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Choosing an Adequate Convex Structure for Controller and Observer Gains in Takagi-Sugeno Control Systems ⋆

Alberto Ortiz *, **Thierry-Marie Guerra** **, **Víctor Estrada-Manzo** *, **Jimmy Lauber** **

* *Dept. of Mechatronics of the Universidad Politécnica de Pachuca, C.P. 43830, Zempoala, Mexico. (e-mail: victor_estrada@upp.edu.mx).*

** *Polytechnic University Hauts-de-France UPHF, LAMIH UMR CNRS 8201, Valenciennes, France. (e-mail: {Thierry.Guerra,Jimmy.Lauber}@uphf.fr).*

Abstract: This work presents a methodology to choose the convex structure for nonlinear gains in approaches based on Takagi-Sugeno models. The proposal faces the problem of numerical complexity in LMI-based approaches within this area; it gives a finer convex structure for controller and observer matrices depending on two criteria: less number of LMI constraints or more decision variables while keeping the same number of LMIs in traditional approaches. The advantages of the proposal are illustrated via numerical examples.

Keywords: Linear matrix inequality, Takagi-Sugeno model, convex structures, numerical complexity, control systems.

1. INTRODUCTION

Nowadays, many theoretical results are available for quasi linear parameter-varying (qLPV) models or the so-called Takagi-Sugeno (TS) ones (Tanaka and Wang, 2001), both in continuous and discrete time; there are works on stability, state feedback, estimation, output feedback with or without observer design, with robustness issues or performances such as H_2 or H_∞, input/output saturation, see for example recent overviews (Guerra et al., 2015; Nguyen et al., 2019). Applications in various fields are also numerous, see for example a review in (Precup and Hellendoorn, 2011). In this work, the approaches under consideration are the ones whose conditions are expressed in terms of linear matrix inequalities (LMIs), which can be efficiently solved using convex optimization algorithms (Boyd et al., 1994). In this context, there is an increasing attention paid to transforming control, estimation, diagnosis problems into LMI constraints, very often using technical lemmas such as Schur's complement, Finsler's lemma, S-procedure and among others (Boyd et al., 1994).

Although, these theoretical results are available, they are generally not compatible with "real" systems, excepted low order state, few inputs or outputs and "not-so-many" nonlinearities, delays, saturation issues. These facts may lead unfeasible LMI problems even if the problem itself is doable. Unfeasible solutions may have different sources (González et al., 2016): the first one comes from the fact that a simple quadratic Lyapunov function is unable to solve the entire class of models described, and in a sense, the more they are complex, the less a quadratic feasible solution may be reached (Khalil, 2002); in this case, the results have little interest as they may never be applicable. Another source of unfeasibility is coming from the conditions themselves; e.g., some theoretical results show that according to a given parameter $d \in \mathbb{N}$ when $d \to \infty$, asymptotically necessary and sufficient conditions are reached (Scherer, 2005; Sala and Ariño, 2007; Kruszewski et al., 2009). Although very interesting on a theoretical point of view, generally it faces computational problems, e.g., even for "small" systems feasible solutions are only available for low values of the parameter $d \leq 6 \sim 10$. On the opposite, the real setups that need solutions can be more and more complex, thus resulting in a contradiction that becomes in many cases a serious issue. To exhibit an example, in (Guerra et al., 2020), the problem of explaining how a disabled stays sitting necessitates a 6-order nonlinear descriptor model with 3 inputs and 3 outputs and $2^9 = 512$ vertices for an exact convex representation by means of the sector nonlinearity approach (Ohtake et al., 2001). It renders LMI problems with million unknowns and no conditions are available to get a result. Thus, this situation illustrates that for a large number of LMI constraints and many LMI slack variables the designer is faced to numerical problems since solvers reach their computational limits.

This work proposes a simple and efficient way to select the convex structure of the decision variables that allows an optimization in the sense of complexity, that can be applied directly to any LMI problem. The rest of the paper is organized as follows: Section 2 recalls the notations and useful tools and gives the motivation through examples. Section 3 gives the principal result and the way to optimize the number of slack variables without increasing

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the complexity in terms of number of LMI constraints. In Section 4illustrates with examples the interest to consider this path before running the convex optimization algorithms. Finally, Section 5 gathers some conclusion and future work.

