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Local Sampled-Data Gain-Scheduling Control of quasi-LPV Systems ⋆

Pedro H. S. Coutinho * **Márcia L. C. Peixoto** *
Miguel Bernal ** **Anh-Tu Nguyen** ***
Reinaldo M. Palhares ****

* *Federal University of Minas Gerais, Graduate Program in Electrical Engineering, Brazil (e-mails: {phcoutinho, marciapeixoto}@ufmg.br)*
** *Department of Electrical and Electronics Engineering, Sonora Institute of Technology, Mexico (e-mail: miguel.bernal@itson.edu.mx)*
*** *Université Polytechnique Hauts-de-France, LAMIH UMR CNRS 8201, France (e-mail: nguyen.trananhtu@gmail.com)*
**** *Federal University of Minas Gerais, Department of Electronics Engineering, Brazil (e-mail: rpalhares@ufmg.br)*

Abstract: This paper deals with the problem of sampled-data gain-scheduling control design for affine quasi-LPV systems. As the control implementation is based on sample-data and its update occurs only at specific sampling instants, the state-dependent scheduling functions of the controller are piecewise continuous. This characteristic causes a mismatch between the system model and the controller parameters, namely asynchronous scheduling functions of the controller with respect to the plant during the inter-sampling behavior. To cope with this phenomenon, a polytopic description of the inter-sampling is constructed based on bounding assumptions on the scheduling functions and their time derivatives. Then, regarding a time-delay approach, a constructive and numerically implementable LMI-based synthesis condition is derived, and a convex procedure is proposed for enlarging the estimate of the region of attraction of the origin of the closed-loop system. The time-derivative bounds of the scheduling functions are explicitly accounted into the local analysis to ensure the method's implementation. A numerical example is provided to illustrate the proposed methodology.

Keywords: Digital control, Nonlinear systems, Gain scheduling, Convex optimization, Linear matrix inequalities.

1. INTRODUCTION

Sampled-data control systems are characterized by the presence of sampling mechanisms connecting the plant (physical system) to the controller, as in embedded and networked control systems data is transmitted over digital communication channels. The existing methods to tackle digital control problems generally consider a discrete-time counterpart of both the plant and the controller (Jungers et al., 2017), but deriving exact discrete models for nonlinear systems is still a challenging problem. For this reason, approximate discretization of the plant (Coutinho et al., 2020), or sampled-data models describing inter-sampling behavior (Nešić and Postoyan, 2015) are often employed. In the particular context of sampled-data models, aperiodic sampling has been considered as a modeling abstraction to describe sampling jitters, packet dropouts, or fluctuations caused by real-time scheduling protocols (Hetel et al., 2017), thus providing more appropriate results than standard fixed sampling. The main approaches to tackle the problem of control under aperiodic sampling are based on the time-delay framework (Fridman, 2010; Seuret and Gouaisbaut, 2013) and hybrid systems theory (Nesic et al., 2009); see (Hetel et al., 2017) for a recent survey on the topic.

The time-delay framework has shown to be useful to derive constructive analysis and synthesis conditions in the form of linear matrix inequalities (LMIs) for linear systems (Fridman, 2010; Seuret and Gouaisbaut, 2013), Lur'e-type systems (Shang-Guan et al., 2017), nonlinear systems under differential algebraic representations (Moreira et al., 2019), and, based on gain-scheduling control techniques, for Takagi-Sugeno (T-S) (Lopes et al., 2020) and quasi-linear parameter-varying (quasi-LPV) models (Palmeira et al., 2020). In the latter case, gain-scheduling controllers are parameterized in terms of state-dependent scheduling functions, or premise variables of T-S models. However, as in digital control implementations the state is available only at specific sampling instants, the parameters of the controller often differ from those of the T-S or quasi-LPV models of the nonlinear plant, leading to the so-called asynchronous parameters. However, as pointed out by Palmeira et al. (2020), most works deal with the asynchronous phenomenon by assuming the existence of bounds for the parameters without providing formal guarantees of the existence of these bounds during operation.

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The lack of such guarantees has motivated efforts to derive local stability and stabilization conditions for sampled-data quasi-LPV or T-S fuzzy systems (Palmeira et al., 2018; Lopes et al., 2020; Palmeira et al., 2020). The importance of local analysis is the characterization of a region in the state-space where the state-dependent parameters satisfy bounding assumptions. Palmeira et al. (2018) proposed local stability analysis conditions for sampled-data quasi-LPV systems and the results have been recently extended to the synthesis problem (Palmeira et al., 2020). However, as this representation is based on prescribed bounds for the time derivative of state-dependent scheduling functions, it is difficult to employ such a relaxation when time derivatives of scheduling functions depend on the control input signal. To deal with this case, Palmeira et al. (2020) proposed a particular result that does not take the auxiliary polytope into account. However, it introduces conservativeness since the gain-scheduling control structure is possibly reduced to a linear one.

This work intends to provide a less conservative condition for stabilization of sampled-data quasi-LPV control systems. More precisely, the synthesis condition is derived based on the time-delay approach using the Wirtinger-based integral inequality and a methodology is provided to deal with the case of time derivatives of scheduling functions depending on the control input signal. For the latter, the region in the state space where the prescribed time-derivative bounds are satisfied is fully accounted in the synthesis condition, such that estimates of the region of attraction are obtained under this extra constraint. As a result, the maximum allowable sampling period (MASP) can be enlarged by adjusting the prescribed time-derivative bounds. Moreover, a trade-off between the time-derivative bounds and the estimate of the region of attraction can be established.

This paper is organized as follows. In Section 2, the sampled-data control problem for the considered class of nonlinear systems is presented. The equivalent local polytopic description is revisited in Section 3. The proposed condition derived regarding Lyapunov-Krasovskii and looped-functional arguments is provided in Section 4. A numerical example illustrating the advantages of the proposal is provided in Section 5 while conclusions and future research directions are discussed in Section 6.

