_{f_1}u_i = \mu u_i, \quad K_{f_2}u_i = 0, \\ K_{f_1}\omega_1^\alpha = -\frac{\mu}{2}\omega_1^\alpha, \quad K_{f_1}\omega_2^\alpha = -\frac{\mu}{2}\omega_2^\alpha, \quad K_{f_2}\omega_1^\alpha = \frac{\sqrt{3}\mu}{2}\omega_2^\alpha, \quad K_{f_2}\omega_1^\alpha = -\frac{\sqrt{3}\mu}{2}\omega_1^\alpha, \\ K_{u_i}u_j = \mu\varepsilon_0\varepsilon_i\delta_{ij}f_1, \quad K_{\omega_1^\alpha}\omega_2^\alpha = \frac{\sqrt{3}}{2}\mu\varepsilon_0 f_2, \quad K_{\omega_1^\alpha}\omega_1^\alpha = L(\omega_1^\alpha,\omega_1^\alpha) - \frac{\mu}{2}\varepsilon_0 f_1, \\ K_{\omega_2^\alpha}\omega_2^\alpha = L(\omega_1^\alpha,\omega_1^\alpha) + \frac{\mu}{2}\varepsilon_0 f_1, \quad K_{\omega_k^\alpha}\omega_l^\beta = L(\omega_k^\alpha,\omega_l^\beta), k,l \in 1,2, 1 \le \alpha \ne \beta \le r. \end{cases}$$

Therefore, in order to determine the difference tensor explicitly, we only need to determine all the terms $L(\omega_k^\alpha,\omega_l^\beta), k,l \in \{1,2\}, 1 \le \alpha,\beta \le r$. In order to do so we will summarize the above properties in a more invariant way.

Let I be the identity map and define for any $w \in W$

$$(15) \qquad Tw = \frac{2}{\sqrt{3}\lambda}\left(K_{e_1} + \frac{1}{2}\lambda I\right)w.$$

We can easily check that T satisfies

$$\begin{array}{llll} T\omega_1^\alpha = \omega_2^\alpha, & T\omega_2^\alpha = -\omega_1^\alpha, & T^2w = -w, & h(Tv,w) = h(v,Tw), \\ T\omega_1^\alpha = \omega_2^\alpha, & T\omega_2^\alpha = -\omega_1^\alpha, & h(Tw,Tv) = -h(w,v), & h(Tv,w) = h(v,Tw), \end{array}$$

for $w, v \in W$. In addition, from (11) it follows that $L(w, Tv) = -L(v, Tw)$ and $L(Tw, Tv) = L(v, w)$. We also have that L satisfies an isotropy condition. Indeed, let $w = \sum_{\alpha=1}^{r} a_\alpha \omega_2^\beta + \sum_{\beta=1}^{r} b_\beta \omega_2^\beta$. By using (13) in lemma 7 we have

$$\begin{aligned} h(K_w w, e_1) &= \sum_{\alpha,\beta=1}^{n} a_\alpha a_\beta h(K_{e_1}\omega_1^\alpha, \omega_1^\beta) + \sum_{\alpha,\beta=1}^{n} b_\alpha b_\beta h(K_{e_1}\omega_2^\alpha, \omega_2^\beta) \\ &+ \sum_{\alpha,\beta=1}^{n} a_\alpha b_\beta h(K_{e_1}\omega_1^\alpha, \omega_2^\beta) + \sum_{\alpha,\beta=1}^{n} b_\alpha a_\beta h(K_{e_1}\omega_2^\alpha, \omega_1^\beta) \\ &= -\frac{\lambda}{2}\sum_{\alpha,\beta=1}^{n}(a_\alpha a_\beta - b_\alpha b_\beta)\delta_{\alpha\beta} - \frac{\sqrt{3}\lambda}{2}\sum_{\alpha,\beta=1}^{n}(a_\alpha b_\beta + b_\alpha a_\beta)\delta_{\alpha\beta} \\ &= -\frac{\lambda}{2}h(w,w) + \frac{\sqrt{3}\lambda}{2}h(w,Tw). \end{aligned} \tag{16}$$

Similarly, we obtain

$$h(K_w w, e_2) = \lambda^2 h(w,w) + \sqrt{3}\lambda^2 h(w, Tw). \tag{17}$$

By combining (10), (16) and (17) we get

$$\begin{aligned} h(L(w,w), L(w,w)) &= h(K_w w, K_w w) + \frac{1}{2\lambda^2} h(K_w w, e_1) h(K_w w, e_2) \\ &= \lambda h(w,w)^2 + \frac{1}{2\lambda^2}\left(-\frac{1}{2}\lambda^3 h(w,w)^2 + \frac{3}{2}\lambda^3 h(w,Tw)^2\right) \\ &= \frac{3}{4}\lambda(h(w,w)^2 + h(w,Tw)^2). \end{aligned} \tag{18}$$

Liniarizing the previous expression for arbitrary vectors $W_1, W_2, W_3, W_4 \in W$, we obtain:

$$\begin{aligned} &h(L(W_1,W_2), L(W_2,W_3)) + h(L(W_1,W_3), L(W_2,W_4)) + h(L(W_1,W_4), L(W_2,W_3)) \\ &= \frac{3}{4}(h(W_1,W_2)h(W_3,W_4) + h(W_1,W_3)h(W_2,W_4) + h(W_1,W_4)h(W_2,W_3) \\ &+ h(W_1,TW_2)h(W_3,TW_4) + h(W_1,TW_3)h(W_2,TW_4) + h(W_1,TW_4)h(W_2,TW_3)). \end{aligned} \tag{19}$$

Note that given a metric of neutral signature on $\{f_1, f_2\}^\perp$ and operators T and L satisfying the previous conditions, we can define a frame such that (14) holds. We start with a vector $u \in \{f_1, f_2\}^\perp$ with length 1. Then Tu has length -1. We now write $w = au + bTu$. The fact that w has length 1 and is orthogonal to Tw. Then implies that

$$(a^2 - b^2) + 2ab < u, Tu >= 1$$
$$(a^2 - b^2) < u, Tu > -2ab = 0,$$

which determines a and b uniquely upto sign. It is then sufficient to take $w_1^1 = w$ and $w_2^1 = Tw$ and to complete the construction is an inductive way.

In what follows we are going to determine the possible dimensions of the studied submanifold M^n. In order to do this, we will use a well known result from the theory of composition of quadratic forms, namely the '1,2,4,8 Theorem' proved by Hurwitz in 1898. One can find it for example in [15]. It states that there exists an n-square identity over the complex numbers of the form

$$(x_1^2 + \ldots + x_n^2)(y_1^2 + \ldots + y_n^2) = z_1^2 + \ldots + z_n^2, \tag{20}$$

where $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_n)$ are systems of indeterminates and each $z_k = z_k(X, Y)$ is a bilinear form in X and Y, if and only if $n = 1, 2, 4$ or 8. We are going to see how this result applies in our case and then determine the values of L on the components of the basis in order to determine the difference tensor of our immersion.

In order to apply the *1,2,4,8 Theorem*, we are going to find conveniently defined complex vector spaces and an operator which preserves lengths.
First, we denote by $U^{\mathbb{C}}$ the complex linear extension of V and by $W^{\mathbb{C}}$ the complex linear extension of W. We now take

$$\mathcal{W}_1 = \{v + iTv | v \in W\},$$
$$\mathcal{W}_2 = \{w - iTw | w \in W\}.$$

Note that these are indeed complex linear vector spaces as $i(v \pm iTv) = \mp(Tv \mp iv) = (\mp Tv \pm iT(\mp Tv))$ and we complexify the metric and the previously defined operator L. Note that L is symmetric and that from the properties of L and T it follows that the restriction of L to $\mathcal{W}_1 \times \mathcal{W}_1$ and $\mathcal{W}_2 \times \mathcal{W}_2$ vanishes identically. Therefore in order to determine L it is sufficient to study L on

$$\begin{cases} L : \mathcal{W}_1 \times \mathcal{W}_2 \to U^{\mathbb{C}} \\ L(\omega, \tilde{\omega}) = K_\omega \tilde{\omega} + \frac{1}{4\lambda^2} h(K_\omega \tilde{\omega}, e_2) e_1 + \frac{1}{4\lambda^2} h(K_\omega \tilde{\omega}, e_1) e_2, \end{cases} \tag{21}$$

where $U^{\mathbb{C}} := span\{u_1, \dots, u_r\}$ over $\mathbb{C}$.

Proposition 11. *The operator L defined in* (21) *satisfies:*

(1) For any vectors $x \in \mathcal{W}_1$ and $y \in \mathcal{W}_2$ we have

$$h(L(x, y), L(x, y)) = \frac{3\mu^2}{4} h(x, x) h(y, y); \tag{22}$$

(2) Given x_0 in $\mathcal{W}_1$ such that $h(x_0, x_0) = 1$, we have that $L(x_0, -)$ preserves norms in the sense that

$$h(L(x_0, y), L(x_0, y)) = \frac{3}{2}\mu^2 h(y, y), \forall y \in \mathcal{W}_2;$$

(3) Given x_0 a non-null vector, we have that $L(x_0, -) : \mathcal{W}_2 \mapsto \mathcal{U}^{\mathbb{C}}$ is a bijective operator;

(4) For any $x, x' \in \mathcal{W}_1, y, z \in \mathcal{W}_2$ we have that

$$h(L(x, y), L(x', z)) + h(L(x', y), L(x, z)) = \frac{3}{2}\mu^2 h(x, x') h(y, z). \tag{23}$$

Proof. (1) Take $W_1 = W_3 = \omega_1$ and $W_2 = W_4 = \omega_2$ in relation (19), where $\omega_1 := v + iTv \in \mathcal{W}_1$ and $\omega_2 := w - iTw \in \mathcal{W}_2$. Using the properties of T in (15) and the fact that ω_1 and ω_2 are orthogonal, we obtain $h(L(\omega_1, \omega_2), L(\omega_1, \omega_2)) = \frac{3\mu^2}{4} h(\omega_1, \omega_1) h(\omega_2, \omega_2)$.

(2) This property follows directly from the previously proved one.

(3) We linearize in the second argument in property (22), that is $y \rightsquigarrow y + z$, for $y, z \in \mathcal{W}_2$ and we get for arbitrary $x \in \mathcal{W}_1$

$$h(L(x, y), L(x, z)) = \frac{3}{4}\mu^2 h(x, x) h(y, z). \tag{24}$$

Fix $x = x_0$, for x_0 arbitrarily chosen in $\mathcal{W}_1$, and write equation (24) once for $y = y_1$ and once for $y = y_2$. Assuming $L(x_0, y_1) = L(x_0, y_2)$, as h is nondegenerate and x_0 is a non-null vector, we get that $L(x_0, -)$ is injective. This gives $\dim Im(L(x_0, -)) = \dim \mathcal{W}_2 = r$, but, as $\dim U^{\mathbb{C}} = r$, we obtain that L is also surjective.

(4) The property in (23) follows immediately by liniarizing in (24) for $x \rightsquigarrow x + x', \forall x, x' \in \mathcal{W}_1$.

□

Theorem 12. *Let M_k^n be a λ-isotropic affine hypersurface. Assume that $\lambda \neq 0$. Then either $n = 2, 5, 8, 14$ or 26.*

Proof. We assume that $n > 2$. We can write out equation (22) for the elements of the bases. For more convenience, choose $\{e_i\}_{i=\{1,\ldots,r\}}, \{f_j\}_{j=\{1,\ldots,r\}}, \{g_k\}_{k=\{1,\ldots,r\}}$ bases for $\mathcal{U}, \mathcal{W}_1, \mathcal{W}_2$, respectively, and let $u = \sum_{i=1}^{r} u_i e_i, v = \sum_{j=1}^{r} v_j f_j$. With this choice, relation (22) becomes

$$(u_1^2 + \ldots + u_r^2)(v_1^2 + \ldots + v_r) = z_1^2 + \ldots + z_r^2, \tag{25}$$

where $L(e_i, f_j) = l_{ij}^k g_k$ and $z_k = \sum_{i,j=1}^{n} u_i v_j l_{ij}^k$. Equation (25) yields an r-square quadratic equation. Thus, we may apply now the theorem of Hurwitz and obtain $r = 1, 2, 4, 8$, which implies that $n = 5, 8, 14, 26$. □

4. Isotropic affine hyperspheres

From now on, we will assume that M is a λ-isotropic affine hypersphere with $\lambda \neq 0$.

