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Abstract. In this paper, we study locally strongly convex Tchebychev hypersurfaces, namely the *centroaffine totally umbilical hypersurfaces*, in the $(n+1)$-dimensional affine space $\mathbb{R}^{n+1}$. We first make an ordinary-looking observation that such hypersurfaces are characterized by having a Riemannian structure admitting a canonically defined closed conformal vector field. Then, by taking the advantage of properties about Riemannian manifolds with closed conformal vector fields, we show that the ellipsoids are the only centroaffine Tchebychev hyperovaloids. This solves the longstanding problem of trying to generalize the classical theorem of Blaschke and Deicke on affine hyperspheres in equiaffine differential geometry to that in centroaffine differential geometry.

1. Introduction

In this paper, we study locally strongly convex centroaffine hypersurfaces, i.e., the hypersurfaces of the $(n+1)$-dimensional affine space $\mathbb{R}^{n+1}$ with centroaffine normalization. It is well-known that in both Euclidean and equiaffine differential geometry, the Weingarten (shape) operator contains essential geometric information about a hypersurface. This is different in centroaffine differential geometry, where one studies the properties of hypersurfaces in $\mathbb{R}^{n+1}$ which are invariant under the centroaffine transformation group $G = GL(n+1, \mathbb{R})$, where G keeps the origin of $\mathbb{R}^{n+1}$ fixed. Since the centroaffine normalization induces the identity as Weingarten operator, from the point of view of relative differential geometry any nondegenerate hypersurface with centroaffine normalization is a relative hypersphere (see sections 6.3 and 7.2 of [21]); thus in centroaffine differential geometry *the usually induced Weingarten operator* contains no further geometric information.

In such situation, C.P. Wang [27] made a breakthrough by giving the reasonable definition for the Weingarten (shape) operator on centroaffine hypersurfaces of the $(n+1)$-dimensional affine space $\mathbb{R}^{n+1}$. Specifically, on a centroaffine hypersurface, besides the centroaffine metric, there exists a canonically defined *Tchebychev vector field* T. Let $\hat{\nabla}$ denote the Levi-Civita connection with respect to the centroaffine metric, then the operator $\mathcal{T} := \hat{\nabla} T$ was introduced to be defined as *the centroaffine shape operator* by Wang [27]. (Note: eversince [27] the centroaffine shape operator $\mathcal{T}$ is also called the *Tchebychev operator*). To justify this terminology, it was shown that the Tchebychev operator $\mathcal{T}$ in centroaffine differential geometry is analogous to the shape operator in the equiaffine differential geometry. Indeed, C.P. Wang [27]

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calculated the first variation formula of the centroaffine area functional and, as an important result, he showed (cf. Theorem 2 of [27]) that the critical hypersurfaces of this functional are exactly hypersurfaces with vanishing *centroaffine mean curvature* $H := \frac{1}{n}\mathrm{Tr}\,\mathcal{T}$; moreover, as in Euclidean and equiaffine differential geometry, C.P. Wang also proved (Theorem 1 in [27]) that the only hyperovaloid in $\mathbb{R}^{n+1}$ with constant centroaffine mean curvature is the ellipsoid centered at the origin of $\mathbb{R}^{n+1}$. It is worthy to note that, as there are no general results about the sign of the second variation of the centroaffine area functional at the critical hypersurfaces, it was suggested in [10] (see also [26]) to call a centroaffine hypersurface with $H = 0$ the *centroaffine extremal hypersurface*.

The centroaffine shape operator $\mathcal{T}$ was studied systematically from Liu and Wang [16]. In particular, there is an important subclass of centroaffine hypersurfaces, namely the *centroaffine totally umbilical hypersurfaces*. By definition, it consists of centroaffine hypersurfaces whose shape operator $\mathcal{T}$ is proportional to the identity isomorphism of the tangent spaces. Following Liu and Wang [16], these centroaffine hypersurfaces are usually referred to as *Tchebychev hypersurfaces*. Obviously, the notion of Tchebychev hypersurfaces generalizes in a natural way the notion of *affine hyperspheres* in equiaffine differential geometry. More to be pointed out is that both, i.e., Tchebychev hypersurfaces in centroaffine differential geometry and affine hyperspheres in equiaffine differential geometry, have exactly the similar structure equations (cf. [9, 12, 16, 18, 21]). Because of such nice similarity, the Tchebychev hypersurfaces have been under extensive study. For references, we refer to [1, 6, 11, 14, 15, 16, 17, 22].

In equiaffine differential geometry, we have the well-known classical theorem of Blaschke and Deicke (cf. Theorem 3.35 in [12]) which states that if a *hyperovaloid* (which means a connected compact locally strongly convex hypersurface without boundary in the $(n + 1)$-dimensional affine space $\mathbb{R}^{n+1}$) is an affine hypersphere, then it is an ellipsoid. The Blaschke and Deicke's theorem and the preceding mentioned similarity between affine hyperspheres and Tchebychev hypersurfaces motivate strongly to study the following problem, which will provide an interesting new global characterization of the ellipsoid as centroaffine Tchebychev hyperovaloid.

PROBLEM ([6]). *Let $x : M^n \to \mathbb{R}^{n+1}$ $(n \geq 2)$ be a centroaffine Tchebychev hyperovaloid. Must $x(M^n)$ be an ellipsoid containing the origin of $\mathbb{R}^{n+1}$?*

The PROBLEM has been considered, first by Liu and Wang [16] but restricts to the case $n = 2$. It was solved affirmatively:

Theorem 1.1 (cf. Theorem 4.3 of [16])**.** *Let $x : M^2 \to \mathbb{R}^3$ be a centroaffine Tchebychev ovaloid. Then $x(M^2)$ is an ellipsoid in $\mathbb{R}^3$ such that the origin of $\mathbb{R}^3$ is in the inside of $x(M^2)$.*

The PROBLEM has been further investigated by many researchers in higher dimensional cases. In [15], Liu, Simon and Wang solved it affirmatively under an additional *nondegenerate equiaffine Gauss map* condition (cf. Theorem 5.2 of [15]). In Theorem 5 of [14], following an argument about affine hyperspheres, M. Li also obtained some partial results from the point of view in relative affine differential geometry. More recently, joint with Z.K. Yao the first two authors of the present article solved the PROBLEM affirmatively under the additional condition *the centroaffine metric having nonnegative sectional curvatures* (Theorem 1.7 of [6]). We would mention that the method of [6] depends heavily on the recent classification of locally strongly convex centroaffine hypersurfaces with parallel traceless difference tensor (cf. [4, 5, 9]).