Notation: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{N}_{\leq p}$ denotes the set of natural numbers less than or equal to $p \in \mathbb{N}$, $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\mathbb{B} = \{0, 1\}$ denotes the Boolean domain, $\mathbb{R}$ denotes the field of real numbers, and $\mathbb{R}_{\geq 0}$ ($\mathbb{R}_{>0}$) denotes the set of all non-negative (positive) real numbers. In a symmetric matrix, the symbol '$\star$' denotes the transpose of the symmetric term and $\text{He}\{X\} = X + X^\top$. $\text{diag}(X_1, \ldots, X_n)$ denotes the block diagonal matrix of matrices $X_1, \ldots, X_n$. Given a multi-index $\mathbf{i} = (i_1, \ldots, i_p) \in \mathbb{B}^p$, where $\mathbb{B}^p = \{\mathbf{i} : i_j \in \mathbb{B}, j \in \mathbb{N}_{\leq p}\}$, it is defined $\mathbb{B}^{p+} = \{\mathbf{i} : i_j \leq i_{j+1}, i_j \in \mathbb{B}, j \in \mathbb{N}_{\leq p-1}\}$. $\mathcal{P}(\mathbf{i})$ is the set of permutations of the entries of $\mathbf{i}$. $\mathbb{K}$ is the set of differentiable functions from $[0, T]$ to $\mathbb{R}^n$, with $T \in \mathbb{R}_{>0}$.

2. PROBLEM FORMULATION

Consider nonlinear systems of the form

$$\dot{x}(t) = A(x)x(t) + B(x)u(t) \tag{1a}$$
$$u(t) = K(x_k)x(t_k), \quad \forall t \in [t_k, t_{k+1}) \tag{1b}$$
$$x(t_k) = x_k, \ \forall k \in \mathbb{N}_0, \quad x(0) = x_0 \text{ given,}$$

where $x(t) \in D \subset \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ is the control input; $A : D \mapsto \mathbb{R}^{n\times n}$ and $B : D \mapsto \mathbb{R}^{n\times m}$ are matrix-valued functions affine in continuously differentiable functions $z_j : D \mapsto \mathbb{R}$, for $j \in \mathbb{N}_{\leq p}$, that is

$$A(x) = A_0 + \textstyle\sum_{j=1}^{p} z_j(x)A_j, \ B(x) = B_0 + \sum_{j=1}^{p} z_j(x)B_j,$$
$$K(x_k) = K_0 + \textstyle\sum_{j=1}^{p} z_j(x_k)K_j,$$

with $A_j \in \mathbb{R}^{n\times n}$, $B_j \in \mathbb{R}^{n\times m}$, $K_j \in \mathbb{R}^{m\times n}$, $j \in \{0\} \cup \mathbb{N}_{\leq p}$. The region D is a convex polytope containing the origin that admits the following half-space representation:

$$D = \{x \in \mathbb{R}^n : b_j^\top x \leq 1, j \in \mathbb{N}_{\leq n_e}\}, \tag{2}$$

where $b_j \in \mathbb{R}^n$, $j \in \mathbb{N}_{\leq n_e}$ define the edges of the polytope. As D is compact and $z_j(x)$ are continuously differentiable functions, then there exist bounds such that

$$z_j^0 \leq z_j(x) \leq z_j^1, \tag{3a}$$
$$-\phi_j \leq \dot{z}_j(x) \leq \phi_j, \tag{3b}$$

for all $x \in D$, $j \in \mathbb{N}_{\leq p}$, and the Jacobian matrix

$$J(x) = ((\nabla z_1(x))^\top, \ldots, (\nabla z_p(x))^\top),$$

is bounded for all $x \in D$.

Based on a digital control implementation, the state $x(t)$ and scheduling functions $z(x(t))$ are transmitted to the controller only at specific sampling instants $t_k \in \mathbb{R}_{\geq 0}$ satisfying to $t_0 = 0$, $t_{k+1} - t_k > 0$, $\lim_{k\to\infty} t_k = \infty$. Then, regarding a zero-order hold mechanism, piecewise-constant signals of $x(t_k)$ and $z(x(t_k))$, $t \in [t_k, t_{k+1})$ are available to the controller. Hence, the following arbitrary sampling problem is addressed in this work.

Problem: Consider the nonlinear sampled-data control system (1)–(3) and a bounded set $\mathcal{T} = (0, \overline{h}]$. Determine the gain-scheduling control law (1b) such that the origin of the closed-loop system (1)–(3) is locally asymptotically stable for any arbitrary time-varying sampling interval $h_k := t_{k+1} - t_k$ with values in $\mathcal{T}$.

The time-delay approach is considered to address this problem, which consists in rewriting the control input as

$$u(t) = K(x(t - \tau(t))x(t - \tau(t)) \tag{4a}$$
$$\tau(t) = t - t_k, \quad \forall t \in [t_k, t_{k+1}), \tag{4b}$$

where $\tau(t)$ is piecewise-linear satisfying to $\dot{\tau}(t) = 1$ for all $t \neq t_k$, $\tau(t_k) = 0$, and $\tau(t) \in [0, \overline{h}]$. Thus, the arbitrary sampling problem can be addressed by designing the gain-scheduled controller such that the origin of the closed-loop system

$$\dot{x}(t) = A(x)x(t) + B(x)K(x_k)x(t - \tau(t)), \tag{5}$$

for all $t \geq 0$, is asymptotically stable for all samplings sequences with $h_k \leq \overline{h}$, $k \in \mathbb{N}$. In particular, it is of interest to determine the region of attraction of the origin of the closed-loop system (5), but analytically obtaining that region may not be an easy task (Palmeira et al., 2018; Coutinho et al., 2020). For this reason, to fully address the control problem, we intend to determine the largest possible estimate of the region of attraction $\Omega \subset D$.