Proposition 13. *Let $n \geq 3$ and M^n be an n-dimensional affine λ-isotropic hypersphere in $\mathbb{R}^{n+1}$. Then M^n is constant isotropic.*

Proof. Let $e_1' := f_1, e_2' := f_2, e_3' := u_1, \ldots, e_{r+2}' := u_r, e_{r+3}' := \omega_1^1, \ldots, e_{2r+2}' := \omega_1^r, e_{2r+3}' := \omega_2^r$. Then $\{e_1', \ldots, e_n'\}$ is an orthogonal basis with $h(e_i', e_j') = \varepsilon_i \delta_{ij}, \varepsilon_i = \pm 1$. We denote by Ric the Ricci tensor of M^n with respect to the affine metric h. As M^n is an affine sphere, we have that the shape operator is a multiple of the identity, say $S = \varepsilon I$. Using as well the Gauss equation in (2), the apolarity condition in proposition 5 and the isotropy condition (4) we have

$$\begin{aligned} Ric(e_j', e_k') &= \sum_{i=1}^{n} \varepsilon_i h(\hat{R}(e_i', e_j')e_k', e_i') \\ &= h(\varepsilon(\varepsilon_j \delta_{jk} e_i' - \varepsilon_i \delta_{ik} e_j') - [K_{e_i'}, K_{e_j'}]e_k', e_i') \\ &= n\varepsilon\varepsilon_j \delta_{jk} - \varepsilon\varepsilon_j \delta_{jk} - \sum_i \varepsilon_i h([K_{e_i}, K_{e_j}]e_k, e_i). \end{aligned} \tag{26}$$

For $k \neq j$ we obtain

$$\begin{aligned} Ric(e_j', e_k') &= -\sum_{i=1}^{n} \varepsilon_i h([K_{e_i'}, K_{e_j'}]e_k', e_i') \\ &= \sum_{i=1}^{n} \varepsilon_i h(K(e_i', e_j'), K(e_i', e_k')) \\ &= -\frac{1}{2} \sum_{i=1}^{n} \varepsilon_i h(K(e_i', e_i'), K(e_j', e_k')) \\ &= 0 \end{aligned} \tag{27}$$

and for $k = j$

$$
\begin{aligned}
Ric(e'_j, e'_j) &= \sum_{i=1}^{n} \varepsilon h(K(e'_i, e'_j), K(e'_i, e'_j)) \\
&= \sum_{i=1}^{n} \frac{1}{2}[-h(K(e'_j, e'_j), K(e'_i, e'_i)) + 2\lambda(p)\delta_{ij} + \lambda(p)\varepsilon_i\varepsilon_j] \\
&= \left(\frac{n}{2} + 1\right) \varepsilon_j \lambda(p).
\end{aligned} \tag{28}
$$

Since $n \geq 3$, by using the fact that the Levi-Civita connection on M^n is torsion free and using the second Bianchi identity, we get that λ is constant. $\square$

Similarly to [9], Proposition 3.6, we can prove the following:

Proposition 14. *Let M^n be an n-dimensional affine submanifold in $\mathbb{R}^{n+1}$. If M^n is constant isotropic with $\lambda \neq 0$, then M^n has parallel difference tensor.*

Proof Since M^n is constant isotropic, we have $\lambda = h(K(v, v), K(v, v))$ and by taking the derivative, we obtain $h(\hat{\nabla}_\chi K(v, v), K(v, v)) = 0, \forall p \in M^n, \forall v, \chi \in T_pM^n, h(v, v) = 1$.
In the isotropy relation (4) we take $X_1 = \hat{\nabla}_\chi v$, $X_2 = X_3 = X_4 = v$ and obtain $h(K(\hat{\nabla}_\chi v, v), K(v, v)) = \lambda h(\hat{\nabla}_\chi v, v)h(v, v) = 0$, for $h(v, v) = 1$. This implies

$$
h((\hat{\nabla} K)(\chi, v, v), h(v, v)) = 0. \tag{29}
$$

for any $v, \chi \in T_pM^n$ such that $h(v, v) = 1$ and in particular, we have

$$
h((\hat{\nabla} K)(v, v, v), h(v, v)) = 0, \tag{30}
$$

Further on, we take the derivative with respect to some vector $U \in T_pM^n$ in equation (4) for $X_1 = X_2 = X_3 = v, X_4 = w$ and for $h(v, w) = 0$ and obtain

$$
h((\hat{\nabla} K)(v, v, v), K(v, w)) - h((\hat{\nabla} K)(v, v, w), K(v, v)) = 0.
$$

As $\hat{\nabla} K$ is totally symmetric, using also (29) we have

$$
h((\hat{\nabla} K)(v, v, v,), K(v, w)) = 0, \tag{31}
$$

for any $v, \chi \in T_pM^n$ such that $h(v, v) = 1$. We can write $K(v, K(v, v)) = av + bw$, for $v \in T_pM^n$, w an $(n-1)$-dimensional tangent vector , $h(v, w) = 0$. Since

$$
\begin{cases}
h(K(v, K(v, v)), v) = h(K(v, v), K(v, v)) = \lambda, \\
h(K(v, K(v, v)), w) = bh(w, w) = 0,
\end{cases}
$$

we get $a = \lambda, b = 0$ so that $K(v, K(v, v)) = \lambda v$. If we take $w = K(v, v)$ in equation (31) we get

$$
\lambda h((\hat{\nabla} K)(v, v, v), v) = 0. \tag{32}
$$

As $\lambda \neq 0$, using (32) and the symmetry of $\hat{\nabla} K$, we also have $\hat{\nabla} K = 0$.

Proposition 15. *Let $n \geq 3$ and M^n be an $n-$dimensional $\lambda-$isotropic affine hypersphere in $\mathbb{R}^{n+1}$, such that $S = \varepsilon I$, with ε constant. Assume that $\lambda \neq 0$. If $\mathbb{R}^{n+1}$ is endowed with an indefinite metric and M^n is not totally geodesic, then M^n is a locally symmetric space and $\lambda = -\frac{1}{2}\varepsilon$.*

Proof. From the previous propositions we conclude that $\hat{\nabla} K = 0$. Hence, by the Gauss equation we have $\hat{\nabla} R = 0$, which means that M^n is a locally symmetric space. Using the Ricci identity, from $\hat{\nabla} K = 0$ we also have $\hat{R}.h = 0$, that is

$$
\hat{R}(X, Y)K(Z, W) - K(\hat{R}(X, Y)Z, W) - K(Z, \hat{R}(X, Y)W) = 0, \tag{33}
$$

for X, Y, Z, W tangent vector fields. If we take $X = Z = W = f_1, Y = f_2$, it implies

$$\hat{R}(f_1, f_2)K(f_1, f_1) = 2K(\hat{R}(f_1, f_2)f_1, f_1) \tag{34}$$

and then, from Gauss equation we have

$$R(f_1, f_2)f_1 = -(\varepsilon + 2\lambda)f_2,$$

which together with (34) implies $\varepsilon + 2\lambda = 0$. □

Proposition 16. *Let $n \geq 3$, $f_1 : M_1^n \to \mathbb{R}^n$ and $f_2 : M_1^n \to \mathbb{R}^n$ be $n-$dimensional $\lambda-$isotropic affine hypersphere in $\mathbb{R}^{n+1}$, such that $S_1 = S_2 = \varepsilon I$, with $\varepsilon = \pm 1$ constant. Let $p_1 \in M_1$ and $p_2 \in M_2$ and assume that there exists an isometry $A : T_{p_1}M_1 \to T_{p_2}M_2$ such that*

$$AK_1(v, w) = K_2(Av, Aw),$$

i.e. A preserves the difference tensor. Then there exists a local isometry $F : (M_1, h_1) \to (M_2, h_2)$ such that

$$dF(K_1(X, Y)) = K_2(dF(X), dF(Y)),$$

for any vector fields X, Y on M_1. Moreover the immersions f_1 and $f_2 \circ F$ are locally congruent.

Proof. From the previous propositions we know that λ is a constant, and that with respect to the Levi Civita connection M_1 and M_2 are locally symmetric spaces whose difference tensor is parallel with respect to the Levi Civita connection.

We take $p_1 \in M_1$ and we take a basis $\{e'_1, \ldots, e'_n\}$ of $T_{p_1}M_1$. As A is an isometry we take as basis of $T_{p_2}M_2$ the vectors $\{Ae'_1, \ldots, Ae'_n\}$. By the initial conditions we have that

$$h_1(e'_i, e'_j) = h_2(Ae'_iAe'_j) \qquad \text{(isometry)}$$
$$h_1(K_1(e'_i, e'_j), e'_k) = h_2(AK_1(e'_i, e'_j), Ae'_k) = h_2(K_2(Ae'_i, Ae'_j), Ae'_k).$$

We now extend $\{e'_1, \ldots, e'_n\}$ to a local differential basis $\{X_1, \ldots, X_n\}$ by parallel translation along geodesics with respect to the Levi Civita connection of the affine metric. In the same way we extend $\{Ae'_1, \ldots, Ae'_n\}$ to local vector fields $\{Y_1, \ldots, Y_n\}$. As the difference tensors are parallel, we have that the components of the difference tensor stay constant along geodesics. Therefore by construction, we have that

$$h_1(X_i, X_j) = h_2(Y_i, Y_j),$$
$$h_1(K_1(X_i, X_j), X_k) = h_2(K_2(Y_i, Y_j), Y_k)$$

Hence by the lemma of Cartan, see [7] we know that there exists a local isometry F such that $dF(X_i) = Y_i$. In order to complete the proof it is now sufficent to apply Theorem 9. □

So in order to complete the classification it is now sufficient to determine, up to isometries, the possible forms of the difference tensor and for each of those forms obtained to determine an explicit example of an affine hypersphere with isotropic difference tensors. This is done explicitly for the 4 remaining dimensions 5, 8, 14 and 26 in the next sections.

5. Affine hyperspheres of dimension 5

5.1. **The form of L, $\dim\mathcal{U} = 1$.** We start with $\omega = v + iTv \in \mathcal{W}_1$, a vector of length 2. As the length of ω is 2 it follows that v has unit length and is orthogonal to Tv. So we can take $\omega_1^1 = v$ and $\omega_2^1 = Tv$. Note that by the properties of L we have that $L(v + iTv, v - iTv)$ is a real vector in $\mathcal{U}^{\mathbb{C}}$ whose square length is $3\mu^2$. Hence we can pick a unit vector u_1 in U such that

$$L(v + iTv, v - iTv) = \sqrt{3}\mu u_1.$$

By the properties of L this implies that

$$L(\omega_1^1, \omega_1^1) = \tfrac{\sqrt{3}}{2}\mu u_1.$$

From the properties of T we see $L(\omega_1^1, \omega_2^1) = 0$ and $L(\omega_1^1, \omega_1^1) = L(\omega_2^1, \omega_2^1)$, hence L is completely determined. Therefore L and also K are completely determined and the signature of the metric, if necessary after replacing ξ by $-\xi$ in order to make $\lambda > 0$, equals 2.

5.2. **A canonical example.**
We consider $\mathbb{R}^6 = s(3)$ as the set of all symmetric 3×3 matrices and we take as hypersurface M those symmetric matrices with determinant 1. We define an action σ of $SL(3, \mathbb{R})$ on M as follows

$$\sigma : \ SL(3, \mathbb{R}) \times \to M, \ \text{such that} \ (g, p) \mapsto \sigma_g(p) = gpg^T.$$

Note that M has two connected components and that the action is transitive on each of the connected components. The connected component of I has been studied in [2] where it was shown that it gives a positive definite isotropic affine hypersurface. It also appears in [8]. Here we are interested in the component of the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, which we denote by M_1. So $M_1 = \{gAg^T | g \in SL(3, \mathbb{R})\}$. The isotropy group of A consists of the matrices g of determinant 1 such that $gAg^T = A$. This Lie group is congruent to $SO(2, 1)$ and therefore, by Theorem 9.2 of [3], we know that M_1 is locally isometric with $\frac{SL(3,\mathbb{R})}{SO(2,1)}$.

Note that, of course, every element of $SL(3, \mathbb{R})$ acts at the same time also on $s(3)$ and that this action belongs to $SL(6, \mathbb{R})$, see [13]. This implies that M_1 is at the same time an homogeneous affine hypersurface and by Proposition 6 an equiaffine sphere centered at the origin.

In order to determine the tangent space at $p = gAg^T$, we look at the curves in M_1

$$\gamma(s) = ge^{sX} A e^{sX^T} g^T.$$

These are indeed curves in M_1, provided that $e^{sX} \in SL(3, \mathbb{R})$ or, equivalently, provided that $\operatorname{Tr} X = 0$. Note that $\gamma'(s) = ge^{sX}(XA + AX^T)e^{sX^T} g^T$, where $v = (XA + AX^T)$ is a symmetric matrix. So by using a dimension argument we see that the tangent space is given by

$$\{gvg^T | v = 2XA, XA = AX^T, \operatorname{Tr} X = 0, X \in \mathbb{R}^{3\times 3}\} = T_pM_1.$$

Working now at the point A, taking $g = I$ and $X \in so(2,1) = \{X \in \mathbb{R}^{3\times 3} | \operatorname{Tr} X = 0, XA = AX^T\}$ we see that

$$\begin{aligned}\nabla_{\gamma'(s)}\gamma'(s) + h(\gamma',\gamma')\gamma &= \gamma''(s)\\ &= e^{sX}(4X^2A)e^{sX^T}\\ &= e^{sX}((4X^2 - \tfrac{4}{3}\operatorname{Tr}(X^2)I)A)e^{sX^T} + \tfrac{4}{3}\operatorname{Tr}(X^2)e^{sX}Ae^{sX^T}\\ &= e^{sX}((4X^2 - \tfrac{4}{3}\operatorname{Tr}(X^2)I)A)e^{sX^T} + \tfrac{4}{3}\operatorname{Tr}(X^2)\gamma(s).\end{aligned}$$

