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Robust Zonotopic Observer Design: Avoiding Unmeasured Premise Variables for Takagi-Sugeno Fuzzy Systems

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Abstract: This paper addresses the design of a robust set-based interval observer for a nonlinear discrete-time system affected by system uncertainty (state disturbance and measurement noise) using a Takagi-Sugeno (TS) fuzzy model including an unmeasurable premise variable. The effect of unmeasurable premise variable and uncertainties is considered as of unknown but bounded nature, i.e., in the set-membership framework. A zonotopic representation of a set towards reducing set operations to simple matrix calculations is used to bound the state estimation provided by the interval observer-based approach. Furthermore, the criterion-based approach and $\mathscr{H}_\infty$ performance technique are considered in order to compute the observer gain to achieve robustness. Finally, an example is employed to both illustrate and discuss the effectiveness of the proposed approach.

Keywords: Takagi-Sugeno fuzzy systems, nonlinear estimation, interval observers, zonotope, $\mathscr{H}_\infty$ performance, linear matrix inequality.

1. INTRODUCTION

In recent years, engineering systems are becoming more and more sophisticated. Then, there has been an increasing interest in the performance analysis of an automatic control system as well as investigation of its safety and reliability (Blanke et al., 2006). Associated with the goal of increasing system performance, one of the significant problems in control theory is the state estimation problem that plays a key role in the fault diagnosis of dynamic systems behavior (Chen and Patton, 2012; Gertler, 1997). Particularly, the state estimation problem is even more challenging facing the nonlinear systems that are composed of hundreds of constitutive elements.

Generally speaking, there are two major classifications of state estimation methods: i) model-based, ii) data-based. In the former class, monitoring the system behavior is done based on the mathematical model of the plant (Gertler, 1997), while the latter class includes those methods that do not use the mathematical model for the same purpose (Basseville and Nikiforov, 1993). Model-based approaches rely on the quality of the mathematical model. However, having the presence of model uncertainty, unknown disturbances, and noises, a mismatch between the actual process behavior and its mathematical model is non-negligible. Therefore, the effect of the uncertainty and noise/disturbance is an important point that must be considered when monitoring the system behavior with the model-based state estimation approaches. Based on the literature, several methods have been introduced to explicitly consider uncertainties in the mathematical model that can be classified into two main paradigms: i) the stochastic approaches that the uncertainties are represented by a random variable (Kalman, 1960; Maybeck, 1982), ii) the deterministic approaches (also called set-membership approach) that the uncertainties are represented as an unknown but bounded variable utilizing a different type of sets, e.g., interval boxes, polytopes, ellipsoids, and zonotopes (Schweppe, 1968; Pourasghar et al., 2020a; Kodakkadan et al., 2017).

On the other hand, there exist many real systems with nonlinear behavior that the established linear system theory cannot be directly applied to deal with them. Several studies have reported different ways of dealing with the nonlinear system. Among them, the Takagi-Sugeno (TS) paradigm is one of the most successful approaches that provides an effective way to describe a class of nonlinear systems. Furthermore, the TS approach can be considered as a kind of bridge between the linear system and nonlinear system theories (Tanaka and Wang, 2004). Preliminary work on TS systems was undertaken by (Takagi and Sugeno, 1985). Generally speaking, providing a way of representing nonlinear systems considering the fuzzy sets, fuzzy rules and a set of local linear models are the basic concepts of TS systems that are smoothly connected by fuzzy membership functions (Nguyen et al., 2019). According to the literature (Tanaka and Wang, 2004), TS fuzzy models can model complex behavior of nonlinear systems due to their approximation regarding any smooth nonlinear function to any degree of accuracy.

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Recent developments have led to a renewed interest in monitoring the nonlinear system behavior based on TS fuzzy-model-based observers. In fact, the TS fuzzy observers are mainly based on TS fuzzy modeling of nonlinear systems consisting of premise variables, i.e., system nonlinearities, that can be either *measurable*, or *unmeasurable* (Bergsten et al., 2002; Pan et al., 2020; Nguyen et al., 2021). In the case of designing the observer for a nonlinear system whose dynamic is modeled by using the TS fuzzy approach with considering all the premise variables are measurable, the main goal will be to reduce the design conservatism which is relatively simple by using different Lyapunov candidate functions and/or introducing slack variables (Guerra et al., 2011). So far, this method has only been applicable for a restrictive class of TS fuzzy systems. However, far too little attention has been paid to the case where premise variables cannot be measured. Moreover, having both the undeniable effect of uncertainties and unmeasured premise variables when designing the TS fuzzy observers has become one of the challenges in the automatic control community.

