

# On the Discretization of Robust Exact Filtering Differentiators

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**Abstract:** This paper deals with the design of discrete-time algorithms for the robust filtering differentiator. Two discrete-time realizations of the filtering differentiator are introduced. The first one, which is based on an exact discretization of the continuous differentiator, is an explicit one, while the second one is an implicit algorithm which enables to remove the numerical chattering phenomenon and to preserve the estimation accuracy properties. Some numerical comparisons between the proposed scheme and an existing discrete-time algorithm show the interest of the proposed implicit discrete-time realization of the filtering differentiator, especially when large sampling periods are considered.

*Keywords:* Nonlinear observers and filter design, Sliding mode control, Observer design.

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## 1. INTRODUCTION

The problems of filtering a noisy signal and differentiation in real-time are crucial issues due to their practical interest in signal processing and control engineering. These problems have been addressed using various methods: Kalman filter (Kalman, 1960), algebraic methods (Mboup et al., 2009), observation techniques (Chitour, 2002; Spurgeon, 2008; Davila et al., 2005) to name a few.

Sliding mode techniques are widely used to design observers due to their exceptional accuracy and robustness properties in the presence of matched perturbations (Edwards and Spurgeon, 1998; Shtessel et al., 2014). However, one of the main disadvantages of these techniques is the chattering phenomenon. High-order sliding mode homogeneous differentiators have been proposed in (Levant, 2003; Levant and Livne, 2011). They give an estimate, in a finite time, of the  $n$  derivatives of a signal if its  $(n + 1)$  order derivative has a known upper bound. Furthermore, they have shown good robustness properties in the presence of noise and exact finite-time convergence in the absence of noise. A filtering differentiator has been investigated in (Levant and Livne, 2019) in order to exactly differentiate a smooth signal while rejecting a larger class of noises.

In practice, observation algorithms are usually discretized in order to be implemented in a digital environment. However, the discrete-time approximations of the continuous algorithms are far from being straightforward. Indeed, for high-gain and sliding mode differentiators, an inadequate discrete-time version of the algorithms may lead to numerical chattering (Drakunov and Utkin, 1990; Utkin, 1994)

i.e., high oscillations only due to the numerical methods used in the discretization scheme.

Several algorithms have been proposed for the implementation of discrete-time sliding mode controllers (Drakunov and Utkin, 1990; Su et al., 2000; Nguyen et al., 2017; Abidi et al., 2007). Concerning the homogeneous differentiator, some explicit discretization algorithms have been derived in (Livne and Levant, 2014; Koch and Reichhartinger, 2018; Koch et al., 2019; Barbot et al., 2020; Levant and Livne, 2019) in order to preserve the estimation accuracy properties. In (Livne and Levant, 2014), a discrete-time realization of the homogeneous differentiator, which preserves the computational simplicity of the one-step Euler scheme, has been introduced. In (Koch and Reichhartinger, 2018), the proposed discrete algorithm is less sensitive to gain overestimation. A discrete-time differentiator, which includes nonlinear higher-order terms, has been derived in (Koch et al., 2019) in order to preserve the asymptotic accuracy properties known from the continuous-time differentiator despite the presence of noise. The work in (Barbot et al., 2020) extends the results from (Livne and Levant, 2014) while also considering non-homogeneous hybrid differentiators. Explicit discrete-time realization of the filtering differentiator has been proposed in (Levant and Livne, 2019).

Recently, some implicit discretization algorithms have been investigated in order to ensure a smooth stabilization of the sliding surface in discrete-time for the case without disturbance (Acary et al., 2011; Brogliato et al., 2019; Huber et al., 2016; Luo et al., 2019). Such algorithms remove the numerical chattering effects due to the time discretization and allow the use of large sampling

periods without reducing too much the performances. However, implicit methods have only applied to first-order sliding mode controllers (Acary et al., 2011), twisting controllers (Huber et al., 2016; Luo et al., 2019) and super-twisting controllers (Brogliato et al., 2019). Nevertheless, an implicit discretization algorithm has been recently proposed in (Carvajal-Rubio et al., 2019) for the homogeneous differentiator.

This paper proposes two discretization algorithms, based on the recent results presented in (Carvajal-Rubio et al., 2019), for the robust filtering differentiator given in (Levant and Livne, 2019). The first one is an explicit exact discrete-time version of the filtering differentiator, while the second one is an implicit discretization algorithm that removes the numerical chattering effects. Some simulations are given to compare the discrete-time algorithm presented in (Levant and Livne, 2019) with the proposed ones (explicit and implicit methods). It will be shown that the proposed scheme provides estimates of the derivatives of a given signal with good accuracy and robustness properties even when a large sampling period is considered.

The rest of the paper is as follows. Section 2 introduces the problem and recalls some preliminaries on the exact filtering differentiator. In Section 3, two discretization algorithms for the robust filtering differentiator are given (i.e., explicit and implicit discrete-time algorithms). At last, in Section 4, some simulations are done to highlight the interest of the proposed scheme when a significant sampling period is considered.