In this paper, as the continuation of [6], we still focus on the above PROBLEM. By adopting a new approach, we first make an ordinary-looking but important observation (Lemma 2.1) that the Tchebychev vector field on centroaffine hypersurfaces is closed, and that a centroaffine hypersurface is a Tchebychev hypersurface if and only if its Tchebychev vector field is a conformal vector field with respect to the centroaffine metric. Then, by taking the advantage of typical properties about a Riemannian manifold with closed conformal vector field, we eventually solve the PROBLEM affirmatively for every dimension $n \geq 3$. Our main result can be stated as follows:

Theorem 1.2. *Let $x : M^n \to \mathbb{R}^{n+1}$ $(n \geq 3)$ be a centroaffine Tchebychev hyperovaloid. Then $x(M^n)$ is an ellipsoid such that the origin of $\mathbb{R}^{n+1}$ is in the inside of $x(M^n)$.*

Remark 1.1. A centroaffine hypersurface $x : M^n \to \mathbb{R}^{n+1}$ is a Tchebychev hypersurface if and only if it satifies $\mathcal{T} = \alpha\,\mathrm{id}$, where α is a smooth function on M^n. However, different from the affine hyperspheres in equiaffine differential geometry, where the equiaffine shape operator must be a constant multiple of the identity isomorphism of the tangent space, here the function α can be not a constant, even if for the ellipsoids or for the general hyperquadrics. This significant difference explains partially why the proof of Theorem 1.2 is complicated and very different from that of Blaschke and Deicke's theorem. To have a better understanding of these respects, we would suggest the readers to compare the proof of Theorem 3.35, p.145 in [12] and that of Theorem 5 in [14].

2. Preliminaries

In this section, we briefly recall some basic facts about centroaffine hypersurfaces. More details are referred to the monographs [12, 18, 21] and the references [13, 27].

Let $\mathbb{R}^{n+1}$ be the $(n+1)$-dimensional affine space equipped with its canonical flat connection D. Let M^n be a connected n-dimensional smooth manifold. An immersion $x : M^n \to \mathbb{R}^{n+1}$ is said to be a centroaffine hypersurface if, for each point $x \in M^n$, the position vector x from the origin $O \in \mathbb{R}^{n+1}$ is transversal to the tangent space T_xM^n of M^n at x. In that situation, the position vector x defines the *centroaffine normalization* modulo orientation. For any vector fields X and Y tangent to M^n, we have the centroaffine formula of Gauss:

$$D_X x_*(Y) = x_*(\nabla_X Y) + h(X,Y)(-\varepsilon x), \tag{2.1}$$

where $\varepsilon = 1$ or -1. Moreover, associated with (2.1) we will call $-\varepsilon x$, ∇ and h the centroaffine normal, the induced (centroaffine) connection and the centroaffine metric, respectively. In this paper, we will consider only locally strongly convex centroaffine hypersurfaces such that the bilinear 2-form h defined by (2.1) is definite; and we will choose ε such that the centroaffine metric h is positive definite.

Let $x : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex centroaffine hypersurface and $\hat{\nabla}$ be the Levi-Civita connection of its centroaffine metric h. Then the tensor K, defined by $K(X,Y) := K_XY := \nabla_XY - \hat{\nabla}_XY$, is called the *difference tensor* of the centroaffine hypersurface. It is symmetric as both connections ∇ and $\hat{\nabla}$ are torsion free. Let $\hat{R}$ denote the Riemannian curvature tensor of the centroaffine metric h, then the following Gauss and Codazzi equations hold:

$$\hat{R}(X,Y)Z = \varepsilon(h(Y,Z)X - h(X,Z)Y) - [K_X, K_Y]Z, \tag{2.2}$$

$$(\hat{\nabla}_Z K)(X,Y) = (\hat{\nabla}_X K)(Z,Y). \tag{2.3}$$

Moreover, we further have the following totally symmetry equation

$$h((\hat{\nabla}_Z K)(X,Y), W) = h((\hat{\nabla}_X K)(Z,W), Y). \tag{2.4}$$

Associated to a centroaffine hypersurface $x : M^n \to \mathbb{R}^{n+1}$, we can define the *Tchebychev form* $T^\sharp$ and the *Tchebychev vector field* T in implicit form by

$$T^\sharp(X) = \tfrac{1}{n}\mathrm{Tr}\,(K_X), \quad h(T,X) = T^\sharp(X), \quad \forall\, X \in TM^n. \tag{2.5}$$

Moreover, using the difference tensor K and the Tchebychev vector field T, one can further define the symmetric *traceless difference tensor* $\tilde{K}$ by

$$\tilde{K}(X,Y) := K(X,Y) - \tfrac{n}{n+2}\big[h(X,Y)T + h(X,T)Y + h(Y,T)X\big]. \tag{2.6}$$

It is well-known that $\tilde{K}$ vanishes if and only if $x(M^n)$ lies in a hyperquadric (cf. Section 7.1 in [21]; Lemma 2.1 and Remark 2.2 in [11]).

As have been stated in the Introduction, the *centroaffine shape operator* $\mathcal{T}$ of a centroaffine hypersurface $x : M^n \to \mathbb{R}^{n+1}$, introduced by C.P. Wang [27] and is also called the *Tchebychev operator*, is a homomorphism mapping $\mathcal{T} : TM \to TM$, defined by

$$\mathcal{T}(X) := \hat{\nabla}_X T, \quad \forall\, X \in TM^n. \tag{2.7}$$

Then, the well-defined function $H := \frac{1}{n}\mathrm{Tr}\,\mathcal{T}$ was named as the *centroaffine mean curvature* of x. This is a meaningful terminology because, according to C.P. Wang [27], the hypersurfaces with $H = 0$ are exactly the critical hypersurfaces of the centroaffine area functional. Moreover, related to the centroaffine shape operator $\mathcal{T}$, it is interesting to consider an important subclass of the centroaffine hypersurfaces, named as the *Tchebychev hypersurfaces*, which is defined as below:

Definiton 2.1 ([16]). *Let $x : M^n \to \mathbb{R}^{n+1}$ be a centroaffine hypersurface such that its Tchebychev operator $\mathcal{T}$ is proportional to the identity isomorphism* $\mathrm{id} : TM^n \to TM^n$*, i.e., $\mathcal{T} = \frac{1}{n}(\mathrm{div}\,T)\,\mathrm{id}$. Then, x is called a Tchebychev hypersurface.*

As it was pointed out in [15] that the Tchebychev hypersurfaces satisfy certain systems of second order PDE, and some of these systems play an important role in the general context of conformal geometry. In this context, we shall further emphasize the following important Riemannian geometric characterization of the Tchebychev hypersurfaces:

Lemma 2.1. *For a centroaffine hypersurface $x : M^n \to \mathbb{R}^{n+1}$, the Tchebychev vector field T is a closed vector field in the sense that the Tchebychev form $T^\sharp$ is a closed form. Moreover, a centroaffine hypersurface $x : M^n \to \mathbb{R}^{n+1}$ is a Tchebychev hypersurface if and only if, associated to the centroaffine metric h, its Tchebychev vector field T is a conformal vector field.*