As the matrix $(4X^2 - \frac{4}{3}\operatorname{Tr}(X^2)I)$ commutes with A, we can decompose the above expression into a tangent part and a part in the direction of the affine normal given by the position vector, and therefore we find that

$$h(\gamma'(s),\gamma'(s)) = \tfrac{4}{3}\operatorname{Tr}(X^2).$$

So we see that s is a constant length parametrisation of the curve γ and therefore we have that $h(\gamma', \hat{\nabla}_{\gamma'}\gamma') = 0$ and

$$h(\gamma', \nabla_{\gamma'}\gamma') = h(\gamma', K(\gamma',\gamma'))$$

In addition, we have

$$\begin{aligned}\gamma'''(s) &= \nabla_{\gamma'(s)}\nabla_{\gamma'(s)}\gamma'(s) + h(\gamma',\gamma')\gamma' + h(\gamma', K(\gamma',\gamma')\gamma\\ &= e^{sX}(8X^3A)e^{sX^T}\\ &= e^{sX}((8X^3 - \tfrac{8}{3}\operatorname{Tr}(X^3)I)A)e^{sX^T} + \tfrac{8}{3}\operatorname{Tr}(X^3)\gamma(s).\end{aligned}$$

So therefore working at $s = 0$ and writing $v = 2XA$ as tangent vector, we obtain that

$$\begin{aligned}h(v,v) &= \tfrac{4}{3}\operatorname{Tr}(X^2),\\ h(v, K(v,v)) &= \tfrac{8}{3}\operatorname{Tr} X^3.\end{aligned}$$

Linearising the above expressions, i.e. writing $v = \alpha_1 v_1 + \alpha_2 v_2$, respectively $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$, for $v = 2X_iA, i = 1,2,3$, and looking at the coefficient of $\alpha_1\alpha_2$, respectively $\alpha_1\alpha_2\alpha_3$, we obtain that

$$\begin{aligned}h(v_1,v_2) &= \tfrac{4}{3}\operatorname{Tr}(X_1X_2) = \tfrac{4}{3}\operatorname{Tr}(X_2X_1)\\ 6h(K(v_1,v_2),v_3) &= \tfrac{8}{3}(\operatorname{Tr} X_1X_2X_3 + \operatorname{Tr} X_3X_1X_2 + \operatorname{Tr} X_2X_3X_1 + \operatorname{Tr} X_1X_3X_2 + \operatorname{Tr} X_3X_2X_1 + \operatorname{Tr} X_2X_1X_3)\\ &= 8(\operatorname{Tr} X_1X_2X_3 + \operatorname{Tr} X_2X_1X_3).\end{aligned}$$

So we see that

$$K(v_1,v_2) = 2(X_1X_2 + X_2X_1 - \tfrac{2}{3}\operatorname{Tr}(X_1X_2)I)A.$$

Indeed, we have that $(X_1X_2 + X_2X_1 - \frac{2}{3}\operatorname{Tr}(X_1X_2)I)$ has vanishing trace, commutes with A and therefore $K(v_1,v_2)$ is indeed the unique tangent vector such that

$$h(K(v_1,v_2),v_3) = \tfrac{4}{3}(\operatorname{Tr}(X_1X_2X_3) + \operatorname{Tr}(X_2X_1X_3)).$$

As by Cayley Hamilton, for a matrix X with vanishing trace, we have that $X^3 = 1/2\operatorname{Tr}(X^2)X + det(X)I$, we deduce that

$$\operatorname{Tr} X^4 = \tfrac{1}{2}(\operatorname{Tr} X^2)^2,$$

and therefore we have that

$$\begin{aligned}h(K(v,v),K(v,v)) &= \tfrac{4}{3}\operatorname{Tr}(2X^2 - \tfrac{2}{3}\operatorname{Tr} X^2 I)^2\\ &= \tfrac{4}{3}(4\operatorname{Tr} X^4 + \tfrac{4}{9}(TrX^2)^2 TrI - \tfrac{8}{3}(\operatorname{Tr} X^2)^2)\\ &= \tfrac{8}{9}(TrX^2)^2\\ &= \tfrac{1}{2}(h(v,v))^2.\end{aligned}$$

Hence M_1 is isotropic with positive λ. A straightforward computation also shows that the index of the metric is 2. Combining therefore the results in this section with Proposition 16 and the classification result of O. Birembaux and M. Djoric, see [2] in the positive definite case, we get the following theorem:

Theorem 17. *Let M be a 5-dimensional λ-isotropic affine hypersphere, with $\lambda \neq 0$. Therefore, if necessary, by changing the sign of the affine normal, we may assume that λ is positive. Then either*

(1) the metric is positive definite, M is isometric with $\frac{SL(3,\mathbb{R})}{SO(3)}$ and M is affine congruent to an open part of the hypersurface $\{gg^T | g \in SL(3,\mathbb{R})\}$ of $\mathbb{R}^6 \equiv s(3) \subset \mathbb{R}^{3\times 3}$, or

(2) the metric has signature 2, M is isometric with $\frac{SL(3,\mathbb{R})}{SO(2,1)}$ and M is affine congruent to an open part of the hypersurface $\{gAg^T | g \in SL(3,\mathbb{R})\}$, where $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, of $\mathbb{R}^6 \equiv s(3) \subset \mathbb{R}^{3\times 3}$.

6. Affine hyperspheres of dimension 8

6.1. **The form of L, $\dim \mathcal{U} = 2$.**
Let $W = span\{\omega_1^1, \omega_2^1, \omega_1^2, \omega_2^2\}$ and $\mathcal{W}_1 = span\{\omega_1^1 + i\omega_2^1, \omega_1^2 + i\omega_2^2\}$, $\mathcal{W}_2 = span\{\omega_1^1 - i\omega_2^1, \omega_1^2 - i\omega_2^2\}$. Remark that all the bases are orthogonal and in addition

$$h(\omega_1^1 + i\omega_2^1, \omega_1^1 + i\omega_2^1) = h(\omega_1^2 + i\omega_2^2, \omega_1^2 + i\omega_2^2) = 2,$$
$$h(\omega_1^1 - i\omega_2^1, \omega_1^1 - i\omega_2^1) = h(\omega_1^2 - i\omega_2^2, \omega_1^2 - i\omega_2^2) = -2.$$

Then, straightforward computations lead to

$$\begin{cases} L(\omega_1^1 + i\omega_2^1, \omega_1^1 - i\omega_2^1) = 2L(\omega_1^1, \omega_1^1), \\ L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2) = 2L(\omega_1^1, \omega_1^2) - 2iL(\omega_1^1, \omega_2^2), \\ L(\omega_1^2 + i\omega_2^2, \omega_1^1 - i\omega_2^1) = 2L(\omega_1^1, \omega_1^2) + 2iL(\omega_1^1, \omega_2^2), \\ L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2) = 2L(\omega_1^2, \omega_1^2). \end{cases} \tag{35}$$

Notice that the vector $L(\omega_1^1 + i\omega_2^1, \omega_1^1 - i\omega_2^1)$ is a real vector of length $3\mu^2$. So we can pick $u_1 \in \mathcal{U}$, $h(u_1, u_1) = 1$ such that

$$L(\omega_1^1 + i\omega_2^1, \omega_1^1 - i\omega_2^1) = \sqrt{3}\mu u_1. \tag{36}$$

With this choice, from property (24) we obtain that $L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2)$ is orthogonal to u_1. Moreover as its length is a real number, we must have that $Re(L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2)$ and $Im(L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2)$ are orthogonal to each other. As they are also both orthogonal to u_1, one of them has to vanish. Therefore, we get two cases:

Case II–1. $Re(L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2)) = 0$

Now we obtain that $L(\omega_1^1, \omega_1^2) = 0$ and $L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2)$ is an imaginary vector of length $3\mu^2$, orthogonal to u_1. Thus, we can pick $u_2 \in \mathcal{U}$ in the direction of $L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2)$ such that $h(u_2, u_2) = -1$ and such that

$$L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2) = i\sqrt{3}\mu u_2. \tag{37}$$

Consider now $L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2)$. It is a real vector orthogonal to u_2, of length $3\mu^2$ and thus we can write $L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2) = \pm\sqrt{3}\mu u_1$.

Furthermore, from (35), (37) and Proposition 11- (4), we obtain

$$\begin{aligned} 3\mu^2 &= -h(L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2), L(\omega_1^2 + i\omega_2^2, \omega_1^1 - i\omega_2^1)) \\ &= h(L(\omega_1^1 + i\omega_2^1, \omega_1^1 - i\omega_2^1), L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2)) \\ &= \sqrt{3}\mu h(u_1, L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2)). \end{aligned}$$

So we get that $L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu u_1$. In this case, the signature of the metric is 4.

Case II–2. $Im(L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2)) = 0$

Reasoning in a similar way, we choose $u_2 \in \mathcal{U}$ a real vector in the direction of $L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2)$, with $h(u_2, u_2) = 1$ such that $L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu u_2$. We find $L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2) = -\sqrt{3}\mu u_1$ and in this case the signature of the metric is 3.

6.2. **Two canonical examples.**

First we consider $\mathbb{R}^9$ as the set of Hermitian symmetric matrics $Y \in \mathbb{C}^{3\times 3}$. We take as hypersurface M those Hermitian symmetric matrices with determinant 1. We define an action σ of $SL(3,\mathbb{C})$ on M as follows

$$\sigma: \ SL(3,\mathbb{C}) \times M \to M, \text{ such that } (g,p) \mapsto \sigma_g(p) = gp\bar{g}^T.$$

Note that M has two connected components and that the action is transitive on each of the connected components. The connected component of I has been studied in [2], where it was shown that it gives a positive definite isotropic affine hypersurface. It also appears in [8]. Here we are interested in the component of the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, which we denote by M_1. So $M_1 = \{gA\bar{g}^T | g \in SL(3,\mathbb{R})\}$. The isotropy group consists of the matrices g of determinant 1 such that $gA\bar{g}^T = A$. This Lie group is congruent to $SU(2,1)$ and therefore, by Theorem 9.2 of [3], we know that M_1 is locally isometric with $\frac{SL(3,\mathbb{C})}{SU(2,1)}$.

Note that of course every element of $SL(3,\mathbb{C})$ acts at the same time also on $\mathbb{R}^9$ in a linear way and that, therefore, this action belongs to $GL(9,\mathbb{R})$. A straightforward computation shows that this action actually belongs to $SL(9,\mathbb{R})$. This implies that M_1 is at the same time an homogeneous affine hypersurface and, by Proposition 6, an equiaffine sphere centered at the origin. So, in order to determine the properties of M_1, it is sufficient to look at a single point.

In order to determine the tangent space at the point $p = gA\bar{g}^T$, we look at the curves in M_1

$$\gamma(s) = ge^{sX}Ae^{s\bar{X}^T}\bar{g}^T.$$

These are indeed curves in M_1 provided that $e^{sX} \in SL(3,\mathbb{C})$ or equivalently provided that $\operatorname{Tr} X = 0$. Note that $\gamma'(s) = ge^{sX}(XA + A\bar{X}^T)e^{s\bar{X}^T}\bar{g}^T$, where $v = (XA + A\bar{X}^T)$ is a Hermitian symmetric matrix. So by using a dimension argument we see that the tangent space is given by

$$\{gv\bar{g}^T | v = 2XA, XA = A\bar{X}^T, \operatorname{Tr} X = 0, X \in \mathbb{C}^{3\times 3}\} = T_pM_1.$$

Working now at the point A, taking $g = I$ and $X \in su(2,1) = \{X \in \mathbb{C}^{3\times 3} | \operatorname{Tr} X = 0, XA = A\bar{X}^T\}$ we see that

$$\begin{aligned}
\nabla_{\gamma'(s)}\gamma'(s) + h(\gamma',\gamma')\gamma &= \gamma''(s) \\
&= e^{sX}(4X^2A)e^{sX^T} \\
&= e^{sX}((4X^2 - \tfrac{4}{3}\operatorname{Tr}(X^2)I)A)e^{s\bar{X}^T} + \tfrac{4}{3}\operatorname{Tr}(X^2)e^{sX}Ae^{s\bar{X}^T} \\
&= e^{sX}((4X^2 - \tfrac{4}{3}\operatorname{Tr}(X^2)I)A)e^{s\bar{X}^T} + \tfrac{4}{3}\operatorname{Tr}(X^2)\gamma(s).
\end{aligned}$$

As the matrix $(4X^2 - \frac{4}{3}\operatorname{Tr}(X^2)I)$ commutes with A, we can decompose the above expression into a tangent part and a part in the direction of the affine normal given by the position vector, and therefore we find that

$$h(\gamma'(s), \gamma'(s)) = \tfrac{4}{3}\operatorname{Tr}(X^2).$$

So we see that s is a constant length parametrisation of the curve γ and therefore we have that $h(\gamma', \hat{\nabla}_{\gamma'}\gamma') = 0$ and

$$h(\gamma', \nabla_{\gamma'}\gamma') = h(\gamma', K(\gamma', \gamma')).$$

As

$$\begin{aligned}\gamma'''(s) &= \nabla_{\gamma'(s)}\nabla_{\gamma'(s)}\gamma'(s) + h(\gamma',\gamma')\gamma' + h(\gamma', K(\gamma',\gamma')\gamma \\ &= e^{sX}(8X^3A)e^{s\bar{X}^T} \\ &= e^{sX}((8X^3 - \tfrac{8}{3}\operatorname{Tr}(X^3)I)A)e^{s\bar{X}^T} + \tfrac{8}{3}\operatorname{Tr}(X^3)\gamma(s),\end{aligned}$$

working at $s = 0$ and writing $v = 2XA$ as tangent vector, we have that

$$\begin{aligned}h(v,v) &= \tfrac{4}{3}\operatorname{Tr}(X^2), \\ h(v, K(v,v)) &= \tfrac{8}{3}\operatorname{Tr} X^3.\end{aligned}$$