3. POLYTOPIC MODELING

This section presents the polytopic modeling of the nonlinear sampled-data system (1a). Due to the aperiodic sampling, the scheduling functions in the plant (1a) and the controller (1b) are asynchronous: in the former case they evolve continuously, in the latter they evolve in a sample-and-hold fashion, being piecewise-continuous signals. To deal with this asynchronous phenomenon, consider the following decomposition (Palmeira et al., 2020):

$$z(x) = z(x_k) + \delta(t), \quad \forall t \in [t_k, t_{k+1}),$$

where $\delta(t)$ denotes the inter-sampling variation of $z(x)$ from $t = t_k$ to $t \in (t_k, t_{k+1})$. It follows from (3) that

$$z_j^0 \leq z_j(x_k) \leq z_j^1, \tag{6a}$$

$$-\overline{h}\phi_j \leq \delta_j(t) \leq \overline{h}\phi_j, \tag{6b}$$

for all $x(t), x(t_k) \in D$, $t \in [t_k, t_{k+1})$. By employing the sector-nonlinearity approach (Tanaka and Wang, 2004), from the bounds defined in (6), each scheduling function can be written as $z_j(x_k) = z_j^0 w_0^j(x_k) + z_j^1 w_1^j(x_k)$, where $w_0^j(x_k) = (z_j^1 - z_j(x_k))/(z_j^1 - z_j^0)$, $w_1^j(x_k) = 1 - w_0^j(x_k)$, and their related variations by $\delta_j(t) = -\overline{h}\phi_j v_0^j(\delta) + \overline{h}\phi_j v_1^j(\delta)$, where $v_0^j(\delta) = (\overline{h}\phi_j - \delta_j(t))/(2\overline{h}\phi_j)$, $v_1^j(\delta) = 1 - v_0^j(\delta)$. Then, the vectors $z(x_k) = (z_1(x_k), \ldots, z_p(x_k))$ and $\delta(t) = (\delta_1(t), \ldots, \delta_p(t))$ belong to convex polytopes with 2^p vertices in $\mathbb{R}^p$, that is

$$z(x_k) = \sum_{\mathbf{i} \in \mathbb{B}^p} \mathrm{w}_{\mathbf{i}}(x_k)\alpha_{\mathbf{i}}, \; \delta(t) = \sum_{\mathbf{j} \in \mathbb{B}^p} \mathrm{v}_{\mathbf{j}}(\delta)\beta_{\mathbf{j}},$$

where $\alpha_{\mathbf{i}} = (z_1^{i_1}, \ldots, z_p^{i_p})$ and $\beta_{\mathbf{j}} = (\delta_1^{j_1}, \ldots, \delta_p^{j_p})$ and $\sum_{\mathbf{i} \in \mathbb{B}^p} \mathrm{w}_{\mathbf{i}}(x_k) = 1$, $\mathrm{w}_{\mathbf{i}}(x_k) \geq 0$, $\sum_{\mathbf{j} \in \mathbb{B}^p} \mathrm{v}_{\mathbf{j}}(\delta) = 1$, $\mathrm{v}_{\mathbf{j}}(\delta) \geq 0$. As a result, the matrix-valued functions in (1) can be represented as follows:

$$\begin{aligned} \left[A(x) \; B(x)\right] &= \sum_{\mathbf{i} \in \mathbb{B}^p} \sum_{\mathbf{k} \in \mathbb{B}^p} \mathrm{w}_{\mathbf{i}}(x_k)\mathrm{v}_{\mathbf{k}}(\delta) \left[\mathbf{A_{ik}} \; \mathbf{B_{ik}}\right] \\ K(x_k) &= \sum_{\mathbf{j} \in \mathbb{B}^p} \mathrm{w}_{\mathbf{j}}(x_k)\mathbf{K_j}, \end{aligned} \tag{7}$$

where

$$\begin{aligned} \mathbf{A_{ik}} &= A_0 + \mathcal{A}\left(\alpha_{\mathbf{i}} \otimes I + \beta_{\mathbf{k}} \otimes I\right), & \mathcal{A} &= [A_1 \, A_2 \cdots A_p] \\ \mathbf{B_{ik}} &= B_0 + \mathcal{B}\left(\alpha_{\mathbf{i}} \otimes I + \beta_{\mathbf{k}} \otimes I\right), & \mathcal{B} &= [B_1 \, B_2 \cdots B_p] \\ \mathbf{K_j} &= K_0 + \mathcal{K}\left(\alpha_{\mathbf{j}} \otimes I\right), & \mathcal{K} &= [K_1 \, K_2 \cdots K_p]. \end{aligned}$$

This representation provides an effective model for inter-sampling variations that leads to asynchronous phenomenon. As this polytopic model is valid only for state trajectories inside of D, an important aspect is obtaining an estimate of the region of attraction $\Omega \subset D$ ensuring the implementation of the designed gain-scheduled controller.

Notice also that this polytopic modeling closely depends on the existence of bounded scheduling function time derivatives, as in (6). Thus, the subset of the state-space in which these bounds are valid, denoted as Δ, should also be accounted in the region of attraction estimation, such that $\Omega \subset D \cap \Delta$. However, it can be hard to determine such a region Δ in the general case when scheduling functions depend on the control input, as can be noticed with

$$\dot{z}_j(x) = \nabla z_j(x)(A(x)x(t) + B(x)u(t)), \tag{8}$$

when $\nabla z_j(x)B(x) \neq 0$, $x \in D$. In principle, unless extra constraints are introduced into the synthesis condition to ensure that the control input is bounded (González et al., 2016), in this case it is not possible to *a priori* establish the bounds in (6) and more conservative results are expected to be obtained, as stated in (Palmeira et al., 2020, Corollary 1). To conclude, the sector-nonlinearity approach is employed to obtain the following polytopic representation for the Jacobian matrix

$$J(x) = \sum_{\mathbf{k} \in \mathbb{B}^\rho} \mathrm{g}_{\mathbf{k}}(x)\gamma_{\mathbf{k}}, \quad \forall x \in D, \tag{9}$$

where $\gamma_{\mathbf{k}} \in \mathbb{R}^{p \times n}$, $\mathbf{k} \in \mathbb{B}^\rho$, being ρ the number of nonlinear terms in $J(x)$. Hereafter the ℓ-th row of the matrix $\gamma_{\mathbf{k}}$ is denoted as $\gamma_{\mathbf{k}(\ell)}$.