Proof. The first statement, which is equivalent to that the Tchebychev operator $\mathcal{T}$ is self-adjoint with respect to the centroaffine metric h, was first shown by C.P. Wang [27]. Next, let $x : M^n \to \mathbb{R}^{n+1}$ be a centroaffine hypersurface with Tchebychev vector field T. If it is a Tchebychev hypersurface, then we have

$$(\mathcal{L}_T h)(X,Y) = h(\hat{\nabla}_X T, Y) + h(X, \hat{\nabla}_Y T) = \tfrac{2}{n}(\mathrm{div}\,T)h(X,Y), \tag{2.8}$$

where $\mathcal{L}_T$ denotes the Lie derivative with respect to the Tchebychev vector field T. This shows that $\mathcal{L}_T h = \frac{2}{n}(\operatorname{div} T)h$. Thus, T is a conformal vector field. Conversely, assume that T is a conformal vector field relative to h, i.e., it holds that

$$(\mathcal{L}_T h)(X,Y) = 2fh(X,Y)$$

for any vector fields X, Y and some smooth function f on M^n. Then, by using that $(\mathcal{L}_T h)(X,Y) = h(\hat{\nabla}_X T, Y) + h(X, \hat{\nabla}_Y T)$ and the self-adjointness of $\mathcal{T}$, namely that

$$h(\hat{\nabla}_X T, Y) = h(\hat{\nabla}_Y T, X), \tag{2.9}$$

we derive $\hat{\nabla}_X T = fX$ for any vector field $X \in TM^n$. It follows that $x : M^n \to \mathbb{R}^{n+1}$ is a Tchebychev hypersurface. □

Before concluding this section, we would emphasize that Riemannian manifolds with closed conformal vector fields have been extensively studied, see the papers e.g. [2, 8, 19, 23, 24, 25]. In next sections, we shall work for the application of the useful characterization of the centroaffine Tchebychev hypersurfaces, established by Lemma 2.1, so as to complete the proof of Theorem 1.2.

3. Local properties of the Tchebychev hypersurfaces

In this section, we will study the local properties of centroaffine Tchebychev hypersurfaces in $\mathbb{R}^{n+1}$. Since our concern is the PROBLEM, and that we already have Theorem 1.1, in sequel we assume that $n \geq 3$.

Recall that in [15], Liu, Simon and Wang established several local geometric characterizations of the Tchebychev hypersurfaces and, as a corollary, they showed that any quadric is a Tchebychev hypersurface.

In view of Lemma 2.1, and according to Lemma 1 of [20] (cf. also Lemma 1 of [7]) which collects some results about Riemannian manifolds admitting closed and conformal vector fields, we immediately obtain the following lemma.

Lemma 3.1 (cf. [20]). *Let $x : M^n \to \mathbb{R}^{n+1}$ be a centroaffine Tchebychev hypersurface with nontrivial Tchebychev vector field T and $\mathcal{T} = \alpha\,\mathrm{id}$. Then we have:*

(i) *The norm $\|T\|$ with respect to the centroaffine metric h, the function α and the curvature tensor $\hat{R}$ satisfy the following relations:*

$$\begin{aligned} \hat{\nabla}\|T\|^2 = 2\alpha T, \quad \|T\|^2\hat{\nabla}\alpha = T(\alpha)T, \\ \|T\|^2\hat{R}(X,Y)T = -T(\alpha)(h(T,Y)X - h(T,X)Y). \end{aligned} \tag{3.1}$$

(ii) *The zeros of T is a discrete set. Moreover, T has nonvanishing divergence at its zeros.*

(iii) *If we denote $\widetilde{M} = \{p \in M^n \mid T(p) \neq 0\}$, then the distribution*

$$p \in \widetilde{M} \to \mathfrak{D}(p) := \{v \in T_pM^n \mid p \in \widetilde{M} \text{ and } h(v,T) = 0\}$$

defines an umbilical foliation on $(\widetilde{M}, h)$. In particular, the functions $\|T\|^2$ and α are constant on the connected leaves of $\mathfrak{D}$.

(iv) *If $g = \|T\|^{-2}h$, then $(\widetilde{M}, g)$ is locally isometric to $(I \times N, dt^2 \oplus g')$ and $T = (\partial/\partial t, 0)$, where I is an open interval in $\mathbb{R}$, $\{t\} \times N$ is a leaf of the foliation $\mathfrak{D}$ for any $t \in \mathbb{R}$.*

Next, closely related to the proof of Theorem 1.2, we first study centroaffine Tchebychev hypersurfaces which satisfy the condition $K_T T = \lambda T$ for some function $\lambda \in C^\infty(M^n)$. Following the notations of Lemma 3.1, we begin with the following lemma.

Lemma 3.2. *Let $x : M^n \to \mathbb{R}^{n+1}$ be a centroaffine Tchebychev hypersurface whose Tchebychev vector field T is nontrivial and satisfies $K_T T = \lambda T$. Then:*

(i) *For any point $p \in \widetilde{M}$, the eigenvalues $\{\lambda_i\}_{2\le i\le n}$ of K_T on $\mathfrak{D}(p)$ satisfy the quadratic equation*

$$\lambda_i^2 - \lambda\lambda_i + \varepsilon\|T\|^2 + \alpha' = 0, \tag{3.2}$$

where $\alpha' := \frac{d\alpha}{dt}$. In particular, at most two of $\{\lambda_i\}_{2\le i\le n}$ are distinct.

(ii) *If $V, W \in \mathfrak{D}$ are eigenvectors of K_T corresponding to different eigenvalues, then $K(V, W) = 0$.*

(iii) *The eigenvalues of K_T are constant on the connected leaves of the foliation $\mathfrak{D}$.*

Proof. At any $p \in \widetilde{M}$, since T is an eigenvector of K_T and K_T is self-adjoint with respect to the centroaffine metric, K_T can be diagonalized on $\mathfrak{D}$. Let $\{X_i\}_{2\le i\le n} \subset \mathfrak{D}$ be the mutually orthogonal eigenvectors of K_T with corresponding eigenvalues $\{\lambda_i\}_{2\le i\le n}$, i.e.,

$$K_T X_i = \lambda_i X_i, \quad i = 2, \ldots, n.$$

Then, the third equation in (3.1) implies that

$$\hat{R}(X_i, T)T = -\alpha' X_i, \quad \hat{R}(X_i, X_j)T = 0. \tag{3.3}$$

On the other hand, by using the Gauss equation, we obtain

$$\begin{aligned} \hat{R}(X_i, T)T &= (\lambda_i^2 - \lambda\lambda_i + \varepsilon\|T\|^2)X_i, \\ \hat{R}(X_i, X_j)T &= (\lambda_i - \lambda_j)K(X_i, X_j). \end{aligned} \tag{3.4}$$

From (3.3) and (3.4), the assertions (i) and (ii) follows.