Linearising the above expressions, i.e. writing $v = \alpha_1 v_1 + \alpha_2 v_2$, respectively $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$, for $v = 2X_iA, i = 1,2,3$, and looking at the coefficient of $\alpha_1\alpha_2$, respectively $\alpha_1\alpha_2\alpha_3$, we obtain that

$$\begin{aligned}h(v_1,v_2) &= \tfrac{4}{3}\operatorname{Tr}(X_1X_2) = \tfrac{4}{3}\operatorname{Tr}(X_2X_1), \\ 6h(K(v_1,v_2),v_3) &= \tfrac{8}{3}(\operatorname{Tr} X_1X_2X_3 + \operatorname{Tr} X_3X_1X_2 + \operatorname{Tr} X_2X_3X_1 + \operatorname{Tr} X_1X_3X_2 + \operatorname{Tr} X_3X_2X_1 + \operatorname{Tr} X_2X_1X_3) \\ &= 8(\operatorname{Tr} X_1X_2X_3 + \operatorname{Tr} X_2X_1X_3).\end{aligned}$$

So we see that

$$K(v_1,v_2) = 2(X_1X_2 + X_2X_1 - \tfrac{2}{3}\operatorname{Tr}(X_1X_2)I)A.$$

Indeed, we have that $(X_1X_2 + X_2X_1 - \frac{2}{3}\operatorname{Tr}(XY)I)$ has vanishing trace, commutes with A and therefore $K(v_1,v_2)$ is indeed the unique tangent vector such that

$$h(K(v_1,v_2),v_3) = \tfrac{4}{3}(\operatorname{Tr}(X_1X_2X_3) + \operatorname{Tr}(X_2X_1X_3)).$$

As by Cayley Hamilton for a matrix X with vanishing trace we have that $X^3 = 1/2\operatorname{Tr}(X^2)X + det(X)I$, we deduce that

$$\operatorname{Tr} X^4 = \tfrac{1}{2}\operatorname{Tr} X^2,$$

and therefore, we have that

$$\begin{aligned}h(K(v,v),K(v,v)) &= \tfrac{4}{3}\operatorname{Tr}(2X^2 - \tfrac{2}{3}\operatorname{Tr} X^2 I)^2 \\ &= \tfrac{4}{3}(4\operatorname{Tr} X^4 + \tfrac{4}{9}(TrX^2)^2 TrI - \tfrac{8}{3}(\operatorname{Tr} X^2)^2) \\ &= \tfrac{8}{9}(TrX^2)^2 \\ &= \tfrac{1}{2}(h(v,v))^2.\end{aligned}$$

Hence M_1 is isotropic with positive λ. A straightforward computation also shows that the index of the metric is 4.

Next, we consider $\mathbb{R}^9 = \mathbb{R}^{3\times 3}$. We take as hypersurface M_2 those matrices with determinant 1. We define an action σ of $SL(3,\mathbb{R})$ on M_2 as follows

$$\sigma: \ SL(3,\mathbb{R}) \times M_2 \to M_2, \text{ such that } (g,p) \mapsto \sigma_g(p) = gp.$$

The isotropy group of the identity matrix consists only of the identity matrix. Therefore, by Theorem 9.2 of [3] we know that M_2 is locally isometric with $SL(3,\mathbb{R})$.

Note that, of course, every element of $SL(3,\mathbb{R})$ acts at the same time also on $\mathbb{R}^9$ in a linear way and that therefore this action belongs to $GL(9,\mathbb{R})$. A straightforward computation shows that this action actually belongs to $SL(9,\mathbb{R})$. This implies that M_2 is at the same time an homogeneous affine hypersurface and, by Proposition 6, an equiaffine sphere centered at the origin. So in order to determine the properties of M_2 it is sufficient to look at a single point.

In order to determine the tangent space at a point p, we look at the curves in M_2

$$\gamma(s) = e^{sX}p.$$

These are indeed curves in M_2, provided that $e^{sX} \in SL(3,\mathbb{R})$ or equivalently, provided that $\operatorname{Tr} X = 0$. Note that $\gamma'(s) = e^{sX}Xp$, so by using a dimension argument we see that the tangent space is given by

$$\{Xp | \operatorname{Tr} X = 0, X \in \mathbb{R}^{3\times 3}\} = T_pM_2.$$

Working now at the point I and $X \in sl(3,\mathbb{R}) = \{X \in \mathbb{R}^{3\times 3} | \operatorname{Tr} X = 0\}$, we see that

$$\begin{aligned}
\nabla_{\gamma'(s)}\gamma'(s) + h(\gamma',\gamma')\gamma &= \gamma''(s)\\
&= e^{sX}X^2\\
&= e^{sX}(X^2 - \tfrac{1}{3}\operatorname{Tr}(X^2)I) + \tfrac{1}{3}\operatorname{Tr}(X^2)e^{sX}\\
&= e^{sX}(X^2 - \tfrac{1}{3}\operatorname{Tr}(X^2)I) + \tfrac{1}{3}\operatorname{Tr}(X^2)\gamma(s).
\end{aligned}$$

As the matrix $(X^2 - \frac{1}{3}\operatorname{Tr}(X^2)I)$ commutes with e^{sX} and has vanishing trace, we can interprete $e^{sX}(X^2 - \frac{1}{3}\operatorname{Tr}(X^2)I)$ as a tangent vector at the point e^{sX}. By decomposing the above expression into a tangent part and a part in the direction of the affine normal given by the position vector, we deduce that

$$h(\gamma'(s),\gamma'(s)) = \tfrac{1}{3}\operatorname{Tr}(X^2).$$

So we see that s is a constant length parametrisation of the curve γ and therefore we have that $h(\gamma', \hat{\nabla}_{\gamma'}\gamma') = 0$ and

$$h(\gamma', \nabla_{\gamma'}\gamma') = h(\gamma', K(\gamma',\gamma')).$$

As

$$\begin{aligned}
\gamma'''(s) &= \nabla_{\gamma'(s)}\nabla_{\gamma'(s)}\gamma'(s) + h(\gamma',\gamma')\gamma' + h(\gamma', K(\gamma',\gamma')\gamma\\
&= e^{sX}X^3\\
&= e^{sX}(X^3 - \tfrac{1}{3}\operatorname{Tr}(X^3)I)e^{sX^T} + \tfrac{1}{3}\operatorname{Tr}(X^3)\gamma(s),
\end{aligned}$$

working at $s = 0$ and writing $v = X$ as tangent vector, we have that

$$\begin{aligned}
h(v,v) &= \tfrac{1}{3}\operatorname{Tr}(X^2),\\
h(v,K(v,v)) &= \tfrac{1}{3}\operatorname{Tr} X^3.
\end{aligned}$$

Linearising the above expressions, i.e. writing $v = \alpha_1 v_1 + \alpha_2 v_2$, respectively $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$, for $v_i = X_i, i = 1,2,3$, and looking at the coefficient of $\alpha_1\alpha_2$, respectively $\alpha_1\alpha_2\alpha_3$ we obtain that

$$\begin{aligned}
h(v_1,v_2) &= \tfrac{1}{3}\operatorname{Tr}(X_1X_2) = \tfrac{1}{3}\operatorname{Tr}(X_2X_1),\\
6h(K(v_1,v_2),v_3) &= \tfrac{1}{3}(\operatorname{Tr} X_1X_2X_3 + \operatorname{Tr} X_3X_1X_2 + \operatorname{Tr} X_2X_3X_1 + \operatorname{Tr} X_1X_3X_2 + \operatorname{Tr} X_3X_2X_1 + \operatorname{Tr} X_2X_1X_3)\\
&= (\operatorname{Tr} X_1X_2X_3 + \operatorname{Tr} X_2X_1X_3).
\end{aligned}$$

So we see that

$$K(v_1,v_2) = \tfrac{1}{2}(X_1X_2 + X_2X_1 - \tfrac{2}{3}\operatorname{Tr}(X_1X_2)I).$$

Indeed, we have that $(X_1X_2 + X_2X_1 - \frac{2}{3}\operatorname{Tr}(X_1,X_2)I)$ has vanishing trace and therefore $K(v_1,v_2)$ is indeed the unique tangent vector such that

$$h(K(v_1,v_2),v_3) = \tfrac{1}{6}(\operatorname{Tr}(X_1X_2X_3) + \operatorname{Tr}(X_2X_1X_3)).$$

As by Cayley Hamilton for a matrix X with vanishing trace we have that $X^3 = 1/2\operatorname{Tr}(X^2)X + det(X)I$, we deduce that

$$\operatorname{Tr} X^4 = \tfrac{1}{2}(\operatorname{Tr} X^2)^2,$$

and therefore we have that

$$\begin{aligned} h(K(v,v),K(v,v)) &= \tfrac{1}{3}\operatorname{Tr}(X^2-\tfrac{1}{3}\operatorname{Tr}X^2 I)^2\\ &= \tfrac{1}{3}(\operatorname{Tr}X^4+\tfrac{1}{9}(TrX^2)^2 TrI-\tfrac{2}{3}(\operatorname{Tr}X^2)^2)\\ &= \tfrac{1}{18}(TrX^2)^2\\ &= \tfrac{1}{2}(h(v,v))^2. \end{aligned}$$

Hence M_1 is isotropic with positive λ. A straightforward computation also shows that the index of the metric is 3.

Combining therefore the results in this section with Proposition 16 and the classification result of O. Birembaux and M. Djoric, see [2] in the positive definite case, we get the following theorem:

Theorem 18. *Let M be a 8-dimensional λ-isotropic affine hypersphere, with $\lambda \neq 0$. Therefore if necessary by changing the sign of the affine normal we may assume that λ is positive. Then either*

(1) *the metric is positive definite, M is isometric with $\frac{SL(3,\mathbb{C})}{SU(3)}$ and M is affine congruent to an open part of the hypersurface $\{g\bar{g}^T | g \in SL(3,\mathbb{C})\}$ of $\mathbb{R}^9$, identified with the space of Hermitian symmetric matrices, or*
(2) *the metric has signature 4, M is isometric with $\frac{SL(3,\mathbb{C})}{SU(2,1)}$ and M is affine congruent to an open part of the hypersurface $\{gA|g^T | g \in SL(3,\mathbb{C})\}$ of $\mathbb{R}^9$, where $A = \begin{pmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}$, identified with the space of Hermitian symmetric matrices, or*
(3) *the metric has signature 3, M is isometric with $SL(3,\mathbb{R})$ and M is affine congruent with $SL(3,\mathbb{R})$ considered as a hypersurface in $\mathbb{R}^9$ identified with $\mathbb{R}^{3\times 3}$.*

7. Affine hyperspheres of dimension 14

7.1. The form of L, $\dim \mathcal{U}^{\mathbb{C}} = 4$.

We start with $w_1 \in \mathcal{W}$ a vector with length 1. As $L(w_1+iTw_1, w_1-iTw_1)$ is a real vector in U with length $3\mu^2$ there exists a real unit length vector u_1 in U such that

$$L(w_1+iTw_1, w_1-iTw_1) = \sqrt{3}\mu u_1.$$

We now complete u_1 to a basis of $\mathcal{U}$ by choosing orthogonal u_2, u_3, u_4 in $\{u_1\}^\perp$ such that $h(u_k,u_k) = \varepsilon_k$, where $\varepsilon_k = \pm 1$. We also introduce δ_k, for k=2,3,4, by

$$\delta_k = i, \qquad \text{if } \varepsilon_k = -1 \quad \text{and} \quad \delta_k = 1, \qquad \text{if } \varepsilon_k = 1.$$

Now we apply Proposition 11, which tells us that we can find vectors w_2, w_3, w_4 such that

$$\begin{aligned} L(w_1+iTw_1, w_2-iTw_2) &= \sqrt{3}\mu\delta_2 u_2,\\ L(w_1+iTw_1, w_3-iTw_3) &= \sqrt{3}\mu\delta_3 u_3,\\ L(w_1+iTw_1, w_4-iTw_4) &= \sqrt{3}\mu\delta_4 u_4. \end{aligned}$$

The first two properties of Proposition 11 then tells us that $\{w_1, Tw_1, \ldots, w_4, Tw_4\}$ is a basis of W, as in Lemma 7. Of course the previous equations also imply that

$$L(w_k+iTw_k, w_1-iTw_1) = \sqrt{3}\mu\bar{\delta}_k u_k.$$

We now look at $L(w_2+iTw_2, w_3-iTw_3)$. From the last part of Proposition 11 it follows that this vector is orthogonal to $L(w_2+iTw_2, w_1-iTw_1)$, $L(w_1+$

$iTw_1, w_3 - iTw_3)$ and $L(w_1 + iTw_1, w_1 - iTw_1)$. So this implies that there exists a complex number b_4 such that

$$L(w_2 + iTw_2, w_3 - iTw_3) = b_4 u_4.$$

Similarly, we have that

$$L(w_2 + iTw_2, w_4 - iTw_4) = b_3 u_3,$$
$$L(w_3 + iTw_3, w_4 - iTw_4) = b_2 u_2.$$

Using again Proposition 11 we see that there exists real numbers c_k such that

$$h(L(w_k + iTw_k, w_k - iTw_k), L(w_k + iTw_k, w_k - iTw_k)) = c_k u_k.$$

From

$$\begin{aligned} (38) \quad & h(L(w_k + iTw_k, w_k - iTw_k), L(w_1 + iTw_1, w_1 - iTw_1)) = \\ & - h(L(w_k + iTw_k, w_1 - iTw_1), L(w_1 + iTw_1, w_k - iTw_k)), \end{aligned}$$

it follows that $c_k = -\sqrt{3}\mu\varepsilon_k$. Next we use the fact that for different indices k and ℓ we have that

$$\begin{aligned} (39) \quad & h(L(w_k + iTw_k, w_k - iTw_k), L(w_\ell + iTw_\ell, w_\ell - iTw_\ell)) = \\ & - h(L(w_k + iTw_k, w_\ell - iTw_\ell), L(w_\ell + iTw_\ell, w_k - iTw_k)). \end{aligned}$$

Expressing this for the different possibilities for k and ℓ we find that

$$3\mu^2\varepsilon_2\varepsilon_3 = -|b_4|^2\varepsilon_4$$
$$3\mu^2\varepsilon_2\varepsilon_4 = -|b_3|^2\varepsilon_3$$
$$3\mu^2\varepsilon_4\varepsilon_3 = -|b_2|^2\varepsilon_4.$$

Hence, up to permuting the vectors, we see that there are two possibilities. Either $\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = -1$, in which case the index of the metric is 8 or $\varepsilon_2 = -1$ and $\varepsilon_3 = \varepsilon_4 = 1$, in which case the index of the metric is 6.