2. PROBLEM STATEMENT

Let us considered nonlinear systems already in a TS form:

$$\begin{aligned} \delta x(t) &= \sum_{\mathbf{i}=1}^{r} \mu_{\mathbf{i}}(z(t))(A_{\mathbf{i}}x(t) + B_{\mathbf{i}}u(t)) , \\ y(t) &= \sum_{\mathbf{i}=1}^{r} \mu_{\mathbf{i}}(z(t))C_{\mathbf{i}}x(t), \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, $y \in \mathbb{R}^o$ is the output vector, $A_{\mathbf{i}}$ and $B_{\mathbf{i}}$, $\mathbf{i} \in \{1, 2, \ldots, r\}$ are vertex matrices of adequate dimensions, $\mu_{\mathbf{i}}(z(t))$, $\mathbf{i} \in \{1, 2, \ldots, r\}$ are membership functions that hold the convex sum property, i.e., $\sum_{\mathbf{i}=1}^{r} \mu_{\mathbf{i}}(z(t)) = 1$, $0 \le \mu_{\mathbf{i}}(z(t)) \le 1$. The operator $\delta x(t)$ stands for $\dot{x}(t)$ in continuous-time or $x(t + 1)$ for discrete-time systems.

The following notation is employed for convex sums of scalar and/or matrices:

- for simple convex sums

$$a_{ij,z(t)} = \sum_{\mathbf{i}=1}^{r} \mu_{\mathbf{i}}(z(t))a_{ij,\mathbf{i}},$$

$$A_{z(t)} = \sum_{\mathbf{i}=1}^{r} \mu_{\mathbf{i}}(z(t))A_{\mathbf{i}}$$

- for double convex sums

$$a_{ij,z(t)z(t)} = \sum_{\mathbf{i}=1}^{r}\sum_{\mathbf{j}=1}^{r} \mu_{\mathbf{i}}(z(t))\mu_{\mathbf{j}}(z(t))a_{ij,\mathbf{ij}},$$

$$A_{z(t)z(t)} = \sum_{\mathbf{i}=1}^{r}\sum_{\mathbf{j}=1}^{r} \mu_{\mathbf{i}}(z(t))\mu_{\mathbf{j}}(z(t))A_{\mathbf{ij}}$$

- for backward/forward delays:

$$a_{ij,z(t\pm1)} = \sum_{\mathbf{k}=1}^{r} \mu_{\mathbf{k}}(z(t \pm 1))a_{ij,\mathbf{k}},$$

$$A_{z(t\pm1)} = \sum_{\mathbf{k}=1}^{r} \mu_{\mathbf{k}}(z(t \pm 1))A_{\mathbf{k}}.$$

The product of two conformable matrices $A \in \mathbb{R}^{m\times l}$ and $B \in \mathbb{R}^{l\times n}$ will be denoted as $AB = \left[\sum_{k=1}^{l} a_{ik}b_{kj}\right]_{i,j=1}^{m,n}$. Another way is by defining

$$A = [\boldsymbol{a}_1 \ \boldsymbol{a}_2 \ \cdots \ \boldsymbol{a}_l] , B = \begin{bmatrix} \boldsymbol{b}_1^T \\ \boldsymbol{b}_2^T \\ \vdots \\ \boldsymbol{b}_l^T \end{bmatrix},$$

where $\boldsymbol{a}_i \in \mathbb{R}^{m\times 1}$ is the i-th column of A and $\boldsymbol{b}_j^T \in \mathbb{R}^{1\times n}$ is the j-th row of B, then we have

$$AB = \sum_{k=1}^{l} \boldsymbol{a}_k\boldsymbol{b}_k^T, \quad \boldsymbol{a}_k\boldsymbol{b}_k^T \in \mathbb{R}^{m\times n}.$$

Additionally, an asterisk $(*)$ is employed in matrix expressions to denote the transpose of the symmetric element; for in-line expressions it will denote the transpose of the terms on its left side, that is, $A + B + A^T + B^T + C = A + B + (*) + C$.