**Notation.** For  $x \in \mathbb{R}$ , the absolute value of  $x$ , denoted by  $|x|$ , is defined as  $|x| = x$  if  $x \geq 0$  and  $|x| = -x$  if  $x < 0$ . The set-valued function  $\text{sign}(x)$  is defined as  $\text{sign}(x) = 1$  if  $x > 0$ ,  $\text{sign}(x) = -1$  if  $x < 0$ , and  $\text{sign}(x) \in [-1, 1]$  if  $x = 0$ . For  $\gamma \geq 0$ , the signed power  $\gamma$  of  $x$  is defined as  $|x|^\gamma = |x|^\gamma \text{sign}(x)$ . In particular, if  $\gamma = 0$  then  $|x|^\gamma = \text{sign}(x)$ .

## 2. PROBLEM STATEMENT AND PRELIMINARIES

### 2.1 Problem statement

The objective of a differentiator is obtain online the first  $n$  derivatives of a function even if there is noise in the measurement. In this paper, this function is represented as  $f_0(t)$ , where  $f_0 : \mathbb{R} \rightarrow \mathbb{R}$ . It is also assumed that this function is at least  $(n+1)$ -th differentiable and  $|f_0^{(n+1)}(t)| \leq L$  for a known real number  $L > 0$ . Furthermore, the input of the differentiator is defined as  $f(t) = f_0(t) + \Delta(t)$ . It is also assumed that  $\Delta(t)$  is a Lebesgue-measurable bounded noise with  $|\Delta(t)| \leq \delta$  for an unknown real number  $\delta > 0$ .

In order to compute the derivatives  $f_0^{(1)}(t), f_0^{(2)}(t), \dots, f_0^{(n)}(t)$ , a state space representation is used. To obtain this representation, the state variables are defined as  $x_i(t) = f_0^{(i)}(t)$  and  $\mathbf{x} = [x_0 \ x_1 \ x_2 \ \dots \ x_n]^T \in \mathbb{R}^{n+1}$ . Therefore, one can obtain the following representation for the differentiation problem in the state space:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{e}_{n+1}f_0^{(n+1)}(t) \\ y_o(t) &= \mathbf{e}_1^T \mathbf{x}(t) + \Delta(t) \end{aligned} \quad (1)$$

with the canonical vectors  $\mathbf{e}_1 = [1 \ 0 \ \dots \ 0 \ 0]^T$ ,  $\mathbf{e}_{n+1} = [0 \ 0 \ \dots \ 0 \ 1]^T$  and  $\mathbf{A} = [0_{1 \times (n+1)} \ \mathbf{e}_1 \ \mathbf{e}_2 \ \dots \ \mathbf{e}_n]$ , which is a nilpotent matrix of appropriate dimensions. The representation (1) is interesting in the sense that the successive time derivatives of  $f_0(t)$  can be obtained through the design of a state observer.

### 2.2 Homogeneous high-order differentiator

In order obtain the first  $n$  derivatives of a signal  $f_0(t)$ , a continuous-time observer has been proposed in (Levant, 2003). For  $\Delta(t) = 0$ , it can be represented in the non-recursive form:

$$\dot{\mathbf{z}} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u}(\sigma_0) \quad (2)$$

where  $\mathbf{u}(\sigma_0) = [\Psi_{0,n}(\sigma_0) \ \Psi_{1,n}(\sigma_0) \ \dots \ \Psi_{n,n}(\sigma_0)]^T$ ,  $\Psi_{i,n}(\cdot) = -\lambda_{n-i}L^{\frac{i+1}{n+1}} [\cdot]^{\frac{n-i}{n+1}}$ ,  $\mathbf{B}$  is identity matrix of appropriate dimensions,  $\sigma_0 = z_0 - x_0$  and  $\mathbf{z} = [z_0 \ z_1 \ z_2 \ \dots \ z_n]^T$  is the finite-time estimate of the state vector  $\mathbf{x}$  using adequate parameters  $\lambda_i > 0$  (see (Reichhartinger et al., 2017; Levant, 2018) for instance). Since the function  $[z_0 - f(t)]^0$  is discontinuous at  $z_0 = f$ , the solutions of system (2) are understood in the Filippov sense (Filippov, 2013).

### 2.3 Finite-time-exact robust filtering differentiator (FTER)

Although, differentiator (2) offers good performance when there exists a Lebesgue-measurable bounded noise  $\Delta(t)$  such that  $|\Delta(t)| \leq \delta$  with small in average  $\delta$ , its performance becomes significantly reduced when  $\delta$  is large. Due to this reason, in Levant (2018), a new finite-time exact robust filtering differentiator has been proposed, with the following structure:

$$\begin{aligned} \dot{\omega}_{i_f} &= -\lambda_{m+1-i_f}L^{\frac{i_f}{m+1}} [\omega_1]^{\frac{m+1-i_f}{m+1}} + \omega_{i_f+1} \\ \dot{\omega}_{n_f} &= -\lambda_{n+1}L^{\frac{n_f}{m+1}} [\omega_1]^{\frac{n+1}{m+1}} + z_0 - g(t) \\ \dot{z}_{i_d} &= -\lambda_{m-i_d}L^{\frac{n_f+1+i_d}{m+1}} [\omega_1]^{\frac{n-i_d}{m+1}} + z_{i_d+1} \\ i_f &= 1, 2, \dots, n_f - 1. \quad i_d = 0, 1, 2, \dots, n. \end{aligned} \quad (3)$$