So far, however, several studies have reported state estimation for TS fuzzy systems, but there is still insufficient study for considering both effects of uncertainties and unmeasured premise variables. Moreover, when designing the observer, additionally to the problem of considering the mentioned uncertainties or unmeasured premise variable, another important problem is how to design the observer gain to be as robust as possible against these effects. Different manners of the computation of the observer gain have been reported to minimize the effect of uncertainties (Pourasghar et al., 2019, 2020b). But, considerably more work will need to be done to determine a robust observer. Then, the main contribution of this paper is to design a robust observer approach for a class of TS fuzzy systems whose observer gain is computed to be as robust as possible against the effect of uncertainty. In this regard, system uncertainty and unmeasurable premise variable are considered to be unknown but bounded using a zonotopic representation of a set. Moreover, two different manners of the computation of the observer gain will be proposed using the criterion-based method and $\mathscr{H}_\infty$ performance technique to achieve the robustness.

Notation. $\mathbb{R}^n$ denotes the set of n-dimensional real numbers and $\oplus$ denotes the Minkowski sum. The matrices are written using capital letter, e.g., A and the calligraphic notation is used for denoting sets, e.g., $\mathcal{X}$. The set of nonnegative integers is denoted by $\mathbb{Z}_+$ and $\mathcal{I}_r = \{1, 2, \dots, r\} \subset \mathbb{Z}_+$.

For $i \in \mathcal{I}_r$, we denote $\mu_r(i) = [0, \dots, 0, \overbrace{1}^{i\text{th}}, 0, \dots, 0]^\top \in \mathbb{R}^r$ a vector of the canonical basis of $\mathbb{R}^r$. For a vector x, x_i denotes its ith entry. For two vectors $x, y \in \mathbb{R}^n$, the convex hull of these vectors is denoted as $\text{co}(x, y) = \{\lambda x + (1-\lambda)y : \ \lambda \in [0,1]\}$. For a matrix X, $X^\top$ denotes its transpose, $X \succ 0$ means X is symmetric positive definite, and $\text{He}X = X + X^\top$. I denotes the identity matrix of appropriate dimension. In block matrices, the symbol $\star$ stands for the terms deduced by symmetry. Arguments are omitted when their meaning is clear.

2. PRELIMINARIES

This section recalls some basic concepts on matrix calculus and zonotopes.

Definition 2.1. (Zonotope). A zonotope $\langle c_z,\ R_z \rangle \subset \mathbb{R}^n$ with the center $c \in \mathbb{R}^n$ and the generator matrix $R \in \mathbb{R}^{n \times p}$ is a polytopic set defined as a linear image of the unit hypercube $[-1,\ 1]^n$:

$$\langle c_z,\ R_z \rangle = \{c_z + R_z s, \|s\|_\infty \le 1\}.$$

$\langle R_z \rangle = \langle 0,\ R_z \rangle$ denotes a centered zonotope. Any permutation of the columns of R leaves it invariant. □

Definition 2.2. (Minkowski Sum). Considering two sets $\mathcal{A}$ and $\mathcal{B}$, their Minkowski sum is a set defined as $\mathcal{A} \oplus \mathcal{B} = \{a + b |\ a \in \mathcal{A},\ b \in \mathcal{B}\}$. Furthermore, the Minkowski sum of the zonotopes $\mathcal{Z}_1 = \langle c_{z_1}, R_{z_1} \rangle$ and $\mathcal{Z}_2 = \langle c_{z_2}, R_{z_2} \rangle$ is $\mathcal{Z}_1 \oplus \mathcal{Z}_2 = \langle c_{z_1} + c_{z_2}, [R_{z_1},\ R_{z_2}] \rangle$. □

Property 2.1. (Linear Image). The linear image of a zonotope $\mathcal{Z} = \langle c, R \rangle$ by a compatible matrix L is $L \odot \langle c, R \rangle = \langle Lc, LR \rangle$. □

Property 2.2. (Reduction Operator). A reduction operator denoted $\downarrow_q$ permits to reduce the number of generators of a zonotope $\langle c,\ R \rangle$ to a fixed number q while preserving the inclusion property $\langle c,\ R \rangle \subset \langle c,\ \downarrow_q \{R\} \rangle$. A simple yet efficient solution to compute $\downarrow_q\ \{R\}$ is given in (Combastel, 2003). It consists in sorting the columns of R on decreasing Euclidean norm and enclosing the influence of the smaller columns only into an easily computable interval hull, so that the resulting matrix $\downarrow_q\ \{R\}$ has no more than q columns. □