2.1 Motivation

Complexity can be seen in different ways: The first problem comes from the model itself: for actual solvers, generally a large number of states, inputs and outputs may lead to numerical problems; moreover, using a classical sector nonlinearity approach (Ohtake et al., 2001), the number of vertices of a TS model is 2^p, p represents the number of different nonlinearities to take into account, then it rapidly increases. Thus, even for very simple problems such as H_∞ stabilization for a 10-order system with 2 inputs, 2 outputs, 6 nonlinearities, i.e., $2^6 = 64$ vertices, the problem very easily reaches the limits of LMI solvers.

A second problem comes from the Lyapunov function (especially in the discrete case) and the so-called multidimensional summation conditions. For example, in the discrete TS framework, a possibility is to use multiple summations for the Lyapunov functions and/or of the slack variables (Ding et al., 2006): $V(x) = x^T P_{[d]}x$, $P_{[d]} = \sum_{\mathbf{i_1}=1}^{r}\sum_{\mathbf{i_2}=1}^{r}\cdots\sum_{\mathbf{i_d}=1}^{r} \mu_{\mathbf{i_1}}\mu_{\mathbf{i_2}}\cdots\mu_{\mathbf{i_d}}P_{\mathbf{i_1i_2}\cdots\mathbf{i_d}} > 0$. Delayed Lyapunov functions also can come at hand (Lendek et al., 2015) introducing more possibilities and therefore, more complexity: $\mathcal{P} = \sum_{\mathbf{i_1}=1}^{r}\sum_{\mathbf{i_2}=1}^{r}\cdots\sum_{\mathbf{i_d}=1}^{r} \mu_{\mathbf{i_1}}(z(t))\mu_{\mathbf{i_2}}(z(t-1))\cdots\mu_{\mathbf{i_d}}z(t-l)P_{\mathbf{i_1i_2}\cdots\mathbf{i_d}} > 0$. Nevertheless, excepted for low order systems, let say 2 or 3 states, they are only doable for $d < 10$. As for multidimensional summation without changing the Lyapunov function, there are theoretical establishing that there is a parameter $d \in \mathbb{N}$ such that the greater d, the best the result is (ideally, in some cases when $d \to \infty$ the conditions are asymptotically necessary and sufficient) at the price of increasing the complexity. For example, asymptotically exact conditions can be derived for specific control problems (Scherer, 2005; Sala and Ariño, 2007) using polynomial roots properties (Polya, 1928): if $\sum_{\mathbf{i}=1}^{r}\sum_{\mathbf{j}=1}^{r} \mu_{\mathbf{i}}\mu_{\mathbf{j}}\Upsilon_{\mathbf{ij}} \ge \gamma > 0$, then there exists d such that $\left(\sum_{\mathbf{k}=1}^{r} \mu_{\mathbf{k}}\right)^d \sum_{\mathbf{i}=1}^{r}\sum_{\mathbf{j}=1}^{r} \mu_{\mathbf{i}}\mu_{\mathbf{j}}\Upsilon_{\mathbf{ij}} > 0$.

In view of these issues, the numerical capabilities for actual LMI solvers together with “real” applications (many states, several inputs and nonlinearities), results that many theoretical results are, nowadays, impossible to apply to these systems. Thus, there is a need for a step between TS modelling and coding LMI constraints. Several possibilities exist, model reduction, e.g., via singular value decomposition (Yam et al., 1999), transforming some nonlinearities into uncertainties to reduce the number of vertices with a reasonable complexity increase (Bouarar et al., 2010), descriptor forms (Estrada-Manzo et al., 2016), among others. The proposal is a simple optimization in terms convex summations, i.e., introducing the maximum of slack variables while keeping the number of LMI constraints constant; this can be applied to any LMI/TS problem already formulated in the literature. Indeed, it resumes to work on terms such as:

$$A_{z(t)}H_{z(t)} = \sum_{\mathbf{i}=1}^{r}\sum_{\mathbf{j}=1}^{r} \mu_{\mathbf{i}}(z(t))\mu_{\mathbf{j}}(z(t))A_{\mathbf{i}}H_{\mathbf{j}},$$

where one of the expression is a convex summation of known matrices $A_{\mathbf{i}}$ and the other is a convex combination of unknown matrices $H_{\mathbf{j}}$ (decision variables).