where  $m = n + n_f$ ,  $n_f \geq 0$ ,  $n_f$  is the filtering order and the parameters  $\lambda_i$  are selected as in (2). Moreover,  $g(t) = f_0(t) + v(t)$ , where  $v(t)$  is comprised of  $n_f + 1$  components,  $v(t) = v_0(t) + v_1(t) + \dots + v_{n_f}(t)$ ,  $v_j(t)$  is a signal of the global filtering order  $j$  and the  $j$ th-order integral magnitude  $\epsilon_j \geq 0$  with  $j = 0, 1, \dots, n_f$ . More details can be founded in (Levant and Livne, 2019). In (Levant and Livne, 2019), it is shown that differentiator (3) offers the following accuracy:

$$\begin{aligned} |z_i - f_0^{(i)}(t)| &\leq \mu_i L \rho^{n+1-i}, \quad \mu_i > 0, \quad i = 0, 1, 2, \dots, n. \\ \rho &= \max \left[ \left( \frac{\epsilon_0}{L} \right)^{\frac{1}{n+1}}, \left( \frac{\epsilon_1}{L} \right)^{\frac{1}{n+2}}, \dots, \left( \frac{\epsilon_{n_f}}{L} \right)^{\frac{1}{m+1}} \right] \end{aligned}$$

### 2.4 Discretization (FTER-D)

In practice, the differentiation algorithms are usually discretized in order to be implemented in a digital environment. In (Levant, 2018), a discrete-time filtering differentiator is presented as follows:

$$\begin{bmatrix} \mathbf{w}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{C}(\tau)_{n_f \times n_f} & \mathbf{D}(\tau)_{n_f \times (n+1)} \\ \mathbf{0}_{(n+1) \times n_f} & \mathbf{\Phi}(\tau)_{(n+1) \times (n+1)} \end{bmatrix} \begin{bmatrix} \mathbf{w}_k \\ \mathbf{z}_k \end{bmatrix} + \tau \mathbf{e}_{n_f} g_k + \tau \mathbf{u}_k \quad (4)$$

with  $\tau = t_{k+1} - t_k$ ,  $\mathbf{e}_{n_f} = [0 \cdots 0 \ 1 \ 0 \cdots 0]^T$ ,  $\mathbf{w}_k = \mathbf{w}(\tau k)$ ,  $\mathbf{z}_k = \mathbf{z}(\tau k)$ ,  $g_k = g(\tau k)$ ,  $\mathbf{0}_{(n+1) \times n_f}$  is a matrix whose elements are 0, the matrices  $\mathbf{C}(\tau)_{n_f \times n_f}$ ,  $\mathbf{D}(\tau)_{n_f \times n_f}$  and  $\mathbf{\Phi}(\tau)_{(n+1) \times (n+1)}$  are defined as:

$$\mathbf{C}(\tau)_{n_f \times n_f} = \begin{bmatrix} 1 & \tau & 0 & \cdots & 0 & 0 \\ 0 & 1 & \tau & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \tau \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \mathbf{D}(\tau)_{n_f \times (n+1)} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \tau & 0 & \cdots & 0 \end{bmatrix}$$

$$\mathbf{\Phi}(\tau)_{(n+1) \times (n+1)} = \begin{bmatrix} 1 & \tau & \frac{\tau^2}{2!} & \frac{\tau^3}{3!} & \cdots & \frac{\tau^{n-1}}{(n-1)!} & \frac{\tau^n}{n!} \\ 0 & 1 & \tau & \frac{\tau^2}{2!} & \cdots & \frac{\tau^{n-2}}{(n-2)!} & \frac{\tau^{n-1}}{(n-1)!} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \tau \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Furthermore  $\mathbf{u}_k$  is defined as:

$$\mathbf{u}_k = [\Psi_{0,m}(\omega_{1,k}), \cdots, \Psi_{m,m}(\omega_{1,k})]^T$$

For the differentiator (4),  $f_{0,k} = f_0(\tau k)$  is assumed as in the continuous differentiator (3),  $v_k = v(\tau k)$  is comprised of  $n_f + 1$  components,  $v_k = v_{0,k} + v_{1,k} + \cdots + v_{n_f,k}$ , where  $v_{i,k}$  are of the global sampling filtering order  $j$  and integral magnitude  $\epsilon_j$  with  $j = 0, 1, \cdots, n_f$  (Levant and Livne, 2019). Furthermore, it is assumed that the set of admissible sampling-time sequences contains sequences for any  $\tau > 0$ . According to Levant and Livne (2019), the discrete differentiator (4) provides the following accuracy:

$$|\sigma_{i,k}| \leq \mu_i L \rho^{n+1-i}, \quad \mu_i > 0, \quad \sigma_{i,k} = z_{i,k} - x_{i,k},$$

$$\rho = \max \left[ \tau, \left( \frac{\epsilon_0}{L} \right)^{\frac{1}{n+1}}, \left( \frac{\epsilon_1}{L} \right)^{\frac{1}{n+2}}, \cdots, \left( \frac{\epsilon_{n_f}}{L} \right)^{\frac{1}{m+1}} \right],$$

$$i = 0, 1, 2, \cdots, n.$$

### 3. DISCRETIZATION OF ROBUST EXACT FILTERING DIFFERENTIATOR

In this Section, two discrete-time realizations of the filtering differentiator are proposed. The first one is an explicit one, which is based on an exact discretization, while the second one is an implicit algorithm.