If we derivate $K_T T = \lambda T$ with respect to $X \in TM^n$, we obtain

$$(\hat{\nabla}_X K)(T, T) + 2\alpha K_T X = X(\lambda)T + \alpha\lambda X. \tag{3.5}$$

It follows that for any vector field $Y \in TM^n$ there holds

$$h((\hat{\nabla}_X K)(T, T), Y) + 2\alpha h(K_T X, Y) = X(\lambda)h(T, Y) + \alpha\lambda h(X, Y). \tag{3.6}$$

From (3.6) and noticing that both $h((\hat{\nabla}_\cdot K)(\cdot,\cdot),\cdot)$ and $h(K(\cdot,\cdot),\cdot)$ are totally symmetric, we get $X(\lambda)h(T, Y) = Y(\lambda)h(T, X)$ for any $X, Y \in TM^n$. It follows that

$$X(\lambda)T = h(T, X)\hat{\nabla}\lambda, \quad X \in TM^n. \tag{3.7}$$

Hence, we have $X(\lambda) = 0$ for any $X \in \mathfrak{D}$.

From item (iii) of Lemma 3.1, we know that $X(\alpha) = 0$ for $X \in \mathfrak{D}$. This, together with (iv) of Lemma 3.1, implies that $X(\alpha') = 0$ for any $X \in \mathfrak{D}$. It follows that the solutions of (3.2) are constant on each connected leaves of the foliation $\mathfrak{D}$. Thus the assertion (iii) follows. □

Now, we can further prove the following proposition.

Proposition 3.1. *Let $x : M^n \to \mathbb{R}^{n+1}$ be a centroaffine Tchebychev hypersurface with nontrivial T such that $K_T T = \lambda T$. Then one of the following two cases occurs:*

(i) $\widetilde{M} = M$, *i.e.* T *has no zeros;* $\hat{\nabla}T = 0$, *and* K_T *has exactly two distinct constant eigenvalues on* $\mathfrak{D}$*;*

(ii) $K_T V = \mu V$ *for any* $V \in \mathfrak{D}$*. Moreover, we have*

$$\mu' = (\lambda - \mu)\alpha, \quad n\|T\|^2 = \lambda + (n-1)\mu, \tag{3.8}$$

and that $c_0 := \varepsilon\|T\|^2 - \mu^2 + \alpha^2$ *is a constant.*

Proof. As M^n is connected and the zeros of T is isolated, the subset $\widetilde{M}$ is also connected. By Lemma 3.2, K_T on $\mathfrak{D}$ has at most two distinct eigenvalues. We put

$$M_0 = \{p \in \widetilde{M} \mid K_T \text{ has two distinct eigenvalues on } \mathfrak{D}(p)\}.$$

First of all, we assume that $M_0 \neq \emptyset$. Obviously, M_0 is an open subset of $\widetilde{M}$.

Claim 1. *M_0 is a closed subset of $\widetilde{M}$.*

Let $\{\lambda_i\}_{2\le i\le n}$ be the eigenvalues of K_T on $\mathfrak{D}$ and assume that

$$\mu_1 := \lambda_2 = \cdots = \lambda_m < \lambda_{m+1} = \cdots = \lambda_n =: \mu_2.$$

Since μ_1 and μ_2 are continuous functions on M_0, we see that m is a constant on each connected component of M_0. So

$$\mathfrak{D}_i = \{V \in \mathfrak{D} \mid K_T V = \mu_i V\}, \quad i = 1, 2,$$

define two distributions on the connected components of M_0 and $\mathfrak{D} = \mathfrak{D}_1 \oplus \mathfrak{D}_2$.

For $V \in \mathfrak{D}_1$ and $W \in \mathfrak{D}_2$, by direct calculations and using Lemma 3.2, we obtain

$$\begin{aligned}\hat{\nabla}_W h(K_T V, V) =& h((\hat{\nabla}_W K)(T, V), V) + \alpha h(K_W V, V) + 2h(K_T V, \hat{\nabla}_W V)\\ =& h((\hat{\nabla}_W K)(T, V), V) + 2\mu_1 h(V, \hat{\nabla}_W V),\end{aligned}$$

and

$$\hat{\nabla}_W(\mu_1 h(V, V)) = 2\mu_1 h(V, \hat{\nabla}_W V).$$

Comparing the above equations and using $h(K_T V, V) = \mu_1 h(V, V)$, we get

$$h((\hat{\nabla}_W K)(T, V), V) = 0, \quad V \in \mathfrak{D}_1, W \in \mathfrak{D}_2. \tag{3.9}$$

Similarly, for $V \in \mathfrak{D}_1$ and $W \in \mathfrak{D}_2$, taking the derivative of $h(K_T V, W) = 0$ with respect to $V \in \mathfrak{D}_1$ and using Lemma 3.2, we obtain

$$h((\hat{\nabla}_V K)(T, V), W) = (\mu_1 - \mu_2)h(\hat{\nabla}_V V, W), \quad V \in \mathfrak{D}_1, W \in \mathfrak{D}_2. \tag{3.10}$$

Comparing (3.9), (3.10) and using (2.4), we derive

$$h(\hat{\nabla}_V V, W) = 0, \quad V \in \mathfrak{D}_1, W \in \mathfrak{D}_2. \tag{3.11}$$

On the other hand, the fact $\mathcal{T} = \alpha\,\mathrm{id}$ implies that

$$h(\hat{\nabla}_V V, T) = -h(\hat{\nabla}_V T, V) = -\alpha h(V, V), \quad V \in \mathfrak{D}_1. \tag{3.12}$$

Thus we get

$$\hat{\nabla}_V V = (\hat{\nabla}_V V)^1 - \alpha\|T\|^{-2}h(V, V)T, \quad V \in \mathfrak{D}_1, \tag{3.13}$$

where $(\hat{\nabla}_V V)^1$ denotes the component of $\hat{\nabla}_V V$ on $\mathfrak{D}_1$.