Computing the length of $L(w_2 + iTw_2, w_3 - iTw_3)$ we have in both cases that $b_4^2\varepsilon_4 = 3\mu^2$. So if necessary by changing the sign of u_4 and w_4, we may assume that $b_4 = \sqrt{3}\mu$.

We now complete the argument by looking at

$$h(L(w_2{+}iTw_2, w_3{-}iTw_3), L(w_1{+}iTw_1, w_4{-}iTw_4) = -h(L(w_1{+}iTw_1, w_3{-}iTw_3), L(w_2{+}iTw_2, w_4{-}iTw_4)$$

This yields that $b_3 = -\sqrt{3}\mu$. Interchanging the indices 2 and 3 in the formula above finally gives that $b_2 = -\sqrt{3}\mu$ in the first case, and $-\sqrt{3}\mu i$ in the second case.

7.2. **Two canonical examples.**

First we look at the following example. We identify $\mathbb{R}^{15}$ with the set of all skew symmetric matrices in $\mathbb{R}^{6\times 6}$. So an element $p \in \mathbb{R}^{15}$ is of the form

$$p = \begin{pmatrix} 0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ -a_1 & 0 & a_6 & a_7 & a_8 & a_9 \\ -a_2 & -a_6 & 0 & a_{10} & a_{11} & a_{12} \\ -a_3 & -a_7 & -a_{10} & 0 & a_{13} & a_{14} \\ -a_4 & -a_8 & -a_{11} & -a_{13} & 0 & a_{15} \\ -a_5 & -a_9 & -a_{12} & -a_{14} & -a_{15} & 0 \end{pmatrix}.$$

We take as hypersurface M in $\mathbb{R}^{15}$ the skew symmetric matrices with determinant 1. Let $G = SL(6, \mathbb{R})$. Then, we have an action ρ of G on M by $\rho(g)(p) = gpg^T$

Here we are interested in the connected component of the matrix

$$I_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

If necessary, we restrict now M to the orbit of I_0. Its isotropy group consists of the matrices g of determinant 1 such that $gI_0g^T = I_0$. This Lie group is congruent to $Sp(6)$ and therefore by Theorem 9.2 of [3] we know that M_1 is locally isometric with $\frac{SL(6,\mathbb{R})}{Sp(6)}$.

Note that of course every element of $SL(6,\mathbb{R})$ acts at the same time also on $\mathbb{R}^{15}$ in a linear way and that therefore this action belongs to $GL(15,\mathbb{R})$. A straightforward computation shows that this action actually belongs to $SL(15,\mathbb{R})$. This implies that M is at the same time an homogeneous affine hypersurface and by Proposition 6 an equiaffine sphere centered at the origin. So in order to determine the properties of M it is sufficient to look at a single point.

In order to determine the tangent space at a point $p = gI_0g^T$, we look at the curves in M

$$\gamma(s) = ge^{sX}I_0e^{sX^T}g^T.$$

These are indeed curves in M_1, provided that $e^{sX} \in SL(6,\mathbb{R})$ or equivalently, provided that $\operatorname{Tr} X = 0$. Note that $\gamma'(s) = ge^{sX}(XI_0 + I_0X^T)e^{sX^T}g^T$, where $v = (XI_0 + I_0X^T)$ is a symmetric matrix. So by using a dimension argument we see that the tangent space is given by

$$\{gvg^T | v = 2XI_0, XI_0 = I_0X^T, \operatorname{Tr} X = 0, X \in \mathbb{R}^{6\times 6}\} = T_pM.$$

In fact, such a matrix X is of the form

$$X = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & 0 & -a_5 \\ c_1 & c_2 & -a_1-b_2 & 0 & -b_4 & -a_4 \\ d_1 & d_2 & 0 & -a_1-b_2 & b_3 & a_3 \\ e_1 & 0 & -d_2 & c_2 & b_2 & a_2 \\ 0 & -e_1 & -d_1 & c_1 & b_1 & a_1 \end{pmatrix}.$$

Working now at the point I_0, taking $g = I$ and $X \in \{X \in \mathbb{R}^{6\times 6} | \operatorname{Tr} X = 0, XI_0 = I_0X^T\}$ we see that

$$\begin{aligned} \nabla_{\gamma'(s)}\gamma'(s) + h(\gamma',\gamma')\gamma &= \gamma''(s) \\ &= e^{sX}(4X^2I_0)e^{sX^T} \\ &= e^{sX}((4X^2 - \tfrac{4}{6}\operatorname{Tr}(X^2)I)I_0)e^{sX^T} + \tfrac{4}{6}\operatorname{Tr}(X^2)e^{sX}I_0e^{sX^T} \\ &= e^{sX}((4X^2 - \tfrac{4}{6}\operatorname{Tr}(X^2)I)I_0)e^{sX^T} + \tfrac{4}{6}\operatorname{Tr}(X^2)\gamma(s). \end{aligned}$$

As the matrix $(4X^2 - \frac{4}{6}\operatorname{Tr}(X^2)I)$ commutes with I_0, we can decompose the above expression into a tangent and a part in the direction of the affine normal given by the position vector, and therefore we find that

$$h(\gamma'(s),\gamma'(s)) = \tfrac{4}{6}\operatorname{Tr}(X^2).$$

So we see that s is a constant length parametrisation of the curve γ and therefore we have that $h(\gamma', \hat{\nabla}_{\gamma'}\gamma') = 0$ and

$$h(\gamma', \nabla_{\gamma'}\gamma') = h(\gamma', K(\gamma',\gamma'))$$

As

$$\begin{aligned}\gamma'''(s) &= \nabla_{\gamma'(s)}\nabla_{\gamma'(s)}\gamma'(s) + h(\gamma',\gamma')\gamma' + h(\gamma', K(\gamma',\gamma')\gamma \\ &= e^{sX}(8X^3 I_0)e^{sX^T} \\ &= e^{sX}((8X^3 - \tfrac{8}{6}\operatorname{Tr}(X^3)I)I_0)e^{sX^T} + \tfrac{8}{6}\operatorname{Tr}(X^3)\gamma(s),\end{aligned}$$

working at $s = 0$ and writing $v = 2XI_0$ as tangent vector, we have that

$$\begin{aligned}h(v,v) &= \tfrac{4}{6}\operatorname{Tr}(X^2), \\ h(v, K(v,v)) &= \tfrac{8}{6}\operatorname{Tr} X^3.\end{aligned}$$

Linearising the above expressions, i.e. writing $v = \alpha_1 v_1 + \alpha_2 v_2$, respectively $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$, for $v_i = 2XI_0, i = 1,2,3$, and looking at the coefficient of $\alpha_1\alpha_2$, respectively $\alpha_1\alpha_2\alpha_3$ we obtain that

$$\begin{aligned}h(v_1,v_2) &= \tfrac{4}{6}\operatorname{Tr}(X_1X_2) = \tfrac{4}{6}\operatorname{Tr}(X_2X_1), \\ 6h(K(v_1,v_2),v_3) &= \tfrac{8}{3}(\operatorname{Tr} X_1X_2X_3 + \operatorname{Tr} X_3X_1X_2 + \operatorname{Tr} X_2X_3X_1 + \operatorname{Tr} X_1X_3X_2 + \operatorname{Tr} X_3X_2X_1 + \operatorname{Tr} X_2X_1X_3) \\ &= 4(\operatorname{Tr} X_1X_2X_3 + \operatorname{Tr} X_2X_1X_3).\end{aligned}$$

So we see that

$$K(v_1,v_2) = 2(X_1X_2 + X_2X_1 - \tfrac{2}{6}\operatorname{Tr}(X_1X_2)I)I_0.$$

Indeed we have that $(X_1X_2 + X_2X_1 - \frac{2}{6}\operatorname{Tr}(X_1X_2)I)$ has vanishing trace, commutes with I_0 and therefore $K(v_1,v_2)$ is indeed the unique tangent vector such that

$$h(K(v_1,v_2),v_3) = \tfrac{2}{3}(\operatorname{Tr}(X_1X_2X_3) + \operatorname{Tr}(X_2X_1X_3)).$$

By straightforward computations we deduce that

$$\operatorname{Tr} X^4 = \tfrac{1}{4}(\operatorname{Tr} X^2)^2,$$

and therefore we have that

$$\begin{aligned}h(K(v,v),K(v,v)) &= \tfrac{4}{6}\operatorname{Tr}(2X^2 - \tfrac{2}{3}\operatorname{Tr} X^2 I)^2 \\ &= \tfrac{4}{6}(4\operatorname{Tr} X^4 + \tfrac{4}{9}(TrX^2)^2 TrI - \tfrac{8}{3}(\operatorname{Tr} X^2)^2) \\ &= \tfrac{2}{9}(TrX^2)^2 \\ &= \tfrac{1}{2}(h(v,v))^2.\end{aligned}$$

Hence M_1 is isotropic with positive λ. A straightforward computation also shows that the index of the metric is 4.

Next, the following example ilustrates the case when the signature of the indefinite metric on M^{14} is 8. First we identify $\mathbb{R}^{15}$ with the set of matrices $a = \{\begin{pmatrix} E & F \\ -\bar{F} & \bar{E}\end{pmatrix}, E = \bar{E}^T, F = -F^T\} \subset \mathbb{C}^{6\times 6}$. An element in a is of the form

$$p := \begin{pmatrix} a_1 & a_2 + ia_3 & a_4 + ia_5 & 0 & a_6 + ia_7 & a_8 + ia_9 \\ a_2 - ia_3 & a_{10} & a_{11} + ia_{12} & -a_6 - ia_7 & 0 & a_{13} + ia_{14} \\ a_4 - ia_5 & a_{11} + ia_{12} & a_{15} & -a_8 - ia_9 & -a_{13} - ia_{14} & 0 \\ 0 & -a_6 + ia_7 & -a_8 + ia_9 & a_1 & a_2 - ia_3 & a_4 - ia_5 \\ a_6 - ia_7 & 0 & -a_{13} + ia_{14} & a_2 + ia_3 & a_{10} & a_{11} - ia_{12} \\ a_8 - ia_9 & a_{13} - ia_{14} & 0 & a_4 + ia_5 & a_{11} + ia_{12} & a_{15}\end{pmatrix}.$$

We take as hypersurface M in $\mathbb{R}^{15}$ all such matrices with determinant 1. Let $G = SU^*(6)$. Then, we have an action ρ of G on M by $\rho(g)(p) = gp\bar{g}^T$. Note that M has two connected components and that the action is transitive on each of the connected components. The connected component of I has been studied in [2], where it was shown that it gives a positive definite isotropic affine hypersurface. Here we are interested in the connected component M_1 containing the matrix

$$I_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

Its isotropy group consists of the matrices g of determinant 1 such that $gI_0\bar{g}^T = I_0$. This Lie group is congruent to $Sp(1,2)$ and therefore, by Theorem 9.2 of [3], we know that M is locally isometric with $\frac{SU^*(6)}{Sp(1,2)}$.

Note that of course every element of $SU^*(6)$ acts at the same time also on $\mathbb{R}^{15}$ in a linear way and that therefore this action belongs to $GL(15,\mathbb{R})$. A straightforward computation shows that this action actually belongs to $SL(15,\mathbb{R})$. This implies that M is at the same time an homogeneous affine hypersurface and, by Proposition 6, an equiaffine sphere centered at the origin. So in order to determine the properties of M_1, it is sufficient to look at a single point.