#### 3.1 Explicit Discretization of the robust exact filtering differentiator (FTE-E)

Applying the procedure presented in (Carvajal-Rubio et al., 2019) to the system (3), the following discrete-time realization of the differentiator is obtained:

$$\begin{bmatrix} \mathbf{w}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \mathbf{\Phi}(\tau)_{(m+1) \times (m+1)} \begin{bmatrix} \mathbf{w}_k \\ \mathbf{z}_k \end{bmatrix} + \mathbf{h}(\tau) g_k + \mathbf{B}^*(\tau) \mathbf{u}_k \quad (5)$$

Here,

$$\mathbf{h}(\tau) = \left[ \frac{\tau^{n_f}}{n_f!} \cdots \frac{\tau^2}{2!} \tau \ 0 \cdots 0 \right]^T$$

and

$$\mathbf{B}^*(\tau) = \begin{bmatrix} \tau & \frac{\tau^2}{2!} & \frac{\tau^3}{3!} & \cdots & \frac{\tau^m}{m!} & \frac{\tau^{m+1}}{(m+1)!} \\ 0 & \tau & \frac{\tau^2}{2!} & \cdots & \frac{\tau^{m-1}}{(m-1)!} & \frac{\tau^m}{m!} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \tau & \frac{\tau^2}{2!} \\ 0 & 0 & 0 & \cdots & 0 & \tau \end{bmatrix}$$

Using Taylor series expansion with Lagrange's remainders (see (Firey, 1960)) on system (1) the following discrete-time system is obtained:

$$\mathbf{x}_{k+1} = \mathbf{\Phi}(\tau)_{(n+1) \times (n+1)} \mathbf{x}_k + \mathbf{H}_{0,k} \quad (6)$$

with  $\mathbf{H}_{0,k} = \left[ \frac{\tau^{n+1}}{(n+1)!} f_0^{(n+1)}(\rho_n) \cdots \tau f_0^{(n+1)}(\rho_0) \right]^T$ ,  $\rho_i \in (t_k, t_{k+1})$ ,  $\mathbf{x}_k = \mathbf{x}(\tau k)$ , and  $|f_0^{(n+1)}(\rho_i)| \leq L$ .

Then, the vector  $[\mathbf{w}_{k+1}^T \ \boldsymbol{\sigma}_{k+1}^T]^T$ , with  $\boldsymbol{\sigma}_k = \boldsymbol{\sigma}(\tau k)$  can be represented as:

$$\begin{bmatrix} \mathbf{w}_{k+1} \\ \boldsymbol{\sigma}_{k+1} \end{bmatrix} = \mathbf{\Phi}(\tau) \begin{bmatrix} \mathbf{w}_k \\ \boldsymbol{\sigma}_k \end{bmatrix} + \mathbf{B}^*(\tau) \mathbf{u}_k - \mathbf{H}_k + \cdots + \begin{bmatrix} \mathbf{0}_{n_f \times n_f} & \mathbf{E}(\tau)_{n_f \times (n+1)} \\ \mathbf{0}_{(n+1) \times n_f} & \mathbf{0}_{(n+1) \times (n+1)} \end{bmatrix} \begin{bmatrix} \mathbf{0}_{(n_f \times 1)} \\ \mathbf{x}_k \end{bmatrix} \quad (7)$$

where  $\mathbf{H}_k = [0 \ 0 \ \cdots \ 0 \ 0 \ \mathbf{H}_{0,k}^T]^T$ , the vector of errors is defined as  $\boldsymbol{\sigma}_k = [\sigma_0(\tau k), \sigma_1(\tau k), \cdots, \sigma_n(\tau k)]$  and

$$\mathbf{E}(\tau)_{(n_f \times (n+1))} = \begin{bmatrix} 0 & \frac{\tau^{(n_f+1)}}{(n_f+1)!} & \frac{\tau^{(n_f+2)}}{(n_f+2)!} & \cdots & \frac{\tau^m}{m!} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{\tau^2}{2!} & \frac{\tau^3}{3!} & \cdots & \frac{\tau^{(n+1)}}{(n+1)!} \end{bmatrix}$$

Due to the non-zero elements of  $\mathbf{E}(\tau)_{n_f \times (n+1)}$ , differentiator (5) does not guarantee convergence for functions with unbounded first  $n$  derivatives. Therefore, in order to avoid the last term of the error system (7), the following discretization is proposed based on the structure of (5):

$$\begin{bmatrix} \mathbf{w}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}(\tau)_{n_f \times n_f} & \mathbf{G}(\tau)_{n_f \times (n+1)} \\ \mathbf{0}_{(n+1) \times n_f} & \mathbf{\Phi}(\tau)_{(n+1) \times (n+1)} \end{bmatrix} \begin{bmatrix} \mathbf{w}_k \\ \mathbf{z}_k \end{bmatrix} + \mathbf{h}(\tau) g_k + \mathbf{B}^*(\tau) \mathbf{u}_k \quad (8)$$

where  $\mathbf{G}(\tau)_{n_f \times (n+1)}$  and  $\mathbf{u}_k$  are defined as:

$$\mathbf{G}(\tau)_{(n_f \times (n+1))} = \begin{bmatrix} \frac{\tau^{n_f}}{n_f!} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{\tau^2}{2!} & 0 & \cdots & 0 & 0 \\ \tau & 0 & \cdots & 0 & 0 \end{bmatrix} \mathbf{u}_k = \begin{bmatrix} \Psi_{0,m}(\omega_{1,k}) \\ \Psi_{1,m}(\omega_{1,k}) \\ \vdots \\ \Psi_{m,m}(\omega_{1,k}) \end{bmatrix}$$