For instance, let us recall conditions for the stabilization of (1) in discrete-time given by Lendek et al. (2015), the control law under design is:

$$u(t) = F_{z(t)z(t-1)} H^{-1}_{z(t)z(t-1)} x(t), \tag{2}$$

where $F_{z(t)z(t-1)} \in \mathbb{R}^{m\times n}$ and $H_{z(t)z(t-1)} \in \mathbb{R}^{n\times n}$ are fuzzy gains to be designed by means of the following Lyapunov function candidate

$$V(x) = x^T(t) P_{z(t-1)} x(t), \tag{3}$$

with $P_{z(t-1)} = \sum_{\mathbf{k}=1}^{r} \mu_{\mathbf{k}}(z(t-1)) P_{\mathbf{k}}$, $P_{\mathbf{k}} > 0$. Via Finsler's Lemma the designing conditions are:

$$\begin{bmatrix} P_{z(t-1)} - H_{z(t)z(t-1)} - H^T_{z(t)z(t-1)} & (*) \\ A_{z(t)} H_{z(t)z(t-1)} + B_{z(t)} F_{z(t)z(t-1)} & -P_{z(t)} \end{bmatrix} < 0. \tag{4}$$

Now, let us consider $A_{z(t)} = \begin{bmatrix} a_{11,z(t)} & 0 & 1 \\ a_{21,z(t)} & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$, then we have $A_{z(t)}H$ is equal to

$$\begin{bmatrix} a_{11,z(t)}h_{11}+h_{31} & a_{11,z(t)}h_{12}+h_{32} & a_{11}h_{13}+h_{33} \\ a_{21,z(t)}h_{11}+h_{21} & a_{21,z(t)}h_{12}+h_{22} & a_{21,z(t)}h_{13}+h_{23} \\ 2h_{11}+h_{21} & 2h_{12}+h_{22} & 2h_{13}+h_{23} \end{bmatrix}.$$

From the above, we can choose an adequate convex structure for the gain matrix $H_{z(t)z(t-1)}$; on the one hand, if less convex sums in (4) with less decision variables is the goal, we can choose

$$H_{[z(t)]z(t-1)} = \begin{bmatrix} h_{11,z(t-1)} & h_{12,z(t-1)} & h_{13,z(t-1)} \\ h_{21,z(t)z(t-1)} & h_{22,z(t)z(t-1)} & h_{23,z(t)z(t-1)} \\ h_{31,z(t)z(t-1)} & h_{32,z(t)z(t-1)} & h_{33,z(t)z(t-1)} \end{bmatrix},$$

where the subscript $[z(t)]$ indicates that not all of the entries of the matrix contain convex structures related to $z(t)$. On the other hand, if the same number of convex sums in (4) but with more decision variables is the objective, then we have $H_{[z(t)]z(t)z(t-1)}$ equal to

$$\begin{bmatrix} h_{11,z(t)z(t-1)} & h_{12,z(t)z(t-1)} & h_{13,z(t)z(t-1)} \\ h_{21,z(t)z(t)z(t-1)} & h_{22,z(t)z(t)z(t-1)} & h_{23,z(t)z(t)z(t-1)} \\ h_{31,z(t)z(t)z(t-1)} & h_{32,z(t)z(t)z(t-1)} & h_{33,z(t)z(t)z(t-1)} \end{bmatrix}.$$

Similar procedure can be performed for $B_{z(t)}F_{z(t)z(t-1)}$. A methodology to apply these ideas is the matter of the following section.