Again, for $V \in \mathfrak{D}_1$ and $W \in \mathfrak{D}_2$, taking the covariant derivative of $K(V, W) = 0$ with respect to V, we obtain

$$(\hat{\nabla}_V K)(V, W) + K(\hat{\nabla}_V V, W) + K(\nabla_V W, V) = 0.$$

Hence, we have

$$h((\hat{\nabla}_V K)(V, W), W) + h(K(\hat{\nabla}_V V, K_W W) = 0, \quad V \in \mathfrak{D}_1, W \in \mathfrak{D}_2. \tag{3.14}$$

Inserting (3.13) into (3.14), we obtain

$$h((\hat{\nabla}_V K)(V,W),W) = \alpha\|T\|^{-2}\mu_2 h(V,V)h(W,W), \quad V \in \mathfrak{D}_1, W \in \mathfrak{D}_2.$$

Then, the fact $h((\hat{\nabla}_W K)(W,V),V) = h((\hat{\nabla}_V K)(V,W),W)$ implies that

$$h((\hat{\nabla}_V K)(V,W),W) = \alpha\|T\|^{-2}\mu_1 h(V,V)h(W,W), \quad V \in \mathfrak{D}_1, W \in \mathfrak{D}_2.$$

Comparing the above computations, we obtain,

$$(\mu_1 - \mu_2)\alpha\|T\|^{-2}h(V,V)h(W,W) = 0, \quad V \in \mathfrak{D}_1, W \in \mathfrak{D}_2.$$

It follows that $\alpha = 0$ on M_0. Hence, we have

$$\hat{\nabla}T = 0, \text{ on } M_0. \tag{3.15}$$

Taking $X = V \in \mathfrak{D}_1$ and $Y = W \in \mathfrak{D}_2$ in (3.6), we get

$$h((\hat{\nabla}_V K)(T,T),W) = 0, \quad V \in \mathfrak{D}_1, W \in \mathfrak{D}_2. \tag{3.16}$$

For $V \in \mathfrak{D}_1$ and $W \in \mathfrak{D}_2$, taking the derivative of $h(K_T V, W) = 0$ with respect to T, we easily obtain

$$h((\hat{\nabla}_V K)(T,T),W) + \mu_2 h(\hat{\nabla}_T V, W) + \mu_1 h(\hat{\nabla}_T W, V) = 0. \tag{3.17}$$

From (3.16) and (3.17), we get

$$(\mu_2 - \mu_1)h(\hat{\nabla}_T V, W) = 0, \quad V \in \mathfrak{D}_1, W \in \mathfrak{D}_2. \tag{3.18}$$

From (3.18) and noting that $h(\hat{\nabla}_T V, T) = -\alpha h(T,V) = 0$, we get

$$\hat{\nabla}_T V \in \mathfrak{D}_1, \quad V \in \mathfrak{D}_1. \tag{3.19}$$

Then, for $V \in \mathfrak{D}_1$, taking the covariant derivative of $K_T V = \mu_1 V$ with respect to T and using (3.15), we obtain

$$(\hat{\nabla}_T K)(T,V) + K(\hat{\nabla}_T V, T) = T(\mu_1)V + \mu_1 \hat{\nabla}_T V. \tag{3.20}$$

Then, by (3.19) and (3.20), we get

$$h((\hat{\nabla}_T K)(T,V),V) = T(\mu_1)h(V,V). \tag{3.21}$$

On the other hand, as $\alpha = 0$ on M_0, from (3.5) and (3.15), we have

$$h((\hat{\nabla}_T K)(T,V),V) = 0. \tag{3.22}$$

Then (3.21) and (3.22) imply that

$$T(\mu_1) = 0 \text{ on } M_0. \tag{3.23}$$

This and (iii) of Lemma 3.2 show that μ_1 is a constant on the component of M_0.

Similarly, we can prove that μ_2 is constant on the component of M_0. So, by continuity of μ_1 and μ_2, K_T restricted on $\mathfrak{D}$ has two distinct eigenvalues on the closure of M_0. Thus, M_0 is a closed subset of $\widetilde{M}$ and Claim 1 is verified.

In summary, we have proved that there are only two possibilities: Either $M_0 = \widetilde{M}$ or $M_0 = \emptyset$.

If $\widetilde{M} = M_0$, then by continuity $\hat{\nabla}T = 0$ and $h(T,T)$ is constant on M. Since by assumption $T \neq 0$, we have proved that T has no zeros and $\widetilde{M} = M$. Thus, case (i) in Proposition 3.1 occurs.

If $M_0 = \emptyset$, K_T restricted on $\mathfrak{D}$ has only one eigenvalue, denoted by μ, which is constant on the leaves of $\mathfrak{D}$. Hence $K_T V = \mu V$ for any $V \in \mathfrak{D}$.

Taking the derivative of $K_T V = \mu V$ with respect to T and noting that $\hat{\nabla}_T V \in \mathfrak{D}$, we obtain

$$(\hat{\nabla}_T K)(T, V) + \alpha\mu V = \mu' V, \quad V \in \mathfrak{D}. \tag{3.24}$$

On the other hand, from (3.5), we obtain

$$(\hat{\nabla}_T K)(T, V) = (\lambda - 2\mu)\alpha V, \quad V \in \mathfrak{D}. \tag{3.25}$$

From (3.24) and (3.25), we get the first equation in (3.8).

For any $p \in \widetilde{M}$, let $\{e_1 = \|T\|^{-1}T, e_2, \ldots, e_n\}$ be an orthonormal basis of T_pM^n. Then, by definition (2.5) we can derive that $T = \frac{1}{n}\sum_{i=1}^{n} K_{e_i}e_i$. It follows that

$$T = \frac{1}{n}\Big(\|T\|^{-2}K_T T + \sum_{i=2}^{n} K_{e_i}e_i\Big). \tag{3.26}$$

Taking the inner product of both sides of (3.26) with T, we get immediately the second equation in (3.8).

By using (3.2), the first two equations in (3.1) and the first equation in (3.8), direct calculations show that

$$\hat{\nabla}(\varepsilon\|T\|^2 - \mu^2 + \alpha^2) = 2\|T\|^{-2}\alpha(\mu^2 - \lambda\mu + \varepsilon\|T\|^2 + \alpha')T = 0.$$

It follows that $\varepsilon\|T\|^2 - \mu^2 + \alpha^2 =: c_0$ is a constant on $\widetilde{M}$.

We have proved that if $M_0 = \emptyset$ then case (ii) in Proposition 3.1 occurs. $\square$

As a crucial step to complete the proof of Theorem 1.2, we intend to derive a locally expression for centroaffine Tchebychev hypersurfaces which are assumed to satisfy property (ii) of Proposition 3.1. To achieve the purpose, we first state the following lemma, whose proof is an easy computation.