4. MAIN RESULTS

4.1 Preliminary Results

The following condition is considered to study the local stability of the origin of the closed-loop system (5).

Lemma 1. Let $V : D \mapsto \mathbb{R}_{\geq 0}$ be a continuously differentiable and radially unbounded function such that

$$V(0) = 0 \quad \text{and} \quad V(x) > 0, \; \forall x \in D \setminus \{0\},$$

and $\mathcal{V}_0 : [0, \overline{h}] \times \mathbb{K} \mapsto \mathbb{R}_{\geq 0}$ be a functional such that

$$\begin{aligned} &\mathcal{V}_0(\tau, \phi) > 0, \; \forall \tau \in (0, h_k), \; \forall h_k \in [\underline{h}, \overline{h}], \\ &\mathcal{V}_0(0, \phi) = \mathcal{V}_0(h_k, \phi) = 0, \quad \forall \phi \in \mathbb{K}, \end{aligned}$$

where $\underline{h} < \overline{h}$, the level set associated with the function V be given by

$$\Omega = \{x \in \mathbb{R}^n : V(x) \leq c, \; c \in \mathbb{R}_{>0}\}, \tag{10}$$

and the region

$$\Delta = \{x \in \mathbb{R}^n : \dot{z}_j(x) \in [-\phi_j, \phi_j]\}. \tag{11}$$

If the following condition hold along the trajectories of (5):

$$\dot{\mathcal{W}}(\tau, x) = \frac{\mathrm{d}}{\mathrm{d}t}[V(x) + \mathcal{V}_0(\tau, x)] < 0, \tag{12}$$

then the origin of (5) is asymptotically stable. In addition, selecting c such that $\Omega \subset D \cap \Delta$, trajectories initiating at Ω never leave that region and converge asymptotically to the origin, thus the bounds (3) are satisfied.

Proof. The result can be proved following similar arguments as (Palmeira et al., 2020, Thm. 1).

4.2 Constructive Design Condition

Consider the following Lyapunov-Krasovskii functional candidate:

$$\mathcal{W}(\tau, x) = V(x) + \mathcal{V}_0(\tau, x), \tag{13}$$

with $V(x) = x(t)^\top P x(t)$,

$$\begin{aligned} \mathcal{V}_0(\tau, x) = &\left(h_k - \tau(t)\right)\left(x(t) - x(t_k)\right)^\top S\left(x(t) - x(t_k)\right) \\ &+ 2\left(h_k - \tau(t)\right)\left(x(t) - x(t_k)\right)^\top Q x(t_k) \\ &+ \left(h_k - \tau(t)\right)\int_{t_k}^{t} \dot{x}^\top(s) R \dot{x}(s) \, \mathrm{d}s \\ &+ \left(h_k - \tau(t)\right)(t - t_k)\tau(t) x^\top(t_k) U x(t_k), \end{aligned}$$

where $P, R \in \mathbb{R}^{n \times n}$ are symmetric and positive definite matrices and $S, Q, U \in \mathbb{R}^{n \times n}$ are symmetric matrices. It can be seen that $V(x)$ and $\mathcal{V}_0(\tau, x)$ satisfy the required properties stated in Theorem 1. More specifically, when $\tau(t) = t - t_k = 0$, that is, $t = t_k$, or when $\tau(t) = t - t_k = h_k$, that is, $t = t_{k+1}$, one has $\mathcal{V}_0(0, x) = \mathcal{V}_0(h_k, x) = 0$. Based on this functional candidate, the following result is proposed.

Theorem 2. Let $\underline{h}, \overline{h}, \epsilon \in \mathbb{R}_{>0}$, with $\underline{h} \leq \overline{h}$, and $\phi_\ell \in \mathbb{R}_{\geq 0}$, $\ell \in \mathbb{N}_{\leq p}$, be given. If there exist matrices $\widetilde{P} > 0$, $\widetilde{R} > 0$, $\widetilde{S} = \widetilde{S}^\top$, $\widetilde{Q} = \widetilde{Q}^\top$, $\widetilde{U} > 0$ and X, all belonging to $\mathbb{R}^{n\times n}$, matrices $\widetilde{K}_j \in \mathbb{R}^{m\times n}$, $j \in \{0\} \cup \mathbb{N}_{\leq p}$, and matrices $\widetilde{Y}_1$ and $\widetilde{Y}_2$ belonging to $\mathbb{R}^{n\times 4n}$ such that inequalities (14)–(18) are satisfied, then the sampled-data system (1), with control gains given by $K_j = \widetilde{K}_j X^{-1}$, $j \in \{0\} \cup \mathbb{N}_{\leq p}$, is asymptotically stable for all sampling sequences $\{h_k\}_{k\in\mathbb{N}_0}$ satisfying $h_k \in [\underline{h}, \overline{h}]$. Moreover, for any initial condition $x(0) \in \Omega$, with $\Omega \subset D \cap \Delta$ as defined in (10), (2), (11) with $c = 1$, the bounds in (3) are satisfied.