Property 2.3. (Zonotope Inclusion). Given a zonotope $\mathcal{Z} = \langle c,\ R \rangle \subset \mathbb{R}^n$, with a vector $c \in \mathbb{R}^n$ denoting the center and an interval matrix $R \in \mathbb{R}^{n \times m} (n \le m)$ denoting the shape of the zonotope, a zonotope inclusion indicated by $\diamond(\mathcal{Z})$ is defined as $\diamond(\mathcal{Z}) = \langle c, [mid(R),\ S] \rangle$, where S is a diagonal matrix that satisfies $S_{ii} = \sum_{j=1}^m \dfrac{diam(R_{ij})}{2}$, $i = 1, 2, \cdots, n$, with $mid(.)$ and $diam(.)$ are the center and diameter of interval matrix, respectively. □

Property 2.4. (State Zonotope Inclusion). Given $\mathcal{X}_{k+1} = A\mathcal{X}_k \oplus Bu_k$, where A and B are interval matrices and u_k is the input at time instant k, considering $\mathcal{X}_k$ as a zonotope with the center $c_{x,k}$ and the shape matrix $R_{x,k}$ such $\mathcal{X}_k = \langle c_{x,k}, R_{x,k} \rangle$, the zonotopic state at the next time instant $k+1$ defined as $\mathcal{X}_{k+1}$ is bounded by a zonotope $\mathcal{X}^e_{k+1} = \langle c_{x,k+1}, R_{x,k+1} \rangle$, with

$$\begin{aligned} c_{x,k+1} &= mid(A)\ c_{x,k} + mid(B)\ u_k, \\ R_{x,k+1} &= \left[\diamond(AR_{x,k}),\ \frac{diam(A)}{2} c_{x,k},\ \frac{diam(B)}{2} u_k \right], \end{aligned}$$

where $\diamond(AR_{x,k})$ shows the shape matrix of the state bounding zonotope. □

3. PROBLEM FORMULATION

We consider the following nonlinear system:

$$\begin{aligned} x_{k+1} &= \Psi(x_k, u_k) + E_\omega \omega_k, \\ y_k &= Cx_k + E_\upsilon \upsilon_k, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the control input, $y \in \mathbb{R}^{n_y}$ is the system output, $\omega \in \mathbb{R}^{n_\omega}$ is the disturbance input, and $\upsilon \in \mathbb{R}^{n_\upsilon}$ is the process noise. The nonlinear function $\Psi(\cdot) \in \mathbb{R}^{n_x \times n_u}$ is differentiable with respect to the state x. The constant matrices C, E_ω and E_υ are with appropriate dimensions.

For observer design, inspired by the TS fuzzy modeling with nonlinear consequents (Coutinho et al., 2020), we reformulate system (1) in the form

$$\begin{aligned} x_{k+1} &= A(\xi_k)x_k + \tau(\xi_k, u_k) + G(\xi_k)\phi(x_k, u_k) + E_\omega \omega_k, \\ y_k &= Cx_k + E_\upsilon \upsilon_k. \end{aligned} \tag{2}$$

In this paper, we assume that the vector of premise variables $\xi_k \in \mathbb{R}^{n_\xi}$ can be measured, i.e., it is a vector-valued function of the elements of the output vector y_k which are not corrupted by the measurement noise υ_k. The nonlinear function $\phi(x, u)$ is differentiable with respect to the state x. Note that the matrix-valued functions $A(\xi)$ and $G(\xi)$, and the vector-valued function $\tau(\xi, u)$ are measurable, whereas the elements of $\phi(x, u)$ cannot be measured from the output. Applying the sector nonlinearity approach (Tanaka and Wang, 2004), the nonlinear system (2) can be represented in the following TS fuzzy form:

$$x_{k+1} = A(h)x_k + \tau(\xi_k, u_k) + G(h)\phi(x_k, u_k) + E_\omega \omega_k, \tag{3}$$

where $[A(h) \;\; G(h)] = \sum_{i=1}^{r} h_i \, [A_i \;\; G_i]$.

The constant matrices A_i and G_i are known, $r = 2^{n_\xi}$ is the number of fuzzy rules. The fuzzy membership functions (MFs) verify the following convex sum property:

$$\sum_{j=1}^{r} h_i(\xi_k) = 1, \quad 0 \le h_i(\xi_k) \le 1, \quad \forall i \in \mathcal{I}_r. \tag{4}$$

Let $\mathcal{H}$ be the set of the membership functions satisfying (4), i.e., $h = [h_1(\xi), \; h_2(\xi), \; \ldots, \; h_r(\xi)]^\top \in \mathcal{H}$.