For the differentiator (8) (i.e., FTE-E),  $\mathbf{E}(\tau)_{n_f \times (n+1)} = \mathbf{0}_{n_f \times (n+1)}$  and

$$\begin{bmatrix} \mathbf{w}_{k+1} \\ \boldsymbol{\sigma}_{k+1} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}(\tau)_{n_f \times n_f} & \mathbf{G}(\tau)_{n_f \times (n+1)} \\ \mathbf{0}_{(n+1) \times n_f} & \mathbf{\Phi}(\tau)_{(n+1) \times (n+1)} \end{bmatrix} \begin{bmatrix} \mathbf{w}_k \\ \boldsymbol{\sigma}_k \end{bmatrix} + \mathbf{B}^*(\tau) \mathbf{u}_k - \mathbf{H}_k \quad (9)$$

### 3.2 Implicit Discretization (FTER-I)

Now, consider the implicit discrete-time algorithm of the robust filtering differentiator. From the differentiator (8), the following algorithm is proposed:

$$\begin{aligned} \begin{bmatrix} \mathbf{w}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} &= \begin{bmatrix} \mathbf{\Phi}(\tau)_{n_f \times n_f} & \mathbf{G}(\tau)_{n_f \times (n+1)} \\ \mathbf{0}_{(n+1) \times n_f} & \mathbf{\Phi}(\tau)_{(n+1) \times (n+1)} \end{bmatrix} \begin{bmatrix} \mathbf{w}_k \\ \mathbf{z}_k \end{bmatrix} \\ &\quad + \mathbf{h}(\tau)g_k + \mathbf{B}^*(\tau)\mathbf{u}_k \\ \mathbf{u}_k &= [\Psi_{0,m}(\omega_{1,k+1}), \dots, \Psi_{m,m}(\omega_{1,k+1})]^T \\ \Psi_{i,m}(\omega_{1,k+1}) &\in -\lambda_{m-i}L^{\frac{i+1}{m+1}} |\omega_{1,k+1}|^{\frac{m-i}{m+1}} \end{aligned} \quad (10)$$

In order to implement the differentiator (10),  $\omega_{1,k+1}$  needs to be calculated at time  $t = t_k$ . Using the difference equation of  $\omega_{1,k+1}$ , the following inclusion is obtained:

$$\begin{aligned} \omega_{1,k+1} + a_m |\omega_{1,k+1}|^{\frac{m}{m+1}} + \dots + a_1 |\omega_{1,k+1}|^{\frac{1}{m+1}} + \dots \\ + b_k \in -a_0 \text{sign}(\omega_{1,k+1}) \end{aligned} \quad (11)$$

where  $b_k = \frac{\tau^{n_f}}{n_f!}(z_{0,k} - g_k) - \sum_{l=1}^{n_f} \frac{\tau^{(l-1)}}{(l-1)!} \omega_{l,k}$  and  $a_l = \frac{\tau^{m-l+1}}{(m-l+1)!} \lambda_l L^{\frac{m-l+1}{m+1}}$ , where  $a_i \in \mathbb{R}^+$  and  $b_k \in \mathbb{R}$ . As in (Carvajal-Rubio et al., 2019; Brogliato et al., 2019), a new support variable is introduced as  $\xi_{k+1} \in \text{sign}(\omega_{0,k+1})$ . Using a similar scheme that the one presented in (Carvajal-Rubio et al., 2019),  $\omega_{1,k+1}$  and  $\xi_{k+1}$  are defined as follows:

- **Case 1:**  $b_k > a_0$ .  $\xi_{k+1} = -1$  and  $\omega_{1,k+1} = -(r_0)^{m+1}$ , where  $r_0$  is the unique positive root of the polynomial:

$$p(r) = r^{m+1} + a_m r^m + \dots + a_1 r + (-b_k + a_0) \quad (12)$$

- **Case 2:**  $b_k \in [-a_0, a_0]$ .  $\omega_{1,k+1} = 0$  and  $\xi_{k+1} = -\frac{b_k}{a_0}$ .
- **Case 3:**  $b_k < -a_0$ .  $\xi_{k+1} = 1$  and  $\omega_{1,k+1} = r_0^{m+1}$ , where  $r_0$  is the positive root of the polynomial:

$$p(r) = r^{m+1} + a_m r^m + \dots + a_1 r + (b_k + a_0) \quad (13)$$

Furthermore, the pair  $\omega_{0,k+1} \in \mathbb{R}$  and  $\xi_{k+1} \in [-1, 1]$  is unique for each set of values of  $a_l$  and  $b_k$ . With the new variable  $\xi_{k+1}$  the differentiator (10) is implemented as follows:

$$\begin{aligned} \begin{bmatrix} \mathbf{w}_{k+1} \\ \mathbf{z}_{k+1} \end{bmatrix} &= \begin{bmatrix} \mathbf{\Phi}(\tau)_{n_f \times n_f} & \mathbf{G}(\tau)_{n_f \times (n+1)} \\ \mathbf{0}_{(n+1) \times n_f} & \mathbf{\Phi}(\tau)_{(n+1) \times (n+1)} \end{bmatrix} \begin{bmatrix} \mathbf{w}_k \\ \mathbf{z}_k \end{bmatrix} \\ &\quad + \mathbf{h}(\tau)g_k + \mathbf{B}^*(\tau)\mathbf{v}_k \\ \mathbf{v}_k &= [\tilde{\Psi}_{0,m}(\omega_{1,k+1}), \dots, \tilde{\Psi}_{m,m}(\omega_{1,k+1})]^T \\ \tilde{\Psi}_{i,m}(\omega_{1,k+1}) &= -\lambda_{m-i}L^{\frac{i+1}{m+1}} |\omega_{1,k+1}|^{\frac{m-i}{m+1}} \xi_{k+1} \end{aligned} \quad (14)$$

*Remark 1.* Since  $\xi_{k+1}$  is defined for any value of  $\omega_{1,k+1}$  and  $\text{sign}(0) \in [-1, 1]$ ,  $\xi_{1,k+1}$  is smoother than the function  $\text{sign}(\omega_{1,k+1})$ .

*Remark 2.* To implement the differentiator (14),  $r_0$  needs to be computed when  $b_k \notin [-a_0, a_0]$ . Hence, a root finding method is needed. Here, the Halley's is used (Scavo and Thoo, 1995).

## 4. SIMULATION RESULTS

In order to analyze and compare the performance of the differentiators (4), (8) and (14), two variables will be used: the mean square error of  $z_i$  in the time interval  $[t_{min}, t_{max}]$  (denoted  $M_i$ ) and  $Y_i$ , which is defined as  $Y_i = \max\{|\sigma_{i,k}| \in \mathbb{R} \mid 10s \leq t_k \leq t_{max}\}$ . In the following simulations, the filtering differentiator has the following parameters  $n = 3$ ,  $n_f = 2$ ,  $\lambda_0 = 1.1$ ,  $\lambda_1 = 6, 75$ ,  $\lambda_2 = 20.26$ ,  $\lambda_3 = 32.24$ ,  $\lambda_4 = 23.72$  and  $\lambda_5 = 7$ . Notice that the parameters  $\lambda_i$  are chosen as in (Levant, 2018). Finally, the initial condition for the differentiator is  $[\omega_0^T \ z_0^T]^T = [0, 0, 0, 0, 0, 0]$ .

For the first scenario,  $f_0(t) = t^4 + \sin(t)$ ,  $L = 25$ ,  $\tau = 0.1s$ ,  $t_{min} = 10s$ ,  $t_{max} = 25s$ . Furthermore, there is no noise input. Figures 1-4 show the corresponding estimation errors using the three differentiators. Variables  $Y_i$  and the  $M_i$  are summarised in the Table 1.

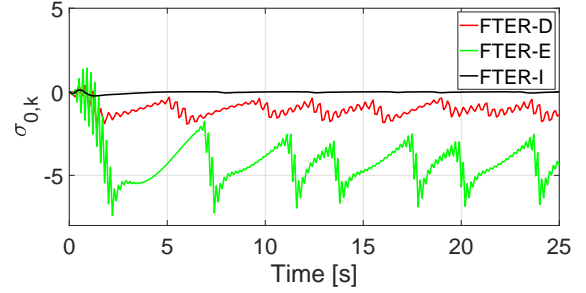


Fig. 1. Estimation error for  $f_0(t)$ .

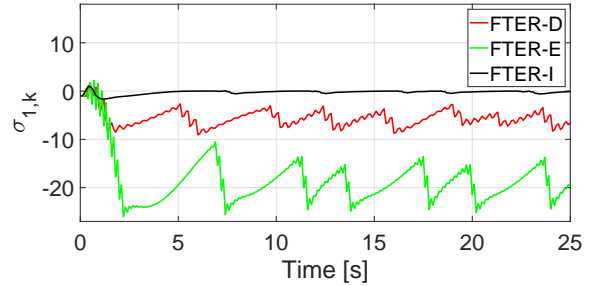


Fig. 2. Estimation error for the first derivative of  $f_0(t)$ .

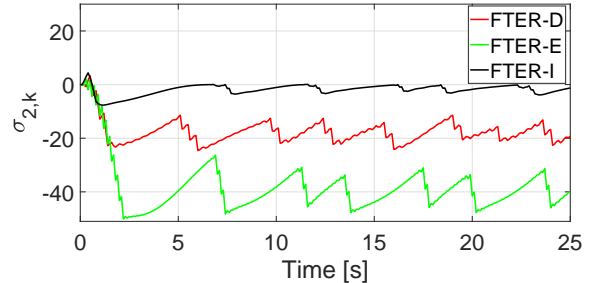


Fig. 3. Estimation error for the second derivative of  $f_0(t)$ .