In what follows, the well-known relaxation scheme from Tuan et al. (2001) will be employed:

Lemma 1. Tuan et al. (2001) Let $\Upsilon_{\mathbf{ijk}} = (\Upsilon_{\mathbf{ijk}})^T$, $(\mathbf{i},\mathbf{j},\mathbf{k}) \in \{1,2,\ldots,r\}^3$ be matrices of proper dimensions. Then

$$\Upsilon_{z(t)z(t)z(t-1)} = \sum_{\mathbf{i}=1}^{r}\sum_{\mathbf{j}=1}^{r}\sum_{\mathbf{k}=1}^{r} \mu_{\mathbf{i}}(z(t))\mu_{\mathbf{j}}(z(t))\mu_{\mathbf{k}}(z(t-1))\Upsilon_{\mathbf{ijk}} < 0$$

holds if the following LMIs hold too

$$\frac{2}{r-1}\Upsilon_{\mathbf{iik}} + \Upsilon_{\mathbf{ijk}} + \Upsilon_{\mathbf{jik}} < 0, \tag{5}$$

for all $(\mathbf{i},\mathbf{j},\mathbf{k}) \in \{1,2,\ldots,r\}^3$.

3. MAIN RESULTS

As motivated in previous section, we deal with terms of the form $A_{z(t)}H_{z(t)}$ for control problem or $G_{z(t)}A_{z(t)}$ for observer counterpart.

3.1 The controller case

Let us consider a double convex sum term

$$A_{z(t)}H_{z(t)} = \sum_{\mathbf{i}=1}^{r}\sum_{\mathbf{j}=1}^{r} \mu_{\mathbf{i}}(z(t))\mu_{\mathbf{j}}(z(t)) A_{\mathbf{i}} H_{\mathbf{j}}. \tag{6}$$

Matrices $A_{z(t)}$ and $H_{z(t)}$ can be viewed as

$$A_{z(t)} = \begin{bmatrix} \boldsymbol{a}_{1,z(t)} & \boldsymbol{a}_{2,z(t)} & \cdots & \boldsymbol{a}_{n,z(t)} \end{bmatrix},\ H_{z(t)} = \begin{bmatrix} \boldsymbol{h}^T_{1,z(t)} \\ \boldsymbol{h}^T_{2,z(t)} \\ \vdots \\ \boldsymbol{h}^T_{n,z(t)} \end{bmatrix}, \tag{7}$$

where $\boldsymbol{a}_{i,z(t)}$ is the i-th column of $A_{z(t)}$ and $\boldsymbol{h}^T_{j,z(t)}$ is the j-th row of $H_{z(t)}$; if one element of the column/row does not depend on nonlinear terms, then the subscript $z(t)$ is eliminated. The following propositions provide a way to change the convex structure of the matrix gain $H_{z(t)}$:

Proposition 1. (Less number of convex sums, less decision variables). The term (6) can be reduced to

$$A_{z(t)} H_{[z(t)]} = \sum_{k=1}^{n} \boldsymbol{a}_{k,[z(t)]} \boldsymbol{h}^T_{k,[z(t)]},$$

with

$$\boldsymbol{a}_{k,[z(t)]} \boldsymbol{h}^T_{k,[z(t)]} = \begin{cases} \boldsymbol{a}_{k,z(t)} \boldsymbol{h}^T_{k}, & \text{if} \quad \boldsymbol{a}_{k,z(t)} \\ \boldsymbol{a}_{k} \boldsymbol{h}^T_{k,z(t)}, & \text{if} \quad \boldsymbol{a}_{k} \end{cases}$$

Proposition 2. (Same number of convex sums, more decision variables). The term (6) can be expressed as

$$A_{z(t)} H_{[z(t)]z(t)} = \sum_{k=1}^{n} \boldsymbol{a}_{k,[z(t)]} \boldsymbol{h}^T_{k,[z(t)]z(t)},$$

with

$$\boldsymbol{a}_{k,[z(t)]} \boldsymbol{h}^T_{k,[z(t)]z(t)} = \begin{cases} \boldsymbol{a}_{k,z(t)} \boldsymbol{h}^T_{k}, & \text{if} \quad \boldsymbol{a}_{k,z(t)} \\ \boldsymbol{a}_{k} \boldsymbol{h}^T_{k,z(t)z(t)}, & \text{if} \quad \boldsymbol{a}_{k} \end{cases}$$