In order to determine the tangent space at a point $p = gI_0\bar{g}^T$, we look at the curves in M

$$\gamma(s) = ge^{sX}I_0e^{s\bar{X}^T}\bar{g}^T.$$

These are indeed curves in M_1, provided that $\operatorname{Tr} X = 0$ and $XJ = J\bar{X}$, for $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Note that $\gamma'(s) = ge^{sX}(XI_0 + I_0\bar{X}^T)e^{s\bar{X}^T}\bar{g}^T$. So by using a dimension argument, we see that the tangent space is given by

$$\{gv\bar{g}^T | v = 2XI_0, XI_0 = I_0\bar{X}^T, \operatorname{Tr} X = 0, XJ = J\bar{X}, X \in \mathbb{C}^{6\times 6}\} = T_pM_1.$$

In fact, such an X if of the form

$$\begin{pmatrix} -x-x_0 & x_1+iy_1 & x_2+iy_2 & 0 & x_3-iy_3 & x_4-iy_4 \\ x_1-iy_1 & x & x_5+iy_5 & -x_3+iy_3 & 0 & -x_6+iy_6 \\ -x_2+iy_2 & -x_5+iy_5 & x_0 & x_4-iy_4 & x_6-iy_6 & 0 \\ 0 & -x_3-iy_3 & -x_4-iy_4 & -x-x_0 & x_1-iy_1 & -x_2-iy_2 \\ x_3+iy_3 & 0 & -x_6-iy_6 & x_1+iy_1 & x & -x_5+iy_5 \\ x_4+iy_4 & x_6+iy_6 & 0 & x_2+iy_2 & x_5+iy_5 & x_0 \end{pmatrix}.$$

Working now at the point I_0, taking $g = I$ and $X \in \mathbb{C}^{6\times 6}$ satisfying $XI_0 = I_0\bar{X}^T, \operatorname{Tr} X = 0, XJ = J\bar{X}$, we see that

$$\begin{aligned} \nabla_{\gamma'(s)}\gamma'(s) + h(\gamma',\gamma')\gamma &= \gamma''(s) \\ &= e^{sX}(4X^2I_0)e^{s\bar{X}^T} \\ &= e^{sX}((4X^2 - \tfrac{4}{6}\operatorname{Tr}(X^2)I)I_0)e^{s\bar{X}^T} + \tfrac{4}{6}\operatorname{Tr}(X^2)e^{sX}I_0e^{s\bar{X}^T} \\ &= e^{sX}((4X^2 - \tfrac{4}{6}\operatorname{Tr}(X^2)I)I_0)e^{s\bar{X}^T} + \tfrac{4}{6}\operatorname{Tr}(X^2)\gamma(s). \end{aligned}$$

As the matrix $(4X^2 - \frac{4}{6}\operatorname{Tr}(X^2)I)$ has the same properties as X, we can decompose the above expression into a tangent part and a part in the direction of the affine normal given by the position vector, and therefore we find that

$$h(\gamma'(s),\gamma'(s)) = \tfrac{4}{6}\operatorname{Tr}(X^2).$$

So we see that s is a constant length parametrisation of the curve γ and therefore we have that $h(\gamma',\hat{\nabla}_{\gamma'}\gamma') = 0$ and

$$h(\gamma',\nabla_{\gamma'}\gamma') = h(\gamma',K(\gamma',\gamma')).$$

As

$$\begin{aligned}\gamma'''(s) &= \nabla_{\gamma'(s)}\nabla_{\gamma'(s)}\gamma'(s) + h(\gamma',\gamma')\gamma' + h(\gamma', K(\gamma',\gamma')\gamma\\ &= e^{sX}(8X^3 I_0)e^{s\bar{X}^T}\\ &= e^{sX}((8X^3 - \tfrac{8}{6}\operatorname{Tr}(X^3)I)I_0)e^{s\bar{X}^T} + \tfrac{8}{6}\operatorname{Tr}(X^3)\gamma(s),\end{aligned}$$

working at $s = 0$ and writing $v = 2XI_0$ as tangent vector, we have that

$$\begin{aligned}h(v,v) &= \tfrac{4}{6}\operatorname{Tr}(X^2),\\ h(v, K(v,v)) &= \tfrac{8}{6}\operatorname{Tr} X^3.\end{aligned}$$

Linearising the above expressions, i.e. writing $v = \alpha_1 v_1 + \alpha_2 v_2$, respectively $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$, for $v_i = 2XI_0, i = 1,2,3$, and looking at the coefficient of $\alpha_1\alpha_2$, respectively $\alpha_1\alpha_2\alpha_3$ we obtain that

$$\begin{aligned}h(v_1,v_2) &= \tfrac{4}{6}\operatorname{Tr}(X_1X_2) = \tfrac{4}{6}\operatorname{Tr}(X_2X_1),\\ 6h(K(v_1,v_2),v_3) &= \tfrac{8}{3}(\operatorname{Tr} X_1X_2X_3 + \operatorname{Tr} X_3X_1X_2 + \operatorname{Tr} X_2X_3X_1 + \operatorname{Tr} X_1X_3X_2 + \operatorname{Tr} X_3X_2X_1 + \operatorname{Tr} X_2X_1X_3)\\ &= 4(\operatorname{Tr} X_1X_2X_3 + \operatorname{Tr} X_2X_1X_3).\end{aligned}$$

So we see that

$$K(v_1,v_2) = 2(X_1X_2 + X_2X_1 - \tfrac{2}{6}\operatorname{Tr}(X_1X_2)I)I_0.$$

Indeed we have that $(X_1X_2 + X_2X_1 - \frac{2}{6}\operatorname{Tr}(X_1X_2)I)$ has vanishing trace, commutes with I_0 and therefore $K(v_1,v_2)$ is indeed the unique tangent vector such that

$$h(K(v_1,v_2),v_3) = \tfrac{2}{3}(\operatorname{Tr}(X_1X_2X_3) + \operatorname{Tr}(X_2X_1X_3)).$$

By straightforward computations we deduce that

$$\operatorname{Tr} X^4 = \tfrac{1}{4}(\operatorname{Tr} X^2)^2,$$

and therefore we have that

$$\begin{aligned}h(K(v,v),K(v,v)) &= \tfrac{4}{6}\operatorname{Tr}(2X^2 - \tfrac{2}{3}\operatorname{Tr} X^2 I)^2\\ &= \tfrac{4}{6}(4\operatorname{Tr} X^4 + \tfrac{4}{9}(TrX^2)^2 TrI - \tfrac{8}{3}(\operatorname{Tr} X^2)^2)\\ &= \tfrac{2}{9}(TrX^2)^2\\ &= \tfrac{1}{2}(h(v,v))^2.\end{aligned}$$

Hence M_1 is isotropic with positive λ. A straightforward computation also shows that the index of the metric is 8.

8. Affine hyperspheres of dimension 26

8.1. The form of L, $\dim\mathcal{U} = 8$.

Before treating each case of the signature for the metric, we first will give some lemmas which will be very useful in order to simplify the proof significantly. We start with an arbitrary vector $w + iTw \in \mathcal{W}_1$ with length 2 and define a real vector $\sqrt{3}\mu u = L(w + iTw, w - iTw)$. We call $w_1^1 = w$ and $w_2^1 = Tw$. Next, we choose arbitrary orthogonal vectors $u_2, \dots, u_8$ such that $u_1, u_2, \dots, u_8$ forms an orthonormal (real) basis in $\mathcal{U}$, that is $h(u_j, u_k) = \varepsilon_j\delta_{jk}$, where $\varepsilon_j = \pm 1$ indicate the length of u_j. As the operator $L(\omega_1^1 + i\omega_2^1, -)$ is bijective, for every u_j we find ω_j^1, ω_j^2, such that

$$L(\omega_1^1 + i\omega_2^1, \omega_1^j - i\omega_2^j) = \sqrt{3}\mu\delta_j u_j, \text{ where } \delta_j = \begin{cases} 1, & \text{if } \varepsilon = 1\\ i, & \text{if } \varepsilon = -1.\end{cases} \tag{40}$$

Lemma 19. *For the previously defined vectors, L satisfies*

$$L(\omega_1^k + i\omega_2^k, \omega_1^k - i\omega_2^k) = -\sqrt{3}\mu\varepsilon_k u_1.$$

Proof. The result is straightforward, by properties (23) and (24):

(41)
$$\begin{aligned}
&h(L(\omega_1^k+i\omega_2^k,\omega_1^k-i\omega_2^k),L(\omega_1^1+i\omega_2^1,\omega_1^j-i\omega_2^j))=\\
&\qquad=-h(L(\omega_1^1+i\omega_2^1,\omega_1^k-i\omega_2^k),L(\omega_1^k+i\omega_2^k,\omega_1^j-i\omega_2^j))\\
&\qquad=-\frac{\delta}{\bar{\delta}}h(L(\omega_1^k+i\omega_2^k,\omega_1^1-i\omega_2^1),L(\omega_1^k+i\omega_2^k,\omega_1^j-i\omega_2^j))\\
&\qquad=-\frac{\delta}{\bar{\delta}}\frac{3\mu^2}{2}h(\omega_1^1+i\omega_2^1,\omega_1^j-i\omega_2^j))\\
&\qquad=\begin{cases}0, & j\neq 1\\ -3\mu^2\varepsilon_k, & j=1.\end{cases}
\end{aligned}$$

□

Lemma 20. *Let u_j and u_k determine ε_j and ε_k such that $\varepsilon_j=\varepsilon_k$, for $k,j>1$. Then $L(\omega_1^k+i\omega_2^k,\omega_1^j-i\omega_2^j)$ is an imaginary vector.*

Proof. Let us define the orthonormal basis of $\mathcal{U}$ given by

$$\begin{cases} u_k^*=\cos(t)u_k+\sin(t)u_j,\\ u_j^*=-sin(t)u_k+cos(t)u_k,\\ u_l^*=u_l, l\neq k,j.\end{cases}$$

By relation (40), we compute

$$L(\omega_1^1+i\omega_2^1,\cos(t)(\omega_1^k-i\omega_2^k)+\sin(t)(\omega_1^j-i\omega_2^j))=\sqrt{3}\mu\delta_j(\cos(t)u_k+\sin(t)u_j)$$

and therefore we find $\omega_1^{*k}+i\omega_2^{*k}=\cos(t)(\omega_1^k+i\omega_2^k)+\sin(t)(\omega_1^j+i\omega_2^j)$ and $\omega_1^{*j}+i\omega_2^{*j}=-\sin(t)(\omega_1^k+i\omega_2^k)+\cos(t)(\omega_1^j+i\omega_2^j)$ such that

$$L(\omega_1^1+i\omega_2^1,\omega_1^{*k}+i\omega_2^{*k})=\sqrt{3}\delta_k u_k^*.$$

Next, by lemma (19) we may write

$$L(\omega_1^{*k}+i\omega_2^{*k},\omega_1^{*k}-i\omega_2^{*k})=-\sqrt{3}\mu\varepsilon_k u_1$$

and using the bilinearity of L, we get the conclusion. □

Lemma 21. *Let u_j and u_k determine ε_j and ε_k such that $\varepsilon_j=-1$ and $\varepsilon_k=1$, for $k,j>1$. Then $L(\omega_1^k+i\omega_2^k,\omega_1^j-i\omega_2^j)$ is a real vector.*

Proof. First, define an orthonormal basis of $\mathcal{U}$ given by

$$\begin{cases} u_k^*=\cosh(t)u_k+\sinh(t)u_j,\\ u_j^*=\sinh(t)u_k+\cosh(t)u_k\\ u*_l=u_l, l\neq k,j\end{cases}$$

and notice that $L(\omega_1^1+i\omega_2^1,\omega_1^k-i\omega_2^k)=\sqrt{3}\mu u_k$ and $L(\omega_1^1+i\omega_2^1,\omega_1^j-i\omega_2^j)\sqrt{3}\mu u_j$. We take a,b,c,d complex functions and find $\omega_1^{*k}-i\omega_2^{*k}=a(\omega_1^k-i\omega_2^k)+b(\omega_1^j-i\omega_2^j)$ and $\omega_1^{*j}-i\omega_2^{*j}=c(\omega_1^k-i\omega_2^k)+d(\omega_1^j-i\omega_2^j)$ to be the unique vectors satisfying

$$L(\omega_1^1+i\omega_2^1,\omega_1^{*k}-i\omega_2^{*k})=\sqrt{3}\mu u_k^* \quad\text{and}\quad L(\omega_1^1+i\omega_2^1,\omega_1^{*j}-i\omega_2^{*j})\sqrt{3}\mu u_j^*.$$

Therefore, we find

$$a=\cosh(t),\quad b=i\sinh(t),\quad c=-i\sinh(t)\quad\text{and}\quad d=\cosh(t).$$

Finally, using the bilinearity of L, the conclusion follows easily from

$$L(\omega_1^{*k}+i\omega_2^{*k},\omega_1^{*k}-i\omega_2^{*k})=-\sqrt{3}\mu u_1.$$

□

In what follows, we will study the remaining cases for the metric on $\mathcal{U}$. First we deal with the case that the signature of the metric is 4, 5 or 6. Let u_1 be defined as in the beginning of this section. Next, choose $u_2 \perp u_1$ such that $h(u_2, u_2) = -1$ and w_1^2 and w_2^2 such that $L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu u_2$ and $u_3 \perp u_1, u_2$ and w_1^3 and w_2^3 such that $h(u_3, u_3) = -1, L(\omega_1^1 + i\omega_2^1, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu u_3$. Then, we look at the vector $L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3)$ and see, by property (23), that it is orthogonal to u_1, u_2 and u_3 and has length $3\mu^2$ and by lemma (20), that it is an imaginary vector. Therefore, we define u_4 of length -1 such that

$$L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu i u_4.$$

Next, by surjectivity of $L(\omega_1^1 + i\omega_2^1, -)$ and by (40) we can pick w_1^4 and w_2^4 such that $L(\omega_1^1 + i\omega_2^1, \omega_1^4 - i\omega_2^4) = \sqrt{3}\mu i u_4$. In the following, we pick $u_5 \perp u_1, u_2, u_3, u_4$ of length 1 and obtain ω_1^5, ω_2^5 such that $L(\omega_1^1 + i\omega_2^1, \omega_1^5 - i\omega_2^5) = \sqrt{3}\mu u_5$. Remark that the vectors $L(\omega_1^2 + i\omega_2^2, \omega_1^5 - i\omega_2^5)$, $L(\omega_1^3 + i\omega_2^3, \omega_1^5 - i\omega_2^5), L(\omega_1^4 + i\omega_2^4, \omega_1^5 - i\omega_2^5)$ are real, of positive length, mutually orthogonal and orthogonal to u_1, u_5. Therefore, the choice of $u_1, \dots, u_5$ implies that the metric on $\{u_2, u_3, u_4\}^\perp$ is positive definite. Therefore, the cases when the metric has signature $4, 5$ or 6 cannot happen.