Lemma 3.3. *Assume that $\lambda(t), \alpha(t)$ and $\mu(t)$ are real-valued functions satisfying*

$$\alpha' = -\mu^2 + \lambda\mu - \varepsilon\|T\|^2, \quad \mu' = (\lambda - \mu)\alpha.$$

Then the ordinary differential equation

$$x_{tt} = (\lambda + \alpha)x_t - \varepsilon|T|^2 x$$

has two linear independent solutions that can be written as follows:

$$x_1 = e^{\int(\alpha+\mu)dt}, \quad x_2 = x_1 \int e^{\int(\lambda - 2\mu - \alpha dt)}dt.$$

Finally, as one main result of this section, we can prove the following

Proposition 3.2. *Let $x : M^n \to \mathbb{R}^{n+1}$ be a centroaffine Tchebychev hypersurface with $\mathcal{T} = \alpha\,\mathrm{id}$ such that $K_T T = \lambda T$ and case (ii) in Proposition 3.1 occurs. Then x can be written as*

$$x = \gamma_1(t)\varphi + \gamma_2(t)C, \tag{3.27}$$

where $\varphi : N \to \mathbb{R}^n$ is an affine hypersphere, C is a nonzero constant vector in $\mathbb{R}^{n+1}$, and

$$\gamma_1(t) = e^{\int(\alpha+\mu)dt}, \ \gamma_2(t) = e^{\int(\alpha+\mu)dt}\int e^{\int(\lambda-2\mu-\alpha)dt}dt. \tag{3.28}$$

Moreover, the difference tensor K of $x : M^n \to \mathbb{R}^{n+1}$ and the difference tensor K' of $\varphi : N \to \mathbb{R}^n$ are related by

$$K(X, Y) = \|T\|^{-2}h(X, Y)\mu T + K'(X, Y), \ X, Y \in \mathfrak{D}. \tag{3.29}$$

Proof. From item (iv) of Lemma 3.1, the Riemannian manifold (M^n, h) is locally isometric to $(I\times N, \|T\|^2(dt^2+g'))$, where N is the integral manifold of $\mathfrak{D}$ and g' is a metric defined on N. Therefore, we can take a local coordinate $(u_1 := t, u_2, \ldots, u_n)$ of $M^n = I \times N$ so that the metric h has the following expression

$$h = \|T\|^2\Big(dt^2 + \sum_{i=2}^{n} g'_{ij}du_idu_j\Big).$$

By using (2.1), and case (ii) of Proposition 3.1, we have

$$x_{tt} = (\lambda + \alpha)x_t - \varepsilon\|T\|^2 x, \tag{3.30}$$

$$x_{tu_i} = (\mu + \alpha)x_{u_i}, \quad i \geq 2. \tag{3.31}$$

In our case, by (3.2), (3.8) and Lemma 3.3, we can solve (3.30) to obtain

$$x = \gamma_1(t)\varphi(u_2, \ldots, u_n) + \gamma_2(t)\psi(u_2, \ldots, u_n). \tag{3.32}$$

where $\varphi(u_2, \ldots, u_n)$ and $\psi(u_2, \ldots, u_n)$ are $\mathbb{R}^{n+1}$-valued functions, $\gamma_1(t)$ and $\gamma_2(t)$ are described by (3.28).

Then, substituting (3.32) into (3.31), we further derive $\frac{\partial\psi}{\partial u_k} = 0$ for $2 \leq k \leq n$. This implies that $\psi(u_2, \ldots, u_n) =: C$ is a constant vector in $\mathbb{R}^{n+1}$. Due to that $x : M^n \to \mathbb{R}^{n+1}$ is a nondegenerate centroaffine hypersurfaces, C must be nonzero (if otherwise, we have $x_t = \gamma_1^{-1}\gamma_1' x$, contradicting to that x is a transversal vector). Now, we have

$$x = \gamma_1(t)\varphi(u_2, \ldots, u_n) + \gamma_2(t)C. \tag{3.33}$$

It follows that

$$\varphi_{u_i} = \gamma_1(t)^{-1}x_{u_i}, \quad i \geq 2. \tag{3.34}$$

Thus, φ defines an immersion from N into $\mathbb{R}^{n+1}$.

Claim 2. $\varphi : N \to \mathbb{R}^{n+1}$ defines a locally strongly convex affine hypersphere in an n-dimensional vector subspace $\mathbb{R}^n$ of $\mathbb{R}^{n+1}$.

To verify Claim 2, noticing that $\|T\|^{-2}h(\partial u_i, \partial u_j) = g'(\partial u_i, \partial u_j)$ for $i, j \geq 2$, and from (3.28) we can derive $x_t = \gamma_1(\alpha + \mu)\varphi + \gamma_2(\alpha + \mu)C + \gamma_1 e^{\int(\lambda-2\mu-\alpha)dt}C$. Then straightforward calculations by using (3.33) and (3.34) give that

$$\begin{aligned}
\varphi_{u_iu_j} &= \gamma_1(t)^{-1}x_{u_iu_j} = \gamma_1(t)^{-1}(x_*(\nabla_{\partial u_i}\partial u_j) - \varepsilon h(\partial u_i, \partial u_j)x)\\
&= \gamma_1(t)^{-1}\Big(x_*(\nabla^T_{\partial u_i}\partial u_j) + h(\hat{\nabla}_{\partial u_i}\partial u_j, \|T\|^{-1}T)\,\|T\|^{-1}x_t\\
&\qquad + h(K_{\partial u_i}\partial u_j, \|T\|^{-1}T)\,\|T\|^{-1}x_t - \varepsilon h(\partial u_i, \partial u_j)x\Big)\\
&= \gamma_1(t)^{-1}\Big(x_*(\nabla^T_{\partial u_i}\partial u_j)\Big) + \Big[c_0\varphi + \Big(c_0\int e^{\int(\lambda-2\mu-\alpha)dt}dt\\
&\qquad + (\mu-\alpha)e^{\int(\lambda-2\mu-\alpha)dt}\Big)C\Big]\,\|T\|^{-2}h(\partial u_i, \partial u_j)\\
&= \varphi_*(\nabla^T_{\partial u_i}\partial u_j) + \Big[c_0\varphi + \Big(c_0\int e^{\int(\lambda-2\mu-\alpha)dt}dt\\
&\qquad + (\mu-\alpha)e^{\int(\lambda-2\mu-\alpha)dt}\Big)C\Big]g'(\partial u_i, \partial u_j), \quad i, j \geq 2,
\end{aligned} \tag{3.35}$$

where $c_0 := \varepsilon\|T\|^2 - \mu^2 + \alpha^2$ is a constant as described in Proposition 3.1, and $\nabla^T_{\partial u_i}\partial u_j$ denotes the tangent component of $\nabla_{\partial u_i}\partial u_j$ in TN .

Now, we consider two possibilities:

Case I. $c_0 = 0$, i.e. $\varepsilon\|T\|^2 - \mu^2 + \alpha^2 = 0$.