For this scenario, the three differentiators converge in finite-time in spite of the unbounded functions  $f_0(t)$ ,  $f_0^{(1)}(t)$ ,  $f_0^{(2)}(t)$  and  $f_0^{(3)}(t)$ . Moreover using the differentiator FTER-I, one obtains the best results as it can be seen in Table 1 and Figures 1-4.

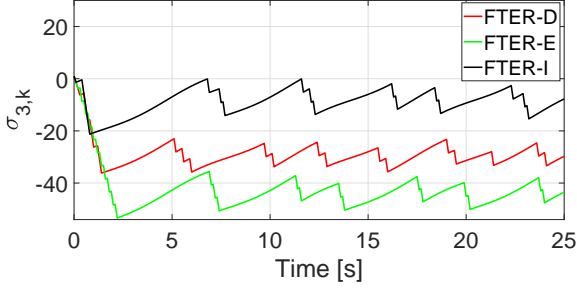


Fig. 4. Estimation error for the third derivative of  $f_0(t)$ .

	FTER-D	FTER-E	FTER-I
$Y_0$	1.8347	6.9573	0.0736
$Y_1$	8.7351	25.1874	0.5835
$Y_2$	24.18	47.741	3.8547
$Y_3$	35.7077	50.3692	15.4041
$M_0$	1.1476	4.4312	0.0256
$M_1$	5.9919	19.3807	0.212
$M_2$	18.4511	39.9016	1.7431
$M_3$	29.7708	43.9609	8.4592

Table 1.  $Y_i$  and  $M_i$  for Scenario I.

In the second scenario,  $f_0(t) = \sin(3t) + \cos(2t) - \sin(t) + \varepsilon_t$ , with  $\varepsilon_t \sim \text{iid}\mathcal{N}(0, 0.1^2)$ ,  $L = 98$ ,  $\tau = 0.1s$ ,  $t_{min} = 10s$  and  $t_{max} = 25s$ . Figures 5-8 show the corresponding estimation errors using the three differentiators. Variables  $Y_i$  and the  $M_i$  are summarised in Table 2.

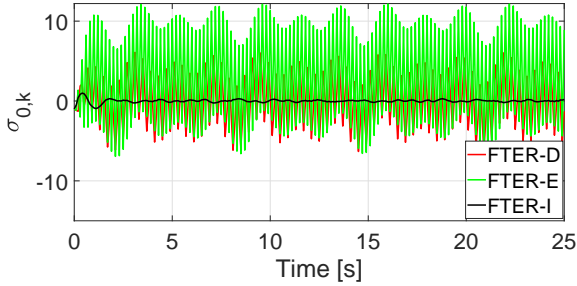


Fig. 5. Estimation error for  $f_0(t)$ .

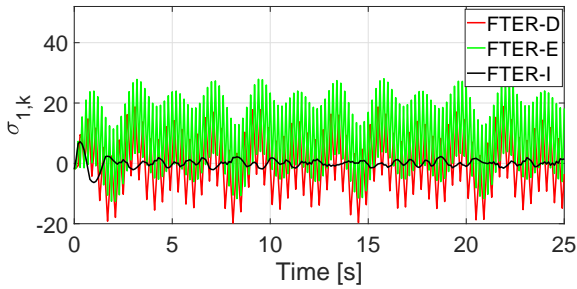


Fig. 6. Estimation error for the first derivative of  $f_0(t)$ .

For this scenario, the best result for the first two derivatives has been obtained using the proposed implicit differentiator, i.e., FTER-I. For the last derivative, the explicit differentiators, i.e., FTER-D and FTER-E, present better indexes  $Y_3$  and  $M_3$  than the implicit one.

For the last scenario, in order to test the differentiator under noise and different sampling times, the parameters

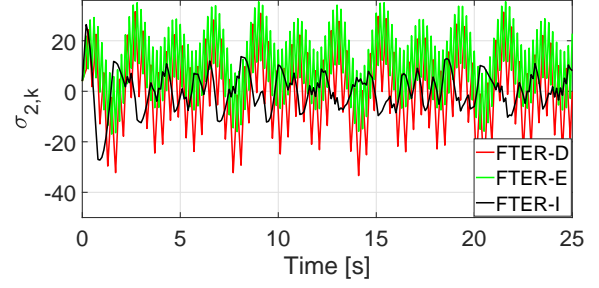


Fig. 7. Estimation error for the second derivative of  $f_0(t)$ .

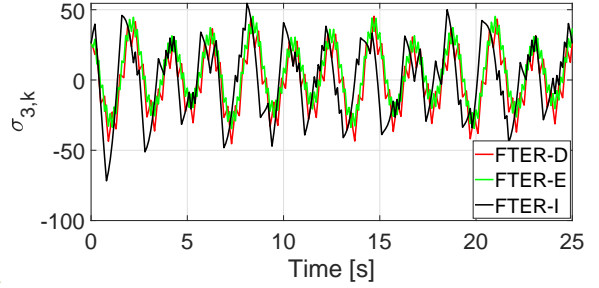


Fig. 8. Estimation error for the third derivative of  $f_0(t)$ .