3.2 The observer case

Within the observer design of discrete-time system, one is face to terms of the form

$$G_{z(t)}A_{z(t)} = \sum_{\mathbf{i}=1}^{r}\sum_{\mathbf{j}=1}^{r} \mu_{\mathbf{i}}(z(t))\mu_{\mathbf{j}}(z(t)) G_{\mathbf{j}} A_{\mathbf{i}} \tag{8}$$

which consists in two sums. Similar to definition (7), now we have

$$G_{z(t)} = \begin{bmatrix} \boldsymbol{g}_{1,z(t)} & \boldsymbol{g}_{2,z(t)} & \cdots & \boldsymbol{g}_{n,z(t)} \end{bmatrix},\ A_{z(t)} = \begin{bmatrix} \boldsymbol{a}^T_{1,z(t)} \\ \boldsymbol{a}^T_{2,z(t)} \\ \vdots \\ \boldsymbol{a}^T_{n,z(t)} \end{bmatrix}, \tag{9}$$

then, the following propositions concerning a finner structure of $G_{z(t)}$:

Proposition 3. (Less number of convex sums, less decision variables). The term (8) can be reduced to

$$H_{[z(t)]}A_{z(t)} = \sum_{k=1}^{n} \boldsymbol{g}_{k,[z(t)]}\boldsymbol{a}_{k,[z(t)]}^T,$$

with

$$\boldsymbol{g}_{k,[z(t)]}\boldsymbol{a}_{k,[z(t)]}^T = \begin{cases} \boldsymbol{g}_k \boldsymbol{a}_{k,z(t)}^T, & \text{if} \quad \boldsymbol{a}_{k,z(t)}^T \\ \boldsymbol{g}_{k,z(t)}\boldsymbol{a}_k^T, & \text{if} \quad \boldsymbol{a}_k^T \end{cases}$$

Proposition 4. (Same number of convex sums, more decision variables). The term (8) can be expressed as

$$H_{[z(t)]}A_{z(t)} = \sum_{k=1}^{n} \boldsymbol{g}_{k,[z(t)]}\boldsymbol{a}_{k,[z(t)]}^T,$$

with

$$\boldsymbol{g}_{k,[z(t)]}\boldsymbol{a}_{k,[z(t)]}^T = \begin{cases} \boldsymbol{g}_k \boldsymbol{a}_{k,z(t)}^T, & \text{if} \quad \boldsymbol{a}_{k,z(t)}^T \\ \boldsymbol{g}_{k,z(t)}\boldsymbol{a}_k^T, & \text{if} \quad \boldsymbol{a}_k^T \end{cases}$$

Remark 1. For comparison purposes Lemma 1 has been chosen; nevertheless, the proposal is not focused on any particular sum-relaxation.

Remark 2. In (Xie et al., 2014) the numerical complexity of the LMI conditions is approximated by $\log_{10}(n_d^3 n_l)$, where n_d is the number of scalar decision variables and n_l is the number of LMI rows. Through the examples, this formula will be employed to highlight the advantages of the proposal.

Remark 3. The applicability of propositions 1 and 3 is that of reducing the number of decision variables, this is specially important when dealing with systems with large number of nonlinearities or vertex models, and thus avoiding numerical problems. On the other hand, propositions 2 and 4 increase of decision variables while keeping the same number of LMIs, this may lead to a reduction of conservativeness at the price of augmenting the numerical complexity. Any of these propositions can be applied to a co-negativity problem with two or more convex sums.

4. EXAMPLES

For the following examples, LMI conditions have been solved in SeDuMi (Sturm, 1999) within YALMIP (Lofberg, 2004)for and MATLAB2015a, the tests have been run in a computer with Intel Xeon 3.40GHz, with 16 GB in RAM.