In case that the index is 0, we proceed as follows. Let u_1 be defined as before, choose $u_2 \perp u_1$ of length 1 and obtain the existence of ω_1^2, ω_2^2 such that $L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu u_2$ and $L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2) = -\sqrt{3}\mu u_1$. Then, choose $u_3 \perp u_1, u_2$ of length 1 and obtain again $L(\omega_1^1 + i\omega_2^1, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu u_3$ and $L(\omega_1^3 + i\omega_2^3, \omega_1^3 - i\omega_2^3) = -\sqrt{3}\mu u_1$. Moreover, the vector $L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3)$ is an imaginary vector, orthogonal on u_1, u_2, u_3 (by relation (23)) and therefore, we get the existence of a unit vector of negative length, u_4, such that $L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu i u_4$. This contradicts the fact that the index equals 0.

Next, we start anew, with different choices of vectors in order to eliminate the case when the signature of the metric is 1.
Let u_1 be defined as before, choose $u_2 \perp u_1$ of length -1 and obtain the existence of ω_1^2, ω_2^2 such that $L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu i u_2$ and $L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu u_1$. Then, choose $u_3 \perp u_1, u_2$ of length 1 and obtain again $L(\omega_1^1 + i\omega_2^1, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu u_3$ and $L(\omega_1^3 + i\omega_2^3, \omega_1^3 - i\omega_2^3) = -\sqrt{3}\mu u_1$. Moreover, the vector $L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3)$ is a real vector, orthogonal on u_1, u_2, u_3 (by relation (23)) and therefore, we get the existence of a unit vector of positive length, u_4, such that $L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu u_4$. Consequently, $L(\omega_1^1 + i\omega_2^1, \omega_1^4 - i\omega_2^4) = \sqrt{3}\mu u_4$ and $L(\omega_1^4 + i\omega_2^4, \omega_1^4 - i\omega_2^4) = -\sqrt{3}\mu u_1$. Next, we pick $u_5 \perp u_1, u_2, u_3, u_4$ of length 1 and find

$$L(\omega_1^1 + i\omega_2^1, \omega_1^5 - i\omega_2^5) = \sqrt{3}\mu u_5 \quad \text{and} \quad L(\omega_1^5 + i\omega_2^5, \omega_1^5 - i\omega_2^5) = -\sqrt{3}\mu u_1.$$

Finally, by lemma (20) and the property in (23), we see that the vectors $L(\omega_1^3 + i\omega_2^3, \omega_1^5 - i\omega_2^5)$ and $L(\omega_1^3 + i\omega_2^3, \omega_1^4 - i\omega_2^4)$ are orthogonal imaginary vectors. This implies that the index of the metric is at least 2.

Now, we will prove that the metric on U cannot have signature 2. Let u_1 be defined as in (**??**), choose $u_2 \perp u_1$ of length -1 and obtain ω_1^2, ω_2^2 such that $L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu i u_2$ and $L(\omega_1^2 + i\omega_2^2, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu u_1$. Then, choose $u_3 \perp u_1, u_2$ of length -1 and obtain again $L(\omega_1^1 + i\omega_2^1, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu i u_3$ and $L(\omega_1^3 + i\omega_2^3, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu u_1$. Remark now that the vector $L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3)$ is an imaginary vector, orthogonal on u_1, u_2, u_3 (by relation (23)). So we have that $L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu i u_4$, where u_4 has negative length and belongs to $\{u_1, u_2, u_3\}^\perp$, where the metric is positive definite, which is a contradiction.

Next we deal with the case that the index of the metric equals 7. So on $\{u_1\}^\perp$ the metric is negative definite. We may take $u_2 \in \mathcal{U}$ such that $h(u_2, u_2) = -1$ and $h(u_1, u_2) = 0$.

As $L(\omega_1^1+i\omega_2^1,-)$ is a surjective operator, we can pick w_1^1 and $w_2^1=Tw_1^1$ such that

$$L(\omega_1^1+i\omega_2^1,\omega_1^2-i\omega_2^2)=\sqrt{3}\mu i u_2, \tag{42}$$

$$L(\omega_1^2+i\omega_2^2,\omega_1^1-i\omega_2^1)=-\sqrt{3}\mu i u_2. \tag{43}$$

and by the lemma we have $L(\omega_1^2+i\omega_2^2,\omega_1^2-i\omega_2^2)=\sqrt{3}\mu u_1$ Next, we take $u_3\in\mathcal{U}$ such that $h(u_3,u_3)=-1$. In a similar way as before, we define ω_1^3 and ω_2^3 and obtain

$$L(\omega_1^1+i\omega_2^1,\omega_1^3-i\omega_2^3)=\sqrt{3}\mu i u_3. \tag{44}$$

By the lemma this implies that $L(\omega_1^3+i\omega_2^3,\omega_1^3-i\omega_2^3)=\sqrt{3}\mu u_1$. Next, we find that $L(\omega_1^3+i\omega_2^3,\omega_1^2-i\omega_2^2)$ is an imaginary vector which is orthogonal to u_1,u_2 and u_3 such that we may write $L(\omega_1^3+i\omega_2^3,\omega_1^2-i\omega_2^2)=\sqrt{3}\mu i u_4$, for some $u_4\in U$, $u_4\perp u_1,u_2,u_3$. Given u_4, we define new ω_1^4 and ω_2^4 in $\mathcal{W}_2$ such that $L(\omega_1^1+i\omega_2^1,\omega_1^4-i\omega_2^4)=\sqrt{3}\mu i u_4$ and we have $L(\omega_1^4+i\omega_2^4,\omega_1^4-i\omega_2^4)=\sqrt{3}\mu u_1$. Next, we want to determine $L(\omega_1^2+i\omega_2^2,\omega_1^4-i\omega_2^4)$. We immediately obtain that it is an imaginary vector of length $3\mu^2$ which is orthogonal to u_1, u_2 and u_4. As

$$h(L(\omega_1^2+i\omega_2^2,\omega_1^4-i\omega_2^4),L(\omega_1^1+i\omega_2^1,\omega_1^3-i\omega_2^3))=-h(L(\omega_1^2+i\omega_2^2,\omega_1^3-i\omega_2^3),L(\omega_1^1+i\omega_2^1,\omega_1^4-i\omega_2^4))=3\mu^2,$$

it follows from the Cauchy-Schwartz inequality on $\{u_1\}^\perp$ that $L(\omega_1^2+i\omega_2^2,\omega_1^4-i\omega_2^4)=\sqrt{3}\mu i u_3$. Similarly it follows that $L(\omega_1^4+i\omega_2^4,\omega_1^3-i\omega_2^3)==\sqrt{3}\mu i u_2$.
Remember that so far we have defined u_1,u_2,u_3 and $u_4\in\mathcal{U}$ and ω_1^1,ω_2^1, ω_1^2,ω_2^2, ω_1^3,ω_2^3 ,$\omega_1^4,\omega_2^4\in\mathcal{W}$. We take now some arbitrary $u_5\in\{u_1,u_2,u_3,u_4\}^\perp$ such that $h(u_5,u_5)=-1$ and use again the surjectivity of $L(\omega_1^1+i\omega_2^1,-)$ to define w_1^5 and $w_2^5=Tw_1^5$ such that $L(\omega_1^1+i\omega_2^1,\omega_1^5-i\omega_2^5)=\sqrt{3}\mu i u_5$ and

$$L(\omega_1^5+i\omega_2^5,\omega_1^5-i\omega_2^5)=\sqrt{3}\mu u_1. \tag{45}$$

Next, we proceed with the computations as we did, for instance, for $L(\omega_1^3+i\omega_2^3,\omega_1^2-i\omega_2^2)$ and define $u_6,u_7,u_8\in\mathcal{U}$ such that

$$L(\omega_1^5+i\omega_2^5,\omega_1^2-i\omega_2^2)=\sqrt{3}\mu i u_6,$$
$$L(\omega_1^5+i\omega_2^5,\omega_1^3-i\omega_2^3)=\sqrt{3}\mu i u_7,$$
$$L(\omega_1^5+i\omega_2^5,\omega_1^4-i\omega_2^4)=\sqrt{3}\mu i u_8.$$

Given u_6,u_7,u_8, we use the surjectivity of $L(\omega_1^1+i\omega_2^1,-)$ and just like previously done, we define $\omega_1^k,\omega_2^k\in\mathcal{U}$, for $k=6,7,8$ and determine

$$L(\omega_1^1+i\omega_2^1,\omega_1^k-i\omega_2^k)=\sqrt{3}\mu i u_k.$$

Next, we find $L(\omega_1^k+i\omega_2^k,\omega_1^k-i\omega_2^k)=\sqrt{3}\mu u_1$ for $k=6,7,8$. Then, we compute similarly as for $L(\omega_1^2+i\omega_2^2,\omega_1^4-i\omega_2^4)$ in order to determine

$$L(\omega_1^6+i\omega_2^6,\omega_1^2-i\omega_2^2)=-\sqrt{3}\mu i u_5. \tag{46}$$

As for the vectors $L(\omega_1^3+i\omega_2^3,\omega_1^6-i\omega_2^6)$, $L(\omega_1^4+i\omega_2^4,\omega_1^6-i\omega_2^6)$ and $L(\omega_1^2+i\omega_2^2,\omega_1^7-i\omega_2^7)$, by using property (24) and the determined vectors so far, we see they are in the directions of u_8,u_7 and u_8 respectively. We can easily determine their components by following the same procedure as for $L(\omega_1^3+i\omega_2^3,\omega_1^2-i\omega_2^2)$. Thus, we may write

$$L(\omega_1^3+i\omega_2^3,\omega_1^6-i\omega_2^6)=\varepsilon u_8,$$

$$L(\omega_1^4+i\omega_2^4,\omega_1^6-i\omega_2^6)=\varepsilon_1 u_7,$$

$$L(\omega_1^2+i\omega_2^2,\omega_1^7-i\omega_2^7)=\varepsilon_2 u_8,$$

where $\varepsilon, \varepsilon_1, \varepsilon_2 = \pm\sqrt{3}\mu i$. Further on, in order to determine $L(\omega_1^3 + i\omega_2^3, \omega_1^7 - i\omega_2^7)$, we first see by property (24) that it is orthogonal to $\{u_1, u_2, u_3, u_4, u_7, u_8\}$. Next, as

$$h(L(\omega_1^3 + i\omega_2^3, \omega_1^7 - i\omega_2^7), L(\omega_1^1 + i\omega_2^1, \omega_1^6 - i\omega_2^6)) + h(L(\omega_1^1 + i\omega_2^1, \omega_1^7 - i\omega_2^7), L(\omega_1^3 + i\omega_2^3, \omega_1^6 - i\omega_2^6)) = 0 \quad (47)$$

and

$$h(L(\omega_1^3 + i\omega_2^3, \omega_1^7 - i\omega_2^7), L(\omega_1^1 + i\omega_2^1, \omega_1^5 - i\omega_2^5)) + h(L(\omega_1^1 + i\omega_2^1, \omega_1^7 - i\omega_2^7), L(\omega_1^3 + i\omega_2^3, \omega_1^5 - i\omega_2^5)) = 0 \quad (48)$$

we find

$$L(\omega_1^3 + i\omega_2^3, \omega_1^7 - i\omega_2^7) = \sqrt{3}\mu i u_5. \quad (49)$$

It is easy to see that $L(\omega_1^4 + i\omega_2^4, \omega_1^7 - i\omega_2^7)$ is coliniar with u_6. From (23) we obtain

$$h(L(\omega_1^1 + i\omega_2^1, \omega_1^6 - i\omega_2^6), L(\omega_1^4 + i\omega_2^4, \omega_1^7 - i\omega_2^7)) + h(L(\omega_1^4 + i\omega_2^4, \omega_1^6 - i\omega_2^6), L(\omega_1^1 + i\omega_2^1, \omega_1^7 - i\omega_2^7)) = 0 \Leftrightarrow \quad (50)$$

$$h(u_6, L(\omega_1^4 + i\omega_2^4, \omega_1^7 - i\omega_2^7)) = \varepsilon_1,$$

so that $L(\omega_1^4 + i\omega_2^4, \omega_1^7 - i\omega_2^7) = -\varepsilon_1 u_6$.
Using similar methods we consecutively obtain that

$$L(\omega_1^2 + i\omega_2^2, \omega_1^8 - i\omega_2^8) = -\sqrt{3}\epsilon_2 \mu i u_7$$
$$L(\omega_1^3 + i\omega_2^3, \omega_1^8 - i\omega_2^8) = -\sqrt{3}\epsilon \mu i u_7.$$

Note that by applying (**??**) on

$$h(L(\omega_1^3 + i\omega_2^3, \omega_1^8 - i\omega_2^8), L(\omega_1^2 + i\omega_2^2, \omega_1^5 - i\omega_2^5)),$$

we see that $\epsilon = -\epsilon_2$. Using similar arguments, we proceed to find that

$$L(\omega_1^4 + i\omega_2^4, \omega_1^8 - i\omega_2^8) = \sqrt{3}\epsilon_1\epsilon_2 \mu i u_5$$
$$\epsilon_1 = \epsilon_2$$
$$L(\omega_1^5 + i\omega_2^5, \omega_1^6 - i\omega_2^6) = -\sqrt{3}\mu i u_2$$
$$L(\omega_1^5 + i\omega_2^5, \omega_1^7 - i\omega_2^7) = -\sqrt{3}\mu i u_3$$
$$L(\omega_1^5 + i\omega_2^5, \omega_1^8 - i\omega_2^8) = -\sqrt{3}\mu i u_4$$
$$L(\omega_1^6 + i\omega_2^6, \omega_1^7 - i\omega_2^7) = \sqrt{3}\mu i \epsilon_2 u_4,$$
$$L(\omega_1^6 + i\omega_2^6, \omega_1^8 - i\omega_2^8) = -\sqrt{3}\epsilon_2 \mu i u_3,$$
$$L(\omega_1^7 + i\omega_2^7, \omega_1^8 - i\omega_2^8) = \sqrt{3}\mu i \epsilon_2 u_2.$$

Moreover it now immediately follows that $\epsilon_2 = 1$.