Remark 3.1. Note that all the unmeasurable premise variables of system (1) are isolated in the nonlinear term $\phi(x_k, u_k)$ in (3). □

For system (3), we assume that the nonlinear function $\phi(x_k, u_k)$ is unknown but bounded and satisfies the following conditions:

$$\frac{\partial \phi_i}{\partial x_j}(x, u) \in \Theta_{ij}, \tag{5}$$

with $\Theta_{ij} = \{\theta_{ij} \in \mathbb{R}^{n_{\theta_{ij}}} : \;\; |\theta_{ij}| \le \bar{\theta}_{ij}, \;\; \bar{\theta}_{ij} \in \mathbb{R}^{n_{\theta_{ij}}}\}$, and $\bar{\theta}$ is a constant vector. Moreover, Θ is considered as convex and compact sets that can be generally expressed as zonotopic approximation reducing set operations to simple matrix calculations, i.e., $\Theta = \langle 0, \; R_\theta \rangle$, where $R_\theta \in \mathbb{R}^{n_\theta \times n_\theta}$ denotes the generator matrix of the set Θ.

Furthermore, the additive uncertainties ω and υ are also assumed to be unknown but bounded, i.e., they belong to the following convex and compact sets:

$$\mathcal{W} = \langle c_\omega, \; R_\omega \rangle, \quad \mathcal{V} = \langle c_\upsilon, \; R_\upsilon \rangle, \tag{6}$$

where c_ω and c_υ denote the centers of the sets $\mathcal{W}$ and $\mathcal{V}$, respectively, with their generator matrices $R_\omega \in \mathbb{R}^{n_\omega \times n_\omega}$ and $R_\upsilon \in \mathbb{R}^{n_\upsilon \times n_\upsilon}$.

Assumption 3.1. The zonotopic sets representing the additive uncertainties represented in (6) are assumed to be bounded by a unit hypercube expressed as the centered zonotopes, i.e., $\forall \; k \ge \; 0$, $\omega = [-1, \; 1] = \langle 0, \; I_{n_\omega} \rangle$ and $\upsilon = [-1, \; 1] = \langle 0, \; I_{n_\upsilon} \rangle$, where I_{n_ω} and I_{n_υ} denote the identity matrices. □

Assumption 3.2. The initial state x_0 belongs to the zonotopic set $\mathcal{X}_0 = \langle c_0, \; R_0 \rangle$, where $c_0 \in \mathbb{R}^{n_x}$ denotes the center and $R_0 \in \mathbb{R}^{n_x \times r_{R_0}}$ is non-empty matrix containing the generators matrix of the initial zonotope $\mathcal{X}_0$. □

Hereafter, the subscript $k+1$ will be replaced by $+$ and k will be omitted for simplicity. Consequently, the dynamical model (3) can be rewritten as

$$\begin{aligned} x_+ &= A(h)x + \tau(\xi, u) + G(h)\phi(x, u) + E_\omega, \omega \\ y &= Cx + E_\upsilon \upsilon. \end{aligned} \tag{7}$$

3.1 Luenberger-Type Observer Structure

For the estimation of the uncertain nonlinear system (7), we consider the following Luenberger observer structure:

$$\begin{aligned} \hat{x}_+ &= A(h)\hat{x} + \tau(\xi, u) + G(h)\phi(\hat{x}, u) + L(h)(y - \hat{y}), \\ \hat{y} &= C\hat{x}, \end{aligned} \tag{8}$$

where $\hat{x}$ and $\hat{y}$ are respectively the state estimation and the output prediction, and $L(h) = \sum_{i=1}^{r} h_i(\xi)L_i$. Let us define the state estimation error as

$$\tilde{x} = x - \hat{x}.$$

Then, the dynamics of the state estimation error can be obtained from (7) and (8) as follows:

$$\tilde{x}_+ = \Big(A(h) - L(h)C\Big)\tilde{x} + G(h)\Delta_\phi + E_\omega \omega - L(h)E_\upsilon \upsilon, \tag{9}$$

with $\Delta_\phi = \phi(x, u) - \phi(\hat{x}, u)$. To deal with the difficulty caused by the nonlinear mismatching term Δ_ϕ, the differential mean value theorem in Lemma A.1 is used to convert Δ_ϕ into a function of the state estimation error $\tilde{x}$ for observer design. Hence, applying Lemma A.1 to the nonlinear function $\phi(x, u)$, there exists a constant vector $\check{s}_i \in \text{co}(x, \hat{x})$, for $\forall i \in \mathcal{I}_{n_\phi}$, such that

$$\Delta_\phi = \left(\sum_{i=1}^{n_\phi} \sum_{j=1}^{n_x} \mu_{n_\phi}(i) \mu_{n_x}^\top(j) \frac{\partial \phi_i}{\partial x_j}(\check{s}_i, u) \right) (x - \hat{x}). \tag{10}$$