$$\begin{bmatrix} \widetilde{Q} & \widetilde{S}^\top - \widetilde{Q} \\ \star & \widetilde{Q} - \widetilde{S} - \widetilde{S}^\top \end{bmatrix} \geq 0, \quad \widetilde{U} \geq 0, \tag{14}$$

$$\sum_{(\mathbf{i},\mathbf{j})\in\mathcal{P}(\mathbf{m},\mathbf{n})} \Theta^1_{\mathbf{ijk}}(h_k) < 0, \tag{15}$$

$$\sum_{(\mathbf{i},\mathbf{j})\in\mathcal{P}(\mathbf{m},\mathbf{n})} \Theta^2_{\mathbf{ijk}}(h_k) < 0, \tag{16}$$

for $\mathbf{m}, \mathbf{n} \in \mathbb{B}^{p+}$, $\mathbf{k} \in \mathbb{B}^p$, $h_k \in \{\underline{h}, \overline{h}\}$,

$$\begin{bmatrix} 1 & b_j^\top X \\ X^\top b_j & \widetilde{P} \end{bmatrix} \geq 0, \tag{17}$$

for $j \in \mathbb{N}_{\leq n_e}$,

$$\sum_{(\mathbf{i},\mathbf{j})\in\mathcal{P}(\mathbf{m},\mathbf{n})} \Theta^3_{\mathbf{ijk},\ell} \leq 0, \tag{18}$$

for $\mathbf{m}, \mathbf{n} \in \mathbb{B}^{p+}$, $\mathbf{k} \in \mathbb{B}^\rho$, $\ell \in \mathbb{N}_{\leq p}$, where

$$\Theta^1_{\mathbf{ijk}}(h) = \widetilde{\Pi}_{1,\mathbf{ijk}} + h(\widetilde{\Pi}_2 + \widetilde{\Pi}_3)$$
$$\Theta^2_{\mathbf{ijk}}(h) = \begin{bmatrix} \widetilde{\Pi}_{1,\mathbf{ijk}} - h\widetilde{\Pi}_3 & h\widetilde{Y}_1^\top & 3h\widetilde{Y}_2^\top \\ \star & -h\widetilde{R} & 0 \\ \star & \star & -3h\widetilde{R} \end{bmatrix}$$
$$\Theta^3_{\mathbf{ijk},\ell} = \begin{bmatrix} -\widetilde{P} & \star \\ \gamma_{\mathbf{k}(\ell)}(\widetilde{\mathbf{A}}_\mathbf{i} X + \widetilde{\mathbf{B}}_\mathbf{i}\widetilde{\mathbf{K}}_\mathbf{j}) & -\phi_\ell^2 \end{bmatrix}$$
$$\widetilde{\Pi}_{1,\mathbf{ijk}} = \widetilde{\Pi}_1^0 + \widetilde{\Pi}^1_{1,\mathbf{ik}} + \widetilde{\Pi}^2_{1,\mathbf{ijk}} - \mathrm{He}\{\widetilde{Y}_1^\top W_1 + 3\widetilde{Y}_2^\top W_2\}$$
$$\widetilde{\Pi}_1^0 = \mathrm{He}\{M_1^\top \widetilde{P} M_4 - W_1^\top \widetilde{Q} M_2 - (M1 + \epsilon(M_2^\top + M_4^\top))XM_4\}$$
$$\widetilde{\Pi}^1_{1,\mathbf{ik}} = \mathrm{He}\{(M_1^\top + \epsilon M_2^\top + \epsilon M_4^\top)\mathbf{A}_{\mathbf{ik}} X M_1\}$$
$$\widetilde{\Pi}^2_{1,\mathbf{ijk}} = \mathrm{He}\{(M_1^\top + \epsilon M_2^\top + \epsilon M_4^\top)\mathbf{B}_{\mathbf{ik}}\widetilde{\mathbf{K}}_\mathbf{j} M_2\}$$
$$\widetilde{\Pi}_2 = M_4^\top \widetilde{R} M_4 + \mathrm{He}\{M_4^\top \widetilde{S} W_1 + M_4^\top \widetilde{Q} M_2\}$$
$$\widetilde{\Pi}_3 = M_2^\top \widetilde{U} M_2, \; \widetilde{\mathbf{A}}_\mathbf{i} = A_0 + \mathcal{A}(\alpha_\mathbf{i} \otimes I), \; \widetilde{\mathbf{B}}_\mathbf{i} = B_0 + \mathcal{B}(\alpha_\mathbf{i} \otimes I)$$
$$\widetilde{\mathbf{K}}_\mathbf{j} = \widetilde{K}_0 + \widetilde{\mathcal{K}}(\alpha_\mathbf{j} \otimes I), \; \widetilde{\mathcal{K}} = [\widetilde{K}_1, \ldots, \widetilde{K}_p]$$
$$M_1 = [I\ 0\ 0\ 0], \; M_2 = [0\ I\ 0\ 0], \; M_4 = [0\ 0\ 0\ I]$$
$$W_1 = [I\ -I\ 0\ 0], \; W_2 = [I\ I\ -2I\ 0].$$

Proof. Initially it is proven that if inequalities (15)–(16) are satisfied then (12) holds. Consider the Lyapunov-Krasovskii candidate given in (13), whose positiveness is ensured from inequalities (14) (Palmeira et al., 2020). Moreover, its time derivative along trajectories of (5) is given by

$$\begin{aligned} \dot{\mathcal{W}}(\tau, x) &= 2x^\top(t)P\dot{x}(t) \\ &- (x(t) - x(t_k))^\top [S(x(t) - x(t_k)) + 2Qx(t_k)] \\ &+ 2(t_{k+1} - t)(\dot{x}(t)S(x(t) - x(t_k)) + \dot{x}(t)Qx(t_k)) \\ &- \int_{t_k}^t \dot{x}^\top(s)R\dot{x}(s)\,\mathrm{d}s + (t_{k+1} - t)\dot{x}^\top(t)R\dot{x}(t) \\ &- (t - t_k)x^\top(t_k)Ux(t_k) + (t_{k+1} - t)x^\top(t_k)Ux(t_k). \end{aligned} \tag{19}$$