	FTER-D	FTER-E	FTER-I
$Y_0$	6.322853	12.145781	0.274985
$Y_1$	19.63017	28.059032	2.268494
$Y_2$	33.374147	35.453408	12.964999
$Y_3$	45.347208	45.279109	50.014388
$M_0$	3.146707	7.543292	0.097569
$M_1$	9.009185	15.711948	0.882423
$M_2$	14.033387	17.492077	5.676374
$M_3$	20.517656	20.648051	23.703817

Table 2.  $Y_i$  and  $M_i$  for Scenario II.

$Y_i$  are given for different constant sampling times in the interval  $\tau \in [0.0001s, 1s]$  with a step of 0.0001s. Furthermore,  $f_0(t) = \sin(3t) + \cos(2t) - \sin(t)$ ,  $L = 98$ ,  $t_{min} = 10s$ ,  $t_{max} = 100s$  and the noise is selected as in (Levant and Livne, 2019),  $v(t) = \cos(10000t + 0.7791) + \varepsilon_t$ , with  $\varepsilon_t \sim \text{iid}\mathcal{N}(0, 0.5^2)$ . The results are summarised in Figures 9-11.

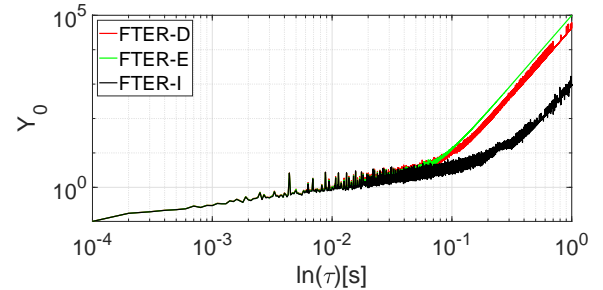
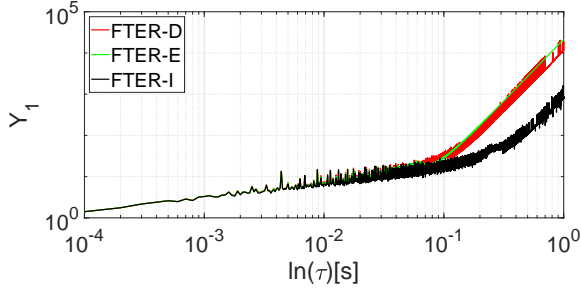
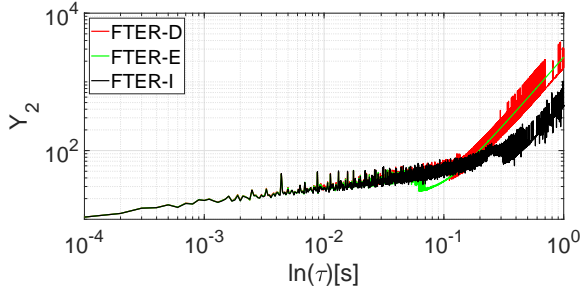
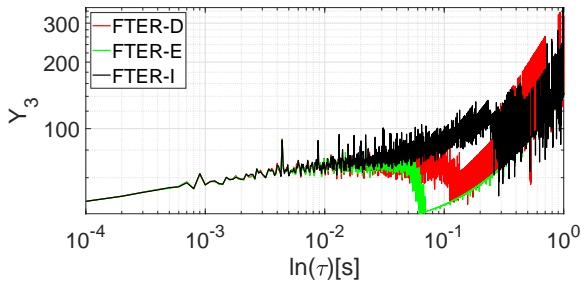


Fig. 9.  $Y_0$  for  $\tau \in [0.0001s, 1s]$ .

From Figures 9-11, one can see that the differentiator FTER-I gives a better performance of the estimation of  $f_0(t)$ ,  $f_0^{(1)}(t)$  and  $f_0^{(2)}(t)$  or at least similar for the different sampling times. Although, Figures could indicate that for low frequencies, the estimation of the second and third

Fig. 10.  $Y_1$  for  $\tau \in [0.0001s, 1s]$ .Fig. 11.  $Y_2$  for  $\tau \in [0.0001s, 1s]$ .Fig. 12.  $Y_3$  for  $\tau \in [0.0001s, 1s]$ .

derivatives of the signal is better for the FTER-D and FTER-E compared with FTER-I.

## 5. CONCLUSION

Two novel discretization algorithms have been presented for the robust filtering differentiator. The first one, which is based on an exact discretization of the continuous differentiator, is an explicit one, while the second one is an implicit algorithm which enables to remove the numerical chattering phenomenon and to preserve the estimation accuracy properties. Both algorithms have shown a competitive performance in simulations for free-noise input and when the first  $n$  derivatives are unbounded. It is also shown a better performance of the current proposal when compared to the discrete version given in (Levant and Livne, 2019). Moreover, in simulations and under noise, the FTER-I presents a better estimation for  $f_0(t)$ ,  $f_0^{(1)}(t)$ , and  $f_0^{(2)}(t)$  than the obtained results using FTER-D and FTER-E. Future works will address convergence and robustness proofs for the proposed discretizations.

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