For simplicity, we denote $\theta_{ij} = \frac{\partial \phi_i}{\partial x_j}(\check{s}_i, u)$, for $\forall (i, j) \in \mathcal{I}_{n_\phi} \times \mathcal{I}_{n_x}$. Note that according to (5), the parameter vector $\theta = \big[\theta_{11}, \; \ldots, \; \theta_{1n_x}, \; \ldots, \; \theta_{n_\phi n_x}\big]$ belongs to the bounded zonotopic convex and compact set Θ.

Considering Δ_ϕ in (10), the estimation error dynamics (9) can be rewritten for $h \in \mathcal{H}$ and $\theta \in \Theta$ as

$$\tilde{x}_+ = \left(\mathscr{A}(h, \theta) - L(h)C\right)\tilde{x} + E_\omega \omega - L(h)E_\upsilon \upsilon, \tag{11}$$

with

$$\mathscr{A}(h, \theta) = \sum_{i=1}^{r} h_i(\xi_k)\mathscr{A}_i(\theta),$$

$$\mathscr{A}_i(\theta) = A_i + G_i \sum_{l=1}^{n_\phi} \sum_{m=1}^{n_x} \mu_{n_\phi}(l) \mu_{n_x}^\top(m) \theta_{lm}.$$

The goal is to determine the observer gain $L(h)$ such that we can achieve desirable bounds of the estimated state $\hat{x}$ despite the effect of bounded uncertainties represented by the disturbance signals ω and υ.

3.2 Zonotopic Interval Observer Structure

Under the consideration of bounded uncertainties and set-zonotopic representation, the state bounding observer corresponding to the nonlinear system (7) can be obtained as a zonotope $\hat{\mathcal{X}} = \langle c_x, \; R_x \rangle$ using the Luenberger-type observer (8) and Proposition 3.1.

Proposition 3.1. (Zonotopic Observer Structure). Consider Assumptions 3.1 and 3.2, and the Luenberger observer structure (8). The center c_x and the shape matrix R_x of $\hat{\mathcal{X}}$ can be recursively computed as

$$\begin{aligned} c_{x,+} &= (A(h) - L(h)C)c_x + \tau(\xi, u) + L(h)y, \\ R_{x,+} &= \left[(A(h) - L(h)C)\bar{R}_x \; G(h)R_\theta \; E_\omega \; L(h)E_v\right], \end{aligned} \quad (12)$$

where $\bar{R}_x =\downarrow_q \{\bar{R}_x\}$. Moreover, the state inclusion property $x \in \langle c_x, R_x\rangle$ in Properties 2.3 and 2.4 holds for all $k \geq 0$.

Proof. Assume $x \in \langle c_x, R_x\rangle$, $\omega \in \langle 0, I_{n_\omega}\rangle$ and $\upsilon \in \langle 0, I_{n_\upsilon}\rangle$, for all $k \geq 0$, where the inclusion property is preserved by using the reduction operator, i.e., $x \in \langle c_x, \bar{R}_x\rangle$. Thus, the state observer (8) can be formulated using the zonotopic representation as

$$\begin{aligned} \hat{x}_+ \in \langle (A(h) - L(h)C)c_x,\ R_{x,+}\rangle = \\ \langle (A(h) - L(h)C)c,\ (A(h) - L(h)C)\bar{R}_x\rangle \\ \oplus \langle \tau(\xi, u),\ 0\rangle \oplus \langle 0,\ G(h)R_\theta\rangle \oplus \langle L(h)y,\ 0\rangle \\ \oplus \langle 0,\ E_\omega\rangle \oplus \langle 0,\ L(h)E_v\rangle . \end{aligned} \quad (13)$$

Then, based on Definition 2.2 and Property 2.1, $c_{x,+}$ and $R_{x,+}$ in (13) can be expressed as in (12), where the center $c_{x,+}$ can be interpreted as a classical punctual state estimate of the unknown state x and the shape matrix $R_{x,+}$ characterizes the zonotopic enclosure of the classical observation error. ■

Since the zonotopic representation of a set is considered, the state estimation error dynamics in (11) can be obtained as a zonotope. Then, the observer gain $L(h)$ is determined such that the error dynamics converges asymptotically into a zonotopic bound instead of the origin. As a result, the zonotopic state estimation error of model (7) can be obtained as a zonotope $\tilde{\mathcal{X}} = \langle c_{\tilde{x}}, R_{\tilde{x}}\rangle$ using the observer (8) and Proposition 3.1.