By defining the vector $\xi(t) = (x(t), x(t_k), \nu_k(\tau), \dot{x}(t))$, with $\nu_k(\tau) = \frac{1}{\tau}\int_{t_k}^t x(s)\,\mathrm{d}s$, it follows from the Wirtinger integral inequality (Seuret and Gouaisbaut, 2013, Corollary 4) that

$$-\int_{t_k}^t \dot{x}^\top(s)R\dot{x}(s)\mathrm{d}s \leq -\frac{1}{\tau}\xi^\top\left(W_1^\top R W_1 + 3W_2^\top R W_2\right)\xi. \tag{20}$$

Then, it follows from the Reciprocally Convex Combination Lemma (Seuret and Gouaisbaut, 2013) that there exist matrices $Y_1, Y_2 \in \mathbb{R}^{n\times 4n}$ such that

$$\begin{aligned} -\tfrac{1}{\tau(t)}W_1^\top R W_1 &\leq -\mathrm{He}\{Y_1^\top W_1\} + \tau(t)Y_1^\top R^{-1}Y_1 \\ -\tfrac{1}{\tau(t)}W_2^\top R W_2 &\leq -\mathrm{He}\{Y_2^\top W_2\} + \tau(t)Y_2^\top R^{-1}Y_2. \end{aligned} \tag{21}$$

Moreover, by introducing the null-term

$$\begin{aligned} &2\xi^\top(t)\left(M_1^\top + \epsilon M_2^\top + \epsilon M_4^\top\right)X^{-\top}A(x)M_1\xi(t) \\ &+2\xi^\top(t)\left(M_1^\top + \epsilon M_2^\top + \epsilon M_4^\top\right)X^{-\top}B(x)K(x_k)M_2\xi(t) \\ &-2\xi^\top(t)\left(M_1^\top + \epsilon M_2^\top + \epsilon M_4^\top\right)X^{-\top}M_4\xi(t) = 0 \end{aligned} \tag{22}$$

it implies from (20), (21), (22) the following upper-bound for (19):

$$\dot{\mathcal{W}}(\tau, x) \leq \xi^\top(t)\Pi(\tau, h_k, x)\xi(t), \tag{23}$$

where

$$\Pi(\tau, h_k, x) = \Pi_1(x) + (h_k - \tau)\Pi_2 + (h_k - 2\tau)\Pi_3 + \tau\Pi_4, \tag{24}$$

with

$$\Pi_1(x) = \Pi_1^0 + \Pi_1^1(x) + \Pi_1^2(x) - \mathrm{He}\{Y_1^\top W_1 + 3Y_2^\top W_2\}$$
$$\Pi_1^0 = \mathrm{He}\{M_1^\top P M_4 - W_1^\top Q M_2 - (M_1^\top + \epsilon M_2^\top + \epsilon M_4^\top)X^{-\top}M_4\}$$
$$\Pi_1^1(x) = \mathrm{He}\{(M_1^\top + \epsilon M_2^\top + \epsilon M_4^\top)X^{-\top}A(x(t))M_1\}$$
$$\Pi_1^2(x) = \mathrm{He}\{(M_1^\top + \epsilon M_2^\top + \epsilon M_4^\top)X^{-\top}B(x(t))K(x(t_k))M_2\}$$
$$\Pi_2 = M_4^\top R M_4 + \mathrm{He}\{M_4^\top S W_1 + M_4^\top Q M_2\}$$
$$\Pi_3 = M_2^\top U M_2, \; \Pi_4 = Y_1^\top R^{-1} Y_1 + Y_2^\top R^{-1} Y_2.$$

As (24) is affine with respect to $\tau(t) \in [0, h_k]$, to ensure that $\Pi(\tau, h_k, x) < 0$, it is sufficient that the following inequalities hold

$$\Pi(0, h_k, x) = \Pi_1(x) + h_k\Pi_2 + h_k\Pi_3 < 0 \tag{25a}$$

$$\Pi(h_k, h_k, x) = \begin{bmatrix} \Pi_1(x) - h_k\Pi_3 & h_k Y_1^\top & 3h_k Y_2^\top \\ \star & -h_k R & 0 \\ \star & \star & -3h_k R \end{bmatrix} < 0. \tag{25b}$$

By pre- and post-multiplying inequality (25a) with $I_4 \otimes X^\top$ and pre- and post-multiplying inequality (25b) with $I_6 \otimes X^\top$, it leads to

$$\Theta_1(h_k, x) = \widetilde{\Pi}_1(x) + h_k\widetilde{\Pi}_2 + h_k\widetilde{\Pi}_3 < 0 \tag{26a}$$

$$\Theta_2(h_k, x) = \begin{bmatrix} \widetilde{\Pi}_1(x) - h_k\widetilde{\Pi}_3 & h_k\widetilde{Y}_1^\top & 3h_k\widetilde{Y}_2^\top \\ \star & -h_k\widetilde{R} & 0 \\ \star & \star & -3h_k\widetilde{R} \end{bmatrix} < 0. \tag{26b}$$

Based on the polytopic representation in (7), it is possible to write

$$\Theta_1(h_k,x) = \sum_{\mathbf{i}\in\mathbb{B}^p}\sum_{\mathbf{j}\in\mathbb{B}^p}\sum_{\mathbf{k}\in\mathbb{B}^p} \mathrm{w}_{\mathbf{i}}(x_k)\mathrm{w}_{\mathbf{j}}(x_k)\mathrm{v}_{\mathbf{k}}(\delta)\Theta^1_{\mathbf{ijk}}(h_k)$$
$$\Theta_2(h_k,x) = \sum_{\mathbf{i}\in\mathbb{B}^p}\sum_{\mathbf{j}\in\mathbb{B}^p}\sum_{\mathbf{k}\in\mathbb{B}^p} \mathrm{w}_{\mathbf{i}}(x_k)\mathrm{w}_{\mathbf{j}}(x_k)\mathrm{v}_{\mathbf{k}}(\delta)\Theta^2_{\mathbf{ijk}}(h_k).$$