Example 1. Let us consider a numerical example borrowed from (Lendek et al., 2015, Example 6), that is a discrete-time TS system (1) with vertex models:

$$A_1 = \begin{bmatrix} 1 & -\beta \\ -1 & -0.5 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 5+\beta \\ 2\beta \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & \beta \\ -1 & -0.5 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 5-\beta \\ -2\beta \end{bmatrix},$$

where $\beta > 0$ is a real-valued parameter. Note that we have

$$A_{z(t)} = \begin{bmatrix} 1 & a_{12,z(t)} \\ -1 & -0.5 \end{bmatrix} \text{ and } B_{z(t)} = \begin{bmatrix} b_{11,z(t)} \\ b_{21,z(t)} \end{bmatrix};$$

thus following Proposition 1, the structure of the convex gain matrices is,

$$H_{[z(t)]z(t-1)} = \begin{bmatrix} h_{11,z(t)z(t-1)} & h_{12,z(t)z(t-1)} \\ h_{21,z(t-1)} & h_{22,z(t-1)} \end{bmatrix} \text{ and } F_{z(t-1)}.$$

So we have conditions

$$\begin{bmatrix} P_{z(t-1)} - H_{[z(t)]z(t-1)} - H_{[z(t)]z(t-1)}^T & (*) \\ A_{z(t)}H_{[z(t)]z(t-1)} + B_{z(t)}F_{z(t-1)} & -P_{z(t)} \end{bmatrix} < 0. \quad (10)$$

For Proposition 2, we have matrices

$$H_{[z(t)]z(t)z(t-1)} = \begin{bmatrix} h_{11,z(t)z(t)z(t-1)} & h_{12,z(t)z(t)z(t-1)} \\ h_{21,z(t)z(t-1)} & h_{22,z(t)z(t-1)} \end{bmatrix}$$

and $F_{z(t)z(t-1)}$. Seeking feasibility for a larger $\beta > 0$, conditions (4) under the relaxation lemma 1 yield feasible up to $\beta = 1.768$, LMI conditinos from (10) (Proposition 1) are feasible up to $\beta = 1.191$; while LMI conditions related to Proposition 2 are feasible up to $\beta = 1.779$. Notice that conditions in (4) consist on 8 LMIs, while the ones associated to propositions 1 and 2 have 4 and 8 LMIs, respectively. Following Remark 2, the numerical complexity is summarized in Table 1; as mentioned in Remark 3 the larger β is achieved with a larger numerical complexity.

Comparison in terms of numerical complexity Example 1

Approach	n_l	n_d	$\log_{10}(n_d^3 n_l)$	β
LMIs from (4)	32	30	5.93	1.768
LMIs from Prop. 1	16	22	5.23	1.191
LMIs from Prop. 2	32	38	6.24	1.779

Example 2. Let us consider a discrete-time system (1) with vertex models:

$$A_1 = \begin{bmatrix} 1.2 & \alpha & 1 & 0.2 \\ 0.5 & 0.1 & 0 & 1 \\ 0 & 0 & -0.5 & 0 \\ 0 & 1 & \beta & -0.3 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 2 \\ 0.5 \\ 1.5 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1.2 & -\alpha & 1 & 0.2 \\ 0.5 & 0.1 & 0 & 1 \\ 0 & 0 & -0.5 & 0 \\ 0 & 1 & -\beta & -0.3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 \\ 3 \\ 0.5 \\ 0.5 \end{bmatrix},$$

where $\alpha = 1$ and $\beta = 1$. For this TS system, the approach used in (Lendek et al., 2015) as well as Propositions 1 and 2 are found feasible; nevertheless we can analyze the time in which each one of them is executed when the number of vertex models artificially increases. To this end, as suggested in (Delmotte et al., 2007), if vertices inside the polytope are equally spaced: $(A_{\delta_k}, B_{\delta_k})$, with $\delta_k = k/(r-1)$, $k \in \{1, 2, \ldots, r-2\}$, they correspond to

$$A_{\delta_k} = (1-\delta_k)A_1 + \delta_k A_2 \quad B_{\delta_k} = (1-\delta_k)B_1 + \delta_k B_2;$$

and then the result remains the same: the TS system is still stabilizable. With this in mind, Table 1 summarizes the results for $\alpha = \beta = 1$ for different number of vertices; it can be seen that Proposition 1 renders a faster solution than the others (the time is obtained with the command `solvertime` from YALMIP; moreover, due to numerical problems (np), both conditions from (Lendek et al., 2015) and Proposition 2 cannot be found feasible for $r \geq 30$. Thus, simple controller gains may lead to feasible problems, especially when the control problem is large in terms of number of states and number of vertex models, see Remark 3.