By direct calculations, we can show that $(\mu-\alpha)e^{\int(\lambda-2\mu-\alpha)dt}$ is a constant. If $(\mu-\alpha)e^{\int(\lambda-2\mu-\alpha)dt}=0$, then $\mu-\alpha=0$ which, together with $c_0=0$, implies that $T=0$. This contradict to the assumption $T\neq 0$. Hence $(\mu-\alpha)e^{\int(\lambda-2\mu-\alpha)dt}\neq 0$. Then, denoting the non-zero constant vector $(\mu-\alpha)e^{\int(\lambda-2\mu-\alpha)dt}C$ still by C, we get the expression

$$\varphi_{u_iu_j}=\varphi_*(\nabla^T_{\partial u_i}\partial u_j)+g'(\partial u_i,\partial u_j)C,\quad i,j\geq 2. \tag{3.36}$$

Case II. $c_0\neq 0$. In this case, we have

$$\begin{aligned}\varphi_{u_iu_j}=&\gamma_1(t)^{-1}\Big(x_*(\nabla^T_{\partial u_i}\partial u_j)\Big)+c_0\Big[\varphi+\Big(\int e^{\int(\lambda-2\mu-\alpha)dt}dt\\&+c_0^{-1}(\mu-\alpha)e^{\int(\lambda-2\mu-\alpha)dt}\Big)C\Big]g'(\partial u_i,\partial u_j),\quad i,j\geq 2.\end{aligned} \tag{3.37}$$

Moreover, direct calculations show that the term

$$\int e^{\int(\lambda-2\mu-\alpha)dt}dt+c_0^{-1}(\mu-\alpha)e^{\int(\lambda-2\mu-\alpha)dt}$$

is a constant. Then, denoting $(\int e^{\int(\lambda-2\mu-\alpha)dt}dt+c_0^{-1}(\mu-\alpha)e^{\int(\lambda-2\mu-\alpha)dt})C$ still by C, we obtain

$$\varphi_{u_iu_j}=\varphi_*(\nabla^T_{\partial u_i}\partial u_j)+c_0(\varphi+C)g'(\partial u_i,\partial u_j),\quad i,j\geq 2. \tag{3.38}$$

Therefore, for both cases, (3.36) (resp. (3.38)) implies that in Case I (resp. Case II) the image of φ is contained in an n-dimensional linear subspace $\mathbb{R}^n$ of $\mathbb{R}^{n+1}$, and the immersion $\varphi:N\to\mathbb{R}^n$ can be interpreted as a relative hypersphere with respect to the relative normal vector field C (resp. $c_0(\varphi+C)$), with induced connection ∇^T and relative metric g' (which by definition is definite), respectively.

Denote by $\hat\nabla'$ the Levi-Civita connection of g'. Using $h=\|T\|^2(dt^2+g')$ and the Koszul's formula, we have the calculation

$$\begin{aligned}\|T\|^{-2}h(\hat\nabla'_{\partial u_i}\partial u_j,\partial u_k)&=g'(\hat\nabla'_{\partial u_i}\partial u_j,\partial u_k)\\&=\|T\|^{-2}h(\hat\nabla_{\partial u_i}\partial u_j,\partial u_k),\quad i,j\geq 2.\end{aligned}$$

This shows that $\hat\nabla'_{\partial u_i}\partial u_j=\hat\nabla^T_{\partial u_i}\partial u_j$ for $i,j\geq 2$, and therefore, the difference tensor K' of $\varphi:N\to\mathbb{R}^n$ is given by

$$K'_{\partial u_i}\partial u_j=K^T_{\partial u_i}\partial u_j,\quad i,j\geq 2, \tag{3.39}$$

where $\hat\nabla^T_{\partial u_i}\partial u_j$ and $K^T_{\partial u_i}\partial u_j$ denote the tangent parts of $\hat\nabla_{\partial u_i}\partial u_j$ and $K_{\partial u_i}\partial u_j$ in TN, respectively.

Let $((g')^{ij})$ (resp. (h^{AB})) denote the inverse matrix of (g'_{ij}) (resp. (h_{AB})), and $h_{AB}=h(\partial u_A,\partial u_B)$ for $A,B\geq 1$. Then, by using (3.39), for any $X\in TN$, we can get the calculation that

$$\begin{aligned}\operatorname{Tr}K'_X&=\sum_{i,j=2}^n(g')^{ij}g'(K'_X\partial u_i,\partial u_j)\\&=\sum_{i,j=2}^n h^{ij}h(K_X\partial u_i,\partial u_j)\\&=nh(T,X)-h^{11}h(K_XT,T)\\&=0.\end{aligned} \tag{3.40}$$

This verifies Claim 2 that $\varphi:N\to\mathbb{R}^n$ is actually an affine hypersphere.

Finally, by (3.39) and the definition of the difference tensor, (3.29) immediately follows. □

4. The Completion of Theorem 1.2's Proof

We first show that for a centroaffine Tchebychev hyperovaloid, the Tchebychev vector field T and the difference tensor field K satisfy an important relation.

Lemma 4.1. *Let $x : M^n \to \mathbb{R}^{n+1}$ be a centroaffine Tchebychev hyperovaloid. Then its Tchebychev vector field T and difference tensor field K satisfy the relation*

$$K_T T = \tfrac{3n}{n+2}\|T\|^2 T. \tag{4.1}$$

Proof. Put $Z := K_T T - \frac{3n}{n+2}\|T\|^2 T$. By a direct calculation, we obtain

$$\begin{aligned} h(\hat{\nabla}_X Z, Y) =& h((\hat{\nabla}_X K)(T,T), Y) + \tfrac{2}{n}(\operatorname{div} T) h(K_X Y, T) \\ & - \tfrac{3}{n+2}(\operatorname{div} T)\big(2h(X,T)h(Y,T) + h(T,T)h(X,Y)\big). \end{aligned} \tag{4.2}$$

From (4.2), it is easily seen that Z is a closed vector field which satisfies $\operatorname{div} Z = 0$. So Z is a harmonic vector field on M^n. Notice that M^n is diffeomorphic to a sphere, whereas on the sphere there are no nontrivial harmonic vector fields. Hence, Z vanishes identically, and we get (4.1). □

It is well known that the centroaffine Tchebychev form $T^\sharp$ can be expressed by the equiaffine support function ρ (cf. [15, 22]):

$$T^\sharp = \tfrac{n+2}{2n} d \ln \rho. \tag{4.3}$$

Put $f := \frac{n+2}{2n} \ln \rho$. Then by (4.3), we can write $T = \hat{\nabla} f$. It follows that

$$\operatorname{Hess} f(X,Y) = \tfrac{1}{n}(\operatorname{div} T) h(X,Y). \tag{4.4}$$

If $T \equiv 0$, then, as a centroaffine hypersurface, x is an affine hypersphere centered at the origin $O \in \mathbb{R}^{n+1}$. By the theorem of Blaschke and Deicke, $x : M^n \to \mathbb{R}^{n+1}$ is an ellipsoid centered at the origin of $\mathbb{R}^{n+1}$.