At last, we will study the solution given by the case when the metric on $\mathcal{U}$ has signature 3. Start with u_1 defined as in (40), choose $u_2 \perp u_1$ of length -1 and by surjectivity of $L(\omega_1^1 + i\omega_2^1, -)$ find ω_1^2, ω_2^2 such that $L(\omega_1^1 + i\omega_2^1, \omega_1^2 - i\omega_2^2) = \sqrt{3}\mu i u_2$. Similarly, choose $u_3 \perp u_1, u_2$ of length -1 and find $L(\omega_1^1 + i\omega_2^1, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu i u_3$. Then, by lemma (20) we can see that the vector $L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3)$ is imaginary, therefore, it defines a unit vector u_4, of length -1, such that $L(\omega_1^2 + i\omega_2^2, \omega_1^3 - i\omega_2^3) = \sqrt{3}\mu i u_4$. Moreover, we find the unique vectors ω_1^4 and ω_2^4 such that $L(\omega_1^1 + i\omega_2^1, \omega_1^4 - i\omega_2^4) = \sqrt{3}\mu i u_4$ and $L(\omega_1^4 + i\omega_2^4, \omega_1^4 - i\omega_2^4) = \sqrt{3}\mu u_1$. Further on, we see that $L(\omega_1^2 + i\omega_2^2, \omega_1^4 - i\omega_2^4)$ and $L(\omega_1^3 + i\omega_2^3, \omega_1^4 - i\omega_2^4)$ are orthogonal to u_1, u_2, u_4 and u_1, u_3, u_4. We compute by property (23) $h(L(\omega_1^2 + i\omega_2^2, \omega_1^4 - i\omega_2^4), L(\omega_1^1 + i\omega_2^1, \omega_1^3 - i\omega_2^3))$ and

$h(L(\omega_1^3+i\omega_2^3,\omega_1^4-i\omega_2^4),L(\omega_1^1+i\omega_2^1,\omega_1^2-i\omega_2^2))$ and, as the metric on $\{u_2,u_3,u_4\}^\perp$ is positive definite, we find

$$L(\omega_1^2+i\omega_2^2,\omega_1^4-i\omega_2^4)=-\sqrt{3}\mu i u_3 \quad \text{and} \quad L(\omega_1^3+i\omega_2^3,\omega_1^4-i\omega_2^4)=\sqrt{3}\mu i u_2.$$

Next, we choose $u_5\perp u_1,u_2,u_3,u_4$ of length 1 and find ω_1^5,ω_2^5 such that $L(\omega_1^1+i\omega_2^1,\omega_1^5-i\omega_2^5)=\sqrt{3}\mu u_5$. Then, we notice by property (23) that $L(\omega_1^2+i\omega_2^2,\omega_1^5-i\omega_2^5)$, $L(\omega_1^3+i\omega_2^3,\omega_1^5-i\omega_2^5)$ and $L(\omega_1^4+i\omega_2^4,\omega_1^5-i\omega_2^5)$ are real vectors and satisfy the orthogonality conditions which allow us to pick u_6,u_7,u_8 of length 1, in their directions respectively, and complete $\{u_1,u_2,u_3,u_4\}$ to an orthonormal basis, that is $L(\omega_1^2+i\omega_2^2,\omega_1^5-i\omega_2^5)=\sqrt{3}\mu u_6$, $L(\omega_1^3+i\omega_2^3,\omega_1^5-i\omega_2^5)=\sqrt{3}\mu u_7$ and $L(\omega_1^4+i\omega_2^4,\omega_1^5-i\omega_2^5)=\sqrt{3}\mu u_8$. Notice that, by lemmas (20) and property (40) we obtain

$$\begin{aligned}
L(\omega_1^6+i\omega_2^6,\omega_1^6-i\omega_2^6)&=-\sqrt{3}\mu u_1, & L(\omega_1^1+i\omega_2^1,\omega_1^6-i\omega_2^6)&=\sqrt{3}\mu u_6,\\
L(\omega_1^7+i\omega_2^7,\omega_1^7-i\omega_2^7)&=-\sqrt{3}\mu u_1, & L(\omega_1^1+i\omega_2^1,\omega_1^7-i\omega_2^7)&=\sqrt{3}\mu u_7,\\
L(\omega_1^8+i\omega_2^8,\omega_1^8-i\omega_2^8)&=-\sqrt{3}\mu u_1, & L(\omega_1^1+i\omega_2^1,\omega_1^8-i\omega_2^8)&=\sqrt{3}\mu u_8.
\end{aligned}$$

In the following, we determine $L(\omega_1^2+i\omega_2^2,\omega_1^6-i\omega_2^6)=-\sqrt{3}\mu u_5$, as it is a real vector of length $3\mu^2$, orthogonal on u_1,u_2,u_3,u_4,u_6, and given that its component in the direction of u_5 is $-\sqrt{3}\mu$ (by property (23)). Furthermore, we find $L(\omega_1^2+i\omega_2^2,\omega_1^7-i\omega_2^7)=\varepsilon_1\sqrt{3}\mu u_8$, as it is orthogonal to $L(\omega_1^2+i\omega_2^2,\omega_1^k-i\omega_2^k)$ and $L(\omega_1^1+i\omega_2^1,\omega_1^7-i\omega_2^7)$, for $k=2,\dots,6$ and $\varepsilon_1=\pm1$. Similarly, we determine for $\varepsilon_j=\pm1$, $j=2,\dots,8$ the following vectors

$$\begin{aligned}
L(\omega_1^2+i\omega_2^2,\omega_1^8-i\omega_2^8)&=\varepsilon_2\sqrt{3}\mu u_7, & L(\omega_1^5+i\omega_2^5,\omega_1^6-i\omega_2^6)&=-i\sqrt{3}\mu u_2\\
L(\omega_1^3+i\omega_2^3,\omega_1^6-i\omega_2^6)&=\varepsilon_3\sqrt{3}\mu u_8, & L(\omega_1^5+i\omega_2^5,\omega_1^7-i\omega_2^7)&=-i\sqrt{3}\mu u_3,\\
L(\omega_1^3+i\omega_2^3,\omega_1^7-i\omega_2^7)&=-\sqrt{3}\mu u_5, & L(\omega_1^5+i\omega_2^5,\omega_1^8-i\omega_2^8)&=\varepsilon_8\sqrt{3}\mu u_4,\\
L(\omega_1^3+i\omega_2^3,\omega_1^8-i\omega_2^8)&=\varepsilon_4\sqrt{3}\mu u_6, & L(\omega_1^6+i\omega_2^6,\omega_1^7-i\omega_2^7)&=-\varepsilon_5 i\sqrt{3}\mu u_4,\\
L(\omega_1^4+i\omega_2^4,\omega_1^6-i\omega_2^6)&=\varepsilon_5\sqrt{3}\mu u_7, & L(\omega_1^6+i\omega_2^6,\omega_1^8-i\omega_2^8)&=-\varepsilon_3 i\sqrt{3}\mu u_3,\\
L(\omega_1^4+i\omega_2^4,\omega_1^7-i\omega_2^7)&=\varepsilon_6\sqrt{3}\mu u_6, & L(\omega_1^7+i\omega_2^7,\omega_1^8-i\omega_2^8)&=-\varepsilon_1 i\sqrt{3}\mu u_2.\\
L(\omega_1^4+i\omega_2^4,\omega_1^8-i\omega_2^8)&=\varepsilon_7\sqrt{3}\mu u_5,
\end{aligned}$$

Then, we can easily find the relations between the coefficients ε_j using property (23): $\varepsilon_2=-\varepsilon_1$, $\varepsilon_4=-\varepsilon_3$, $\varepsilon_6=-\varepsilon_5$ and $\varepsilon_7=-1,\varepsilon_8=-i$. Moreover, we can find $\varepsilon_1=-1,\varepsilon_3=1$ and $\varepsilon_5=-1$ by applying property (23) successively to $L(\omega_1^6+i\omega_2^6,\omega_1^7-i\omega_2^7)$ and $L(\omega_1^2+i\omega_2^2,\omega_1^3-i\omega_2^3)$, $L(\omega_1^2+i\omega_2^2,\omega_1^7-i\omega_2^7)$ and $L(\omega_1^5+i\omega_2^5,\omega_1^4-i\omega_2^4)$, and finally, to $L(\omega_1^3+i\omega_2^3,\omega_1^8-i\omega_2^8)$ and $L(\omega_1^2+i\omega_2^2,\omega_1^5-i\omega_2^5)$.

8.2. **Two canonical examples.**

When the indefinite signature on U is 7, we have the following example.
Let $\mathfrak{h}_3(\mathbb{O})$ denote the set of Hermitian matrices with entries in $\mathbb{O}$, the space of octonions endowed with the Jordan multiplication $\circ$:

$$\mathfrak{h}(\mathbb{O})_3=\{N\in\mathcal{M}_3(\mathbb{O})|\bar{N}^T=N\},$$
$$X\circ Y=\frac{1}{2}(XY+YX).$$

By definition, we have that the determinant of $N\in\mathfrak{h}_3(\mathbb{O})$ is given by

$$\det N=\frac{1}{3}Tr(N\circ N\circ N)-\frac{1}{2}Tr(N\circ N)+\frac{1}{6}(TrN)^3.$$

Remark that a matrix $N\in\mathfrak{h}_3(\mathbb{O})$ is of the form $N=\begin{pmatrix}\xi_1 & x_3 & \bar{x_2}\\ \bar{x_3} & \xi_2 & x_1\\ x_2 & \bar{x_1} & \xi_3\end{pmatrix}$, where $\xi_i\in\mathbb{R},x_i\in\mathbb{O}$. For more details for the space of octonions see [1]. Next, we define

$G = \{N \in \mathfrak{h}(\mathbb{O})_3 | det(N) = 1\}$. We define an action of G on $M_1 = \{\bar{N}AN^T | N \in G\}$ by

$$\rho : G \times M_1 \longrightarrow M_1$$
$$\rho(N)X = \bar{N}XN^T,$$

where $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. By construction, this action is transitive and therefore M_1 is congruent with G/H, where $H = \{N \in G | A\bar{N}AN^T = I\}$. Note that $\rho(N)$ can be seen as a linear transformation acting on $\mathbb{R}^{27}$ and a straightforward computation shows that $\rho(N) \in SL(27, \mathbb{R})$. Therefore, M_1 is an homogeneous affine hypersphere in $\mathbb{R}^{27}$. It is now sufficient to work around a point. We introduce local coordinates around a point $p \in M_1$ by taking $y_1, \cdots, y_{26}$ such that $\xi 1 = 1$,

$$\xi_2 = y_1, \quad \xi_3 = y_2, \quad x_1 = \sum_{i=0}^{7} y_{3+i} e_i, \quad x_2 = \sum_{i=0}^{7} y_{11+i} e_i, \quad x_3 = \sum_{i=0}^{7} y_{19+i} e_i,$$

for $\{e_0, \cdots, e_7\}$ a basis of $\mathbb{O}$. Therefore, the parametrization for our hypersurface is given by

$$\begin{cases} F : \mathbb{R}^{26} \to \mathbb{R}^{27} \\ p \longmapsto g(p)^{-\frac{1}{3}}(1, p), \end{cases}$$

where $p = (y_1, \cdots, y_{26})$ and $g(p) := \det N$. By using the multiplication table for octonions, we can determine $g(p)$ and then, straightforward computations around the point $N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ allow us to find that the isotropy condition holds for $\lambda = \frac{1}{2}$. Thus, the signature of the metric on M is 16.

When the indefinite signature on U is 3, we have the following example.

Consider the set of Hermitian matrices with entries in the split-octonions space endowed with the Jordan multiplication $\circ$, as previously defined. For $\{1, i, j, k, li, lj, lk\}$ an orthogonal basis of the split-octonion space, the length of a vector $x = x_0 + x_1 i + x_2 j + x_3 k + x_4 l + x_5 li + x_6 lj + x_7 lk$ is given by

$$h(x, x) = \bar{x}x = (x_0^2 + x_1^2 + x_2^2 + x_3^2) - (x_4^2 + x_5^2 + x_6^2 + x_7^2).$$

We define the manifold in a similar way as in the previous example and, by similar arguments, we get that M is an isotropic affine hypersphere of dimension 26 for which the signature of the metric is 12.

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