Table 1. Solver time for different approaches

Approach	Solver time (sec)				
	$r=2$	$r=15$	$r=25$	$r=30$	$r=95$
LMIs from (4)	0.20	15.01	78.18	np	np
LMIs from Prop. 1	0.19	0.94	2.61	4.02	45.50
LMIs from Prop. 2	0.22	16.92	107.82	np	np

Example 3. Consider a discrete-time TS system (1) whose vertex are:

$$A_1 = A_3 = \begin{bmatrix} -0.2-\alpha & 0 \\ -0.5 & 1.8 \end{bmatrix}, A_2 = A_4 = \begin{bmatrix} 2.2 & 0 \\ -0.5 & 1.8 \end{bmatrix},$$
$$C_1 = C_2 \begin{bmatrix} 0 \\ 1.2-\beta \end{bmatrix}^T, C_3 = C_4 \begin{bmatrix} 0 \\ 1.2+\beta \end{bmatrix}^T,$$

where $\alpha \in [0,\ 1.5]$ and $\beta \in [-1.2,\ 1.2]$ are real-valued parameters. The task is to compare feasibility regions for observer design by conditions in (Guerra et al., 2012), that is,

$$\begin{bmatrix} -P_{z(t-1)} & (*) \\ G_{z(t)z(t-1)}A_{z(t)} - L_{z(t)z(t-1)}C_{z(t)} & -G_{z(t)z(t-1)} + (*) + P_{z(t)} \end{bmatrix} < 0,$$

and the ones from Proposition 4 with matrices having the structure

$$G_{[z(t)]z(t)z(t-1)} = \begin{bmatrix} g_{11,z(t)z(t-1)} & g_{12,z(t)z(t)z(t-1)} \\ g_{21,z(t)z(t-1)} & g_{22,z(t)z(t)z(t-1)} \end{bmatrix}$$

and $L_{z(t)z(t-1)}$. LMI conditions from both approaches are obtained by means of the relaxation lemma 1. In figure 1 the feasible set solution is shown, it is clear that while both approaches require 64 LMIs, Proposition 4 yields a larger solution set. Regarding numerical complexity for (Guerra et al., 2012) is 8.55 (with $n_d = 112$ and $n_l = 256$) while for Proposition 4 is 9.36 (with $n_d = 208$ and $n_l = 256$).

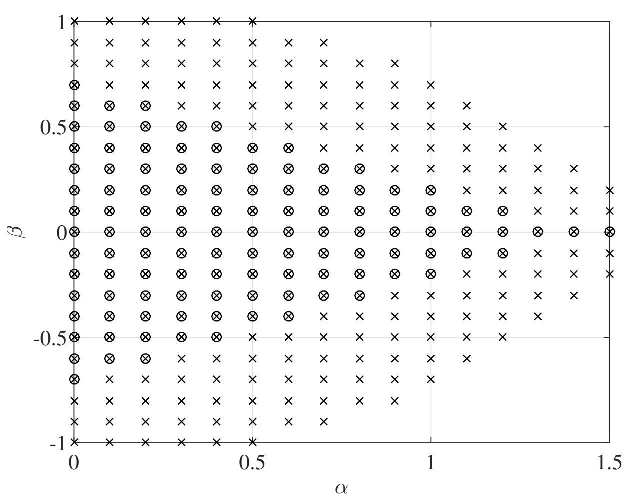

Fig. 1. Feasibility solution set for conditions in (Guerra et al., 2012) marked with ∘ and Proposition 4 marked with ×.

5. CONCLUSIONS

It has been presented a methodology to optimize the convex structure of slack matrices and/or gains in TS control systems. The approach allows reducing conservativeness and therefor feasible solutions can be found whereas traditional convex structures fail. Numerical examples have been used to illustrate how the proposal performs when compare with others. Future research work intends to generalized the given results.

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