Next, we assume that $T \neq 0$. Then, since M^n is compact and is diffeomorphic to a sphere, according to [25] and (4.4) we know that (M^n, h) is conformally equivalent to a round sphere and the number of isolated zeros of T is 2 (cf. also Theorem 4.6 of [15]). On the other hand, from Lemma 4.1 and Proposition 3.1, we see that, in order to complete the proof of Theorem 1.2, we are left to study case (ii) in Proposition 3.1. Then, by using Proposition 3.2, we know that $x : M^n \to \mathbb{R}^{n+1}$ reduces to be

$$x = \gamma_1(t)\varphi + \gamma_2(t) C.$$

where $\varphi : N \to \mathbb{R}^n$ is an affine hypersphere, C is a nonzero constant vector in $\mathbb{R}^{n+1}$, $\gamma_1(t)$ and $\gamma_2(t)$ are described by (3.28).

Since, according to pp.18-19 of [3], the umbilicity of submanifolds is invariant under a conformal transformation of the ambient Riemannian manifold, and in a round sphere the umbilical hypersurfaces are spheres, we obtain that the leaves N of the umbilical foliation $\mathfrak{D}$ are spheres. This implies that $\varphi : N \to \mathbb{R}^n$ is a locally strongly convex affine hypersphere which is compact and without boundary. Then, by the theorem of Blaschke and Deicke $\varphi : N \to \mathbb{R}^n$ is an ellipsoid, and thus $K' = 0$. Hence, from the second equation in (3.8), (3.29) and (4.1), we obtain that

$$\begin{cases} K_T T = \frac{3n}{n+2}\|T\|^2 T, \\ K_T V = \frac{n}{n+2}\|T\|^2 V, \quad V \in \mathfrak{D}, \\ K_V W = \frac{n}{n+2} h(V,W) T, \quad V, W \in \mathfrak{D}. \end{cases} \tag{4.5}$$

Then, by direct calculations, we can show that $\tilde{K} = 0$ on $\widetilde{M}$. By continuity, $\tilde{K} = 0$ holds on the whole M^n. It follows that $x : M^n \to \mathbb{R}^{n+1}$ is an ellipsoid and, as a centroaffine hypersurface, the origin of $\mathbb{R}^{n+1}$ must be in the inside of $x(M^n)$.

This completes the proof of Theorem 1.2. □

References

[1] Th. Binder, *Relative Tchebychev hypersurfaces which are also translation hypersurfaces*, Hokkaido Math. J. **38** (2009), 97–110.

[2] A. Caminha, *The geometry of closed conformal vector fields on Riemannian spaces*, Bull. Braz. Math. Soc. (N.S.) **42** (2011), 277–300.

[3] B.Y. Chen, *Geometry of Submanifolds and Its Applications*, Science University of Tokyo, 1981.

[4] X.X. Cheng and Z.J. Hu, *An optimal inequality on locally strongly convex centroaffine hypersurfaces*, J. Geom. Anal. **28** (2018), 643–655.

[5] X.X. Cheng, Z.J. Hu and M. Moruz, *Classification of the locally strongly convex centroaffine hypersurfaces with parallel cubic form*, Results Math. **72** (2017), 419–469.

[6] X.X. Cheng, Z.J. Hu and Z.K. Yao, *A rigidity theorem for centroaffine Chebyshev hyperovaloids*, Colloq. Math. **157** (2019), 133–141.

[7] I. Castro, C.R. Montealegre and F. Urbano, *Closed conformal vector fields and Lagrangian submanifolds in complex space forms*, Pacific J. Math. **199** (2001), 269–302.

[8] R. Diógenes, E. Ribeiro Jr. and J. Silva Filho, *Gradient Ricci solitons admitting a closed conformal vector field*, J. Math. Anal. Appl. **455** (2017), 1975–1983.

[9] Z.J. Hu, H. Li and L. Vrancken, *Locally strongly convex affine hypersurfaces with parallel cubic form*, J. Diff. Geom. **87** (2011), 239-307.

[10] A.-M. Li, H. Li and U. Simon, *Centroaffine Bernstein problems*, Diff. Geom. Appl. **20** (2004), 331–356.

[11] A.-M. Li, H.L. Liu, A. Schwenk-Schellschmidt, U. Simon and C.P. Wang, *Cubic form methods and relative Tchebychev hypersurfaces*, Geom. Dedic. **66** (1997), 203–221.

[12] A.-M. Li, U. Simon, G.S. Zhao and Z.J. Hu, *Global Affine Differential Geometry of Hypersurfaces*, 2nd ed., de Gruyter Expositions in Mathematics 11, Walter de Gruyter, Berlin/Boston, 2015.

[13] A.-M. Li and C.P. Wang, *Canonical centroaffine hypersurfaces in* $\mathbb{R}^{n+1}$, Results Math. **20** (1991), 660–681.

[14] M. Li, *Rigidity theorems for relative Tchebychev hypersurfaces*, Results Math. **70** (2016), 283–298.

[15] H.L. Liu, U. Simon and C.P. Wang, *Conformal structures in affine geometry: complete Tchebychev hypersurfaces*, Abh. Math. Sem. Hamburg, **66** (1996), 249–262.

[16] H.L. Liu and C.P. Wang, *The centroaffine Tchebychev operator*, Results Math. **27** (1995), 77–92.

[17] H.L. Liu and C.P. Wang, *Relative Tchebychev surfaces in* $\mathbb{R}^3$, Kyushu J. Math. **50** (1996), 533–540.

[18] K. Nomizu and T. Sasaki, *Affine Differential Geometry*, Cambridge University Press, 1994.

[19] M. Obata, *Conformal transformations of Riemannian manifolds*, J. Diff. Geom. **4** (1970), 311–333.

[20] A. Ros and F. Urbano, *Lagrangian submanifolds of* $\mathbb{C}^n$ *with conformal Maslov form and the Whitney sphere*, J. Math. Soc. Jpn. **50** (1998), 203–226.

[21] U. Simon, A. Schwenk-Schellschmidt and H. Viesel, *Introduction to the Affine Differential Geometry of Hypersurfaces*, Lecture Notes, Science University Tokyo, 1991.

[22] U. Simon, H. Trabelsi and L. Vrancken, *Complete hyperbolic Tchebychev hypersurfaces*, J. Geom. **89** (2008), 148–159.
[23] Y. Suyama and Y. Tsukamoto, *Riemannian manifolds admitting a certain confomal transformation group*, J. Diff. Geom. **5** (1971), 415–426.
[24] S. Tanno and W. C. Weber, *Closed conformal vector fields*, J. Diff. Geom. **3** (1969) 361–366.
[25] Y. Tashiro, *Complete Riemannian manifolds and some vector fields*, Trans. Am. Math. Soc. **117** (1965), 251–275.
[26] L. Vrancken, *Centroaffine extremal surfaces*, Soochow J. Math. **30** (2004), 377–390.
[27] C.P. Wang, *Centroaffine minimal hypersurfaces in* $\mathbb{R}^{n+1}$, Geom. Dedicata **51** (1994), 63–74.

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