Thus, based on the LMI-relaxation in Coutinho et al. (2020), the solution of the LMIs in (15) and (16) imply that inequalities in (26a) and (26b) hold, respectively. Therefore, the condition (12) with $\dot{\mathcal{W}}(\tau,x)$ defined in (19) is satisfied along the trajectories of (5) under the exact polytopic representation (7). It follows from Lemma 1 that the origin of the closed-loop system (5) is asymptotically stable.

Now, to ensure that closed-loop trajectories remain in the region in which the polytopic representation hold, one needs to prove that $\Omega \subset D \cap \Delta$. If inequalities (17) hold, it is possible to show that $\Omega \subset D$ (Coutinho et al., 2020). Moreover, if inequalities (18) hold, then

$$\Theta_{3,\ell}(x) = \sum_{\mathbf{i}\in\mathbb{B}^p}\sum_{\mathbf{j}\in\mathbb{B}^p}\sum_{\mathbf{k}\in\mathbb{B}^p}\mathrm{w}_{\mathbf{i}}(x_0)\mathrm{w}_{\mathbf{j}}(x_0)\mathrm{g}_{\mathbf{k}}(x_0)\Theta^3_{\mathbf{ijk},\ell}<0,$$

for all $\ell \in \mathbb{N}_{\leq p}$, which after being pre- and post-multiplied with $\mathrm{diag}(X^{-\top},1)$ results

$$\begin{bmatrix} -P & \star \\ J_{(\ell)}(x_0)\left(A(x_0)+B(x_0)K(x_0)\right) & -\phi_\ell^2 \end{bmatrix} < 0.$$

From Schur complement, it follows that

$$-P + \phi_\ell^{-2}\Pi_5^\top(x_0)\Pi_5(x_0) < 0, \quad \forall \ell \in \mathbb{N}_{\leq p},$$

with $\Pi_5(x(0)) = J_{(\ell)}(x_0)\left(A(x_0)+B(x_0)K(x_0)\right)$, or, equivalently, $\phi_\ell^{-2}\dot{x}^\top(0)\nabla z_j^\top(x_0)\nabla z_j(x_0)\dot{x}(0) < V(x_0)$. For all $x_0 \in \Omega$, it follows that $\dot{z}_j(x_0) \leq \phi_\ell, \forall \ell \in \mathbb{N}_{\leq p}$, which imply $\Omega \subset \Delta_0$, $\Delta_0 = \bigcap_{\ell=1}^p \Delta_0^\ell, \Delta_0^\ell = \{x \in \Omega : |\dot{z}_\ell(x_0)| \leq \phi_\ell\}$ at $t = 0$. As Ω is a positively invariant set, state trajectories starting in Ω at $t = t_0 = 0$ will converge to the origin remaining in Ω. Thus, if $x_0 \in \Omega \subset \Delta_0$ then $x_k \in \Omega \subset \Delta_0 \subseteq \Delta$. Therefore, $\Omega \subset D \cap \Delta$ and the bounds defined in (3) are not violated, ensuring the validity of the polytopic representation (7). This concludes the proof.

Remark 3. The result in Theorem 2 provides an effective way of dealing with the general case in which time derivatives of scheduling functions depend on the control input, as shown in (8), by explicitly characterizing the region Δ and enforcing that $\Omega \subset \Delta$. It allows describing inter-sampling variations regarding the polytopic representation (7), possibly leading to less conservative results than (Palmeira et al., 2020, Corollary 1).

4.3 Enlargement of the Region of Attraction Estimation

Based on the result stated in Theorem 2, the region Ω is contained in the region of attraction of the zero equilibrium of the closed-loop system (5). As a result, it can be used to estimate the region of attraction. Thus, to maximize the region Ω, the following optimization problem is considered:

$$\left.\begin{array}{ll} \underset{\varphi}{\text{minimize}} & \varphi \\ \text{subject to} & \text{(14)–(18)} \\ & \begin{bmatrix} \varphi I & I \\ I & X + X^\top - \widetilde{P} \end{bmatrix} > 0, \end{array}\right\} \quad (27)$$

where $\varphi \in \mathbb{R}_{>0}$. Notice the extra constraint introduced in (27) ensures that $P < \varphi I$, which implies that $\Omega_0 \subset \Omega$, where $\Omega_0 = \{x \in \mathbb{R}^n : x^\top x \leq \varphi^{-1}\}$. Therefore, as long as φ is minimized, it tends to maximize the region Ω.

5. NUMERICAL EXAMPLE

Consider the following nonlinear system (Palmeira et al., 2020, Example 2):

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= (1 + x_1^2(t))x_1(t) + (2 + 8x_2^2(t))x_2(t) + u(t), \end{aligned} \quad (28)$$

where $|x_1| \leq 1$, $|x_2| \leq 1$ define the validity region D. By defining $z(x) = (x_1^2, x_2^2)$, the nonlinear system (28) can be represented as a quasi-LPV model (1a) with

$$A_0 = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 8 \end{bmatrix},$$
$$B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

with $-1 \leq z_1(x) \leq 1$ and $-1 \leq z_2(x) \leq 1$. As $\nabla z_2(x)B(x) = 2x_2 \neq 0, \forall x_2 \neq 0$, it is clear that the time derivative of $z_2(x)$ depends on the control input $u(t)$. In this example, a comparison [1] is made between the proposed condition in Theorem 2 and (Palmeira et al., 2020, Corollary 1), which is the condition that can be applied in this case. In all experiments conducted here, it is assumed $\underline{h} = 0.01$s. The following two aspects are evaluated here: (i) The conservativeness reduction measured in terms of the minimum φ for the largest $\overline{h}$ that the condition (Palmeira et al., 2020, Corollary 1) is feasible. (ii) The relation between the adjustable bounds ϕ_1 and ϕ_2 in (3) with the objective function φ in (27) and the maximum $\overline{h}$ that the optimization with the proposed condition is feasible.