Proposition 3.2. (Zonotopic Estimation Error). We consider Assumptions 3.1 and 3.2, the observer structure (8), and the estimation error dynamics (11). The center $c_{\tilde{x}}$ and the shape matrix $R_{\tilde{x}}$ of $\tilde{\mathcal{X}}$ can be recursively computed as

$$c_{\tilde{x},+} = mid\Big(\mathscr{A}^*(h,\theta)\Big)c_{\tilde{x}} + L(h)y,$$

$$R_{\tilde{x},+} = \left[\diamond\Big(\mathscr{A}^*(h,\theta)\Big)\bar{R}_{\tilde{x}} \;\; \frac{diam\Big(\mathscr{A}^*(h,\theta)\Big)}{2}c_{\tilde{x}} \;\; E_\omega \;\; L(h)E_v\right],$$

where $\mathscr{A}^*(h,\theta) = \mathscr{A}(h,\theta) - L(h)C$ and $\bar{R}_{\tilde{x}} =\downarrow_q \{\bar{R}_{\tilde{x}}\}$. Moreover, the state inclusion property $\tilde{x} \in \langle c_{\tilde{x}}, R_{\tilde{x}}\rangle$ in Properties 2.3 and 2.4 holds for all $k \geq 0$.

Proof. Assume $x \in \langle c_x, R_x\rangle$, $\omega \in \langle 0, I_{n_\omega}\rangle$ and $\upsilon \in \langle 0, I_{n_\upsilon}\rangle$ for all $k \geq 0$, where the inclusion property is preserved by using the reduction operator, i.e., $\tilde{x} \in \langle c_{\tilde{x}}, \bar{R}_{\tilde{x}}\rangle$. Thus, the state estimation error in (11) can be formulated using the zonotopic representation as

$$\begin{aligned} \tilde{x}_+ \in \langle c_{\tilde{x},+},\ R_{\tilde{x},+}\rangle = \Big\langle \big(\mathscr{A}^*(h,\theta)\big)c_{\tilde{x}},\ \big(\mathscr{A}^*(h,\theta)\big)\bar{R}_{\tilde{x}}\Big\rangle \\ \oplus \langle L(h)y,\ 0\rangle \oplus \langle 0,\ E_\omega\rangle \\ \oplus \langle 0,\ L(h)E_v\rangle . \end{aligned} \quad (15)$$

Then, based on Definition 2.2 and Properties 2.1 and 2.4, $c_{\tilde{x},+}$ and $R_{\tilde{x},+}$ in (15) can be expressed as in (14), where the center $c_{\tilde{x},+}$ can be interpreted as a classical punctual state estimate error of the unknown state $\tilde{x}$ and the shape matrix $R_{\tilde{x},+}$ characterizes the zonotopic enclosure of the classical observation error. ■

4. SET-BASED INTERVAL OBSERVER DESIGN

This section presents two approaches for observer gain design: criterion-based approach, and $\mathscr{H}_\infty$-based approach.

4.1 Criterion-Based Observer Design

As can be seen from Proposition 3.1, the zonotopic state bounding observer (12) is parameterized by means of the observer gain $L(h)$, at each time instant k. According to (Alamo et al., 2005; Combastel, 2015), the size of the state bounding zonotope in $\hat{\mathcal{X}} = \langle c_x, R_x\rangle$ can be obtained by minimizing the F-radius of $\hat{\mathcal{X}} = \langle c_x, R_x\rangle$. Then, the corresponding observer gain, denoted by $L^*(h)$, can be computed using Theorem 4.1.

Theorem 4.1. Consider the nonlinear TS fuzzy system (7) and its associated Luenberger-type observer (8). The size of the zonotope defined in (12) can be optimized by using the following observer gain:

$$L^*(h) = \frac{A(h)R_x R_x^\top C^\top}{CR_x R_x^\top C^\top + E_v E_v^\top}.$$

Proof. The proof follows from the application of results presented in (Combastel, 2015). ■

4.2 $\mathscr{H}_\infty$−Based Observer Design

To attenuate the effect of uncertainties and thus to increase the estimation accuracy, the observer gain $L(h)$ of the zonotopic observer Proposition 3.1 can be computed to achieve an $\mathscr{H}_\infty$ performance. The following lemma is utilized to facilitate the observer design.

Lemma 4.1. (Bounded Real Lemma (Boyd et al., 1994)). Consider the following system:

$$\begin{aligned} x_+ &= Ax + Bw, \\ z &= Cx. \end{aligned} \quad (16)$$

Given $\gamma > 0$, if there exist a positive definite matrix $P \succ 0$ such that

$$\begin{bmatrix} A^\top PA + C^\top C - P & A^\top PB \\ * & B^\top PB - \gamma^2 I \end{bmatrix} \prec 0.$$

Then, system (16) is stable. Moreover, the $\mathscr{H}_\infty$−gain performance from w to z is smaller than γ. ■

Based on this lemma, the following theorem is developed to design an $\mathscr{H}_\infty$ observer gain $L(h)$.

Theorem 4.2. If there exists a positive definite matrix $P \in \mathbb{R}^{n_x \times n_x}$, matrices $W_i \in \mathbb{R}^{n_x \times n_y}$, for $i \in \mathcal{I}_r$, and a positive scalar γ, such that

$$\begin{bmatrix} -P + I & \star & \star & \star & \star \\ 0 & -\gamma^2 I & \star & \star & \star \\ 0 & 0 & -\gamma^2 I & \star & \star \\ 0 & 0 & 0 & -\gamma^2 I & \star \\ PA_i - W_i C & PG_i & PE_\omega & -W_i E_v & -P \end{bmatrix} \prec 0, \quad (17)$$

for $\forall i \in \mathcal{I}_r$. Then, the state estimation error dynamics (9) is stable and the $\mathscr{H}_\infty$−gain performance from the unknown-term vector $[\Delta_\phi \; \omega \; \upsilon]$ to the state estimation error $\tilde{x}$ is smaller than γ. Moreover, the gains of the observer can be computed as $L_i = P^{-1}W_i, \quad i \in \mathcal{I}_r$.

Proof. It follows from Lemma 4.1 that an $\mathscr{H}_\infty$−gain performance γ is guaranteed for the estimation error dynamics (9) if

$$\begin{bmatrix} \hat{A}(h)^\top P\hat{A}(h) - P + I & * \\ E_d^\top P\hat{A}(h) & E_d^\top PE_d - \gamma^2 I \end{bmatrix} \prec 0, \qquad (18)$$

where $\hat{A}(h) = A(h) - L(h)C$ and $E_d = [G(h) \;\; E_\omega \;\; -L(h)E_\upsilon]$. Then, by Schur complement lemma, inequality (18) is proved to be equivalent to

$$\begin{bmatrix} -P+I & \star & \star \\ 0 & -\gamma^2 I & \star \\ P\hat{A}(h) & PE_d & -P \end{bmatrix} \prec 0. \qquad (19)$$

Substituting the expression of E_d into (19) yields

$$\begin{bmatrix} -P+I & \star & \star & \star & \star \\ 0 & -\gamma^2 I & \star & \star & \star \\ 0 & 0 & -\gamma^2 I & \star & \star \\ 0 & 0 & 0 & -\gamma^2 I & \star \\ P\hat{A}(h) & PG(h) & PE_\omega & -PL(h)E_\upsilon & -P \end{bmatrix} \prec 0. \qquad (20)$$

With the change of variable $W(h) = PL(h)$, condition (17) can be directly obtained from (20) by convexity. ∎

Note that a minimization of the $\mathscr{H}_\infty$−gain performance level γ allows minimizing the effect of unknowns terms on the state estimated error.

5. ILLUSTRATIVE EXAMPLE

To illustrate the proposed state-bounding observer, we consider the nonlinear system borrowed from (Nguyen et al., 2021), which can be represented by the TS fuzzy model (7) with

$$x_+ = \sum_{i=1}^{2} h_i(\xi)A_i x + \tau(\xi, u) + G(h)\phi(x) + E_\omega \omega,$$
$$y = Cx + E_\upsilon \upsilon,$$

with

$$A_1 = \begin{bmatrix} 1+T & T & 0 & -0.1T \\ T & 1-2T & 0 & 0 \\ T & Ta^2 & 1-0.3T & 0 \\ 0 & 0 & 0 & 1-T \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1+T & T & 0 & -0.1T \\ T & 1-2T & 0 & 0 \\ T & 0 & 1-0.3T & 0 \\ 0 & 0 & 0 & 1-T \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad E_\upsilon = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}, \quad E_\omega = 0.05I,$$

and $T = 0.5$, $\tau(\xi, u) = [T(1+\xi)u \;\; 0 \;\; 0 \;\; 0]^\top$, $\xi = x_1^2$, $G = [T \;\; 0 \;\; 0 \;\; T]^\top$, $\phi(x) = \sin(x_3)$. Moreover, the corresponding membership functions are defined as $h_1(\xi) = \dfrac{x_1^2}{a^2}$ and $h_2(\xi) = 1 - h_1(\xi)$. The parameter vector θ is defined as $\theta_{1j} = 0$, for $i \in \{1, 2, 4\}$, and $\theta_{13} = \cos(x_3)$. Furthermore, the bounded set Θ has two vertices, i.e., $\theta_{13} \in \{0, 1\}$ considering $x_3 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