To address point (i), the MASP obtained for the condition (Palmeira et al., 2020, Corollary 1) is $\overline{h} = 31$ms. For the same $\overline{h}$, the condition in Theorem 2 is solved for different values of ϕ_1 and ϕ_2 as reported in Table 1. It can be observed that smaller values of φ are attained with the proposed condition for the different choices of ϕ_1 and ϕ_2. As ϕ_1 and ϕ_2 are reduced the value of φ increases, indicating the reduction of the region Ω in (10). It illustrates the effect of the region Δ in (11) over the estimate of the region of attraction.

Table 1. Minimum φ for $\overline{h} = 31$ms.

		φ
(Palmeira et al., 2020, Cor. 1)		97.878
Theorem 2 with $\epsilon = 0.1$	$\phi_1 = \phi_2 = 1$	4.0056
	$\phi_1 = \phi_2 = 2$	1.3816
	$\phi_1 = \phi_2 = 3$	1.1087

To address point (ii), the optimization problem (27) is solved for different pairs $\{\phi_1, \phi_2\}$ and, in each case, it is computed the MASP $\overline{h}$ that the optimization is feasible [2]. The results are depicted in Table 2. As the values of

[1] The same polytope construction methodology and LMI relaxations have been employed in all cases for a fair comparison between the conditions. The LMI conditions were solved in Matlab environment using the Yalmip parser and the Mosek solver.

[2] The values of ϵ have been obtained by employing a line search procedure in the interval $\epsilon \in \{0.1, 0.2, \ldots, \}$ to minimize φ.

$\{\phi_1, \phi_2\}$ decrease, the optimization problem is feasible for larger values of $\bar{h}$. This is closely related to the fact that the bounds of $\delta_1(t)$ and $\delta_2(t)$ in (6) are reduced. Nevertheless, to ensure that these bounds are not violated during operation, the estimated region of attraction is reduced due to the constraints imposed by the region Δ in (11). It illustrates a trade-off between the MASP and the estimate of the region of attraction.

Table 2. Minimum φ for the maximum $\bar{h}$ that optimization problem (27) is feasible.

		ϵ	φ	$\bar{h}$ (ms)
$\phi_1 = 1$	$\phi_2 = 1$	0.60	242.986	101
	$\phi_2 = 2$	0.40	63.804	90
	$\phi_2 = 3$	0.35	71.610	83
$\phi_1 = 2$	$\phi_2 = 1$	0.50	478.450	99
	$\phi_2 = 2$	0.41	111.851	89
	$\phi_2 = 3$	0.34	71.074	82
$\phi_1 = 3$	$\phi_2 = 1$	0.47	422.101	97
	$\phi_2 = 2$	0.39	155.470	88
	$\phi_2 = 3$	0.33	55.156	81

To conclude, consider $\{\phi_1, \phi_2\} = \{1, 1\}$ and the MASP of $\bar{h} = 50$ms. In this case, the estimated region of attraction Ω is shown in Figure 1 together with the regions D, which constraints the states, and Δ_0^1, Δ_0^2, constraining the time derivatives of scheduling functions. The control gains are: $K_0 = \begin{bmatrix} -1.2377 & -2.9436 \end{bmatrix}$, $K_1 = \begin{bmatrix} -1.0003 & 0.0003 \end{bmatrix}$, $K_2 = \begin{bmatrix} 0.0023 & -8.0022 \end{bmatrix}$. It can be noticed that the estimate is obtained such that $\Omega \subset D \cap \Delta$, with $\Delta_0^1 \cap \Delta_0^2 \subset \Delta$. Moreover, convergent closed-loop trajectories initiating inside of Ω illustrates the asymptotic stability of the closed-loop equilibrium of the sampled-data control system.

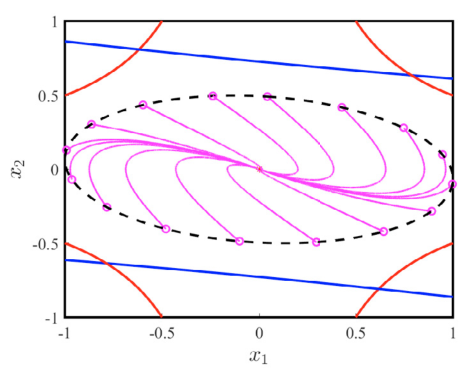

Fig. 1. Closed-loop trajectories, in magenta, for $\{\phi_1, \phi_2\} = \{1, 1\}$, $\epsilon = 0.3$ and $h_k \in [10, 50]$ms. The estimated region of attraction Ω, in dashed black, is contained in the regions D, in solid black, and $\Delta_0 = \Delta_0^1 \cap \Delta_0^2$, with Δ_0^1 in red and Δ_0^2 in blue.

6. CONCLUSION

This paper has proposed a novel synthesis condition for stabilization of sampled-data nonlinear systems represented by quasi-LPV models. The main advantages of the proposal are the use of a less conservative integral inequality and the possibility of describing the inter-sampling behavior in the case of systems whose time derivatives of scheduling functions depend on the control input signal. A local analysis has been performed to ensure the applicability of the proposed method, since it has been proved that the gain-scheduled controller is valid for trajectories initiating inside the estimated region of attraction.

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