The obtained results from the simulation of the state estimation using the zonotopic interval observer approach are presented in Figure 1. In this context, Figure 1 shows the projection of the computed state-bounding zonotope into the state-space when the system is working in its healthy mode, i.e., the system is only affected by the state disturbance and measurement noise and unmeasured premise variable. Moreover, the computation of optimal observer gain for an observation purpose is computed by using Theorem 4.1 that the optimal observer gain L^* is obtained by minimizing the F-radius of the obtained state-bounding zonotope. On the other hand, $\mathscr{H}_\infty$ performance index is considered for the computation of the optimal gain by using Theorem 4.2. As it can be seen from Figure 1, using both methods to compute the observer gain result a quite similar behavior of the observer since the maximum and the minimum bounds of the obtained zonotopic state estimations using both Theorem 4.1 and Theorem 4.2 are quite similar. This fact illustrates that the proposed observer design dealing with TS fuzzy systems with unmeasurable premise variables is well suited.

6. CONCLUSIONS

A robust zonotopic observer design for nonlinear systems represented in TS fuzzy models with nonlinear consequents has been proposed. As a novelty, in the proposed observer design, the propagation of the uncertainties and the effect of unmeasured premise variables are taken into account considering a zonotopic-set representation to reduce the set operations to simple matrix calculations. Moreover, achieving robustness concerning the effect of uncertainties is carried out considering the criterion-based approach, where the optimal observer gain is computed by minimizing the Frobenius norm of the obtained state-bounding zonotope. Alternatively, $\mathscr{H}_\infty$ performance is considered to compute the observer gain by using LMI technique. Finally, an example is used to illustrate the obtained results. For future research, the effectiveness of the proposed results will be investigated to improve the fault sensitivity, rather than only the robust state estimation.

Appendix A

Lemma A.1. (Zemouche et al. (2008)). If function $f(x)$ is differentiable on $\mathrm{co}(a,\ b)$, where $f(x):\ \mathbb{R}^{n_x} \to \mathbb{R}^q$ and $a,\ b \in \mathbb{R}^{n_x}$, then, there exists constant vectors $s_i \in \mathrm{co}(a,\ b)$, $s_i \neq a$ and $s_i \neq b$ for $\forall i \in \mathcal{I}_q$, such that

$$f(a) - f(b) = \left(\sum_{i=1}^{q} \sum_{j=1}^{n} \mu_q(i)\mu_{n_x}^\top(j) \frac{\partial f_i}{\partial x_j}(s_i) \right)(a - b).$$

Note that the differential mean value theorem has been also exploited in the literature for different TS observer designs Nguyen et al. (2021); Pan et al. (2020).

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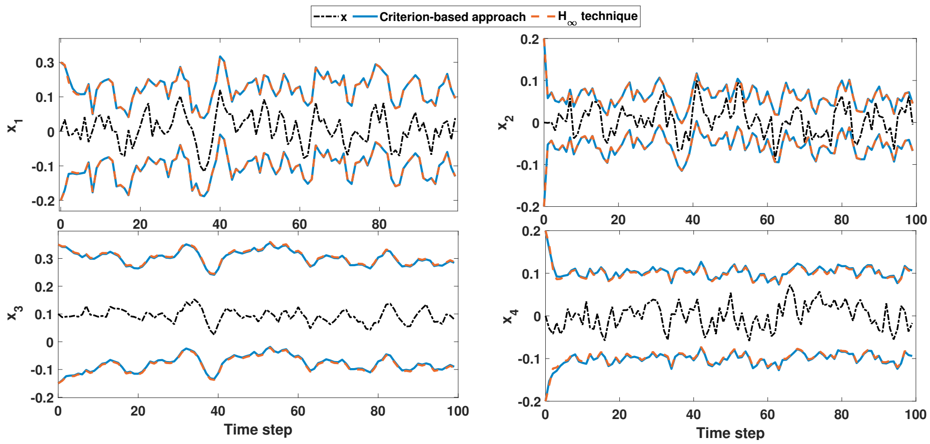

Fig. 1. State estimation using the observer gain with different criterion-based and $\mathscr{H}_\infty$-based gains.

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