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FOUR DIMENSIONAL BIHARMONIC HYPERSURFACES IN NONZERO SPACE FORM HAVE CONSTANT MEAN CURVATURE

ZHIDA GUAN, HAIZHONG LI, AND LUC VRANCKEN

Abstract. In this paper, through making careful analysis of Gauss and Codazzi equations, we prove that four dimensional biharmonic hypersurfaces in nonzero space form have constant mean curvature. Our result gives the positive answer to the conjecture proposed by Balmus-Montaldo-Oniciuc in 2008 for four dimensional hypersurfaces.

1. Introduction

Biharmonic maps were introduced in 1964 by Eells and Sampson [ES] as a generalization of harmonic maps. In their paper, Eells and Sampson suggested considering the bi-energy of a map $\phi : (M^n, g) \to (N^m, h)$ between two Riemannian manifolds defined by

$$E_2(\phi) = \frac{1}{2} \int_{M^n} |\tau(\phi)|^2 d\mu_g, \tag{1.1}$$

where $\tau(\phi)$ is the tension field of ϕ and $d\mu_g$ is the volume element on (M^n, g). Stationary points of the bi-energy functional are called biharmonic maps. Jiang (see [J1], [J2]) is the first mathematician who systematically studied the bi-energy functional, and he computed the first and second variations of E_2. The stationary points of the functional E_2 satisfy the Euler-Lagrange equation

$$-\Delta\tau(\phi) = \sum_{i=1}^{n} R^{N^m}(d\phi(e_i), \tau(\phi))d\phi(e_i), \tag{1.2}$$

where Δ is the Laplacian of (M^n, g). Biharmonic submanifolds have attracted a lot of attentions from mathematicians and many important results on biharmonic submanifolds have been obtained since then (see [C1], [C2], [BMO1], [BMO2], [CMO1], [CMO2], [F1], [F2], [F3], [FH]).

The following conjecture was proposed by Balmus-Montaldo-Oniciuc in 2008 [BMO1] (also see Conjecture 7.2 of page 180 in [OC]).

Conjecture. *Any n-dimensional biharmonic submanifold in $\mathbb{S}^{n+p}$ has constant mean curvature.*

When $n = 2, p = 1$, the conjecture was proved by Caddeo-Montaldo-Oniciuc in [CMO1]; when $n = 3, p = 1$, the conjecture was proved by Balmus-Montaldo-Oniciuc in [BMO2]. In this paper, we prove the conjecture for $n = 4, p = 1$. In fact, we prove the following theorem:

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Theorem 1.1. *Four dimensional biharmonic hypersurfaces in nonzero space form $\mathbb{R}^5(c)(c \neq 0)$ have constant mean curvature.*

In the study of biharmonic submanifolds, there are two other conjectures proposed by Chen in 1991 [C1], and by Caddeo-Montaldo-Oniciuc in 2001 [CMO1], respectively.

Chen's Conjecture. *Every n-dimensional biharmonic submanifold of Euclidean spaces $\mathbb{R}^{n+p}$ is minimal.*

Generalized Chen's Conjecture. *Every n-dimensional biharmonic submanifold of a Riemannian manifold N^{n+p} with non-positive sectional curvature is minimal.*

When $n = 2, p = 1$, Chen's conjecture was proved by Chen and Jiang around 1987 independently; when $n = 3, p = 1$, Chen's conjecture was proved by Hasanis and Vlachos in 1995 [HV]. Recently, Fu-Hong-Zhan [FHZ] have made important progress about Chen's conjecture. In fact, they proved Chen's conjecture for $n = 4, p = 1$.

Ou and Tang [OT] constructed a family of counterexamples, where the generalized Chen's conjecture is false when the ambient space has nonconstant negative sectional curvature. However, the generalized Chen's conjecture remains open when the ambient spaces have constant sectional curvature. In particular, when $p = 1, N^{n+1} = \mathbb{H}^{n+1}$, $n = 2$ and $n = 3$, the generalized Chen's conjecture was proved by Caddeo-Montaldo-Oniciuc [CMO2] and Balmus-Montaldo-Oniciuc [BMO2], respectively. When $c = -1$, our Theorem 1.1 solves the generalized Chen's conjecture for $p = 1, N^{n+1} = \mathbb{H}^{n+1}, n = 4$.

The paper is organized as follows. In Section 2, we recall some fundamental concepts and formulas for n-dimensional biharmonic hypersurfaces in space forms $\mathbb{R}^{n+1}(c)$. In Section 3, for 4-dimensional biharmonic hypersurfaces, we derive some equations and lemmas. In Section 4, we give the proof of Theorem 1.1.

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2. Preliminaries

Let M^n be an n-dimensional hypersurface in $(n+1)$-dimensional space form $\mathbb{R}^{n+1}(c)$ with constant sectional curvature c. When $c = 0$, $\mathbb{R}^{n+1}(c)$ is $(n+1)$-dimensional Euclidean space; when $c = 1$, $\mathbb{R}^{n+1}(c)$ is $(n+1)$-dimensional unit sphere; when $c = -1$, $\mathbb{R}^{n+1}(c)$ is $(n+1)$-dimensional hyperbolic space. Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections of M^n and $\mathbb{R}^{n+1}(c)$. Denote X and Y tangent vector fields of M^n and ξ the unit normal vector field. Then the Gauss formula and Weingarten formula (for example, see [C3]) are

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)\xi, \tag{2.1}$$

$$\tilde{\nabla}_X \xi = -AX, \tag{2.2}$$

where h is the second fundamental form, and A is the Weingarten operator. The mean curvature function H is defined by

$$H = \frac{1}{n} \text{ trace } h. \tag{2.3}$$

Moreover, the Gauss and Codazzi equations are given by

$$R(X,Y)Z = c(\langle Y,Z\rangle X - \langle X,Z\rangle Y) + \langle AY,Z\rangle AX - \langle AX,Z\rangle AY, \tag{2.4}$$

$$(\nabla_X A)\, Y = (\nabla_Y A)\, X, \tag{2.5}$$

where the Riemannian curvature $R(X,Y)Z$ is defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{2.6}$$

We have the following characterization result for M^n to be a biharmonic hypersurface in $\mathbb{R}^{n+1}(c)$ (see [J1], [J2], [BMO2], [CMO2], [F3], [FH]).

Proposition 2.1. A hypersurface M^n in a space form $\mathbb{R}^{n+1}(c)$ is biharmonic if and only if H and A satisfy

$$\Delta H + H \text{ trace } A^2 = ncH, \tag{2.7}$$

$$2A \operatorname{grad} H + nH \operatorname{grad} H = 0, \tag{2.8}$$

where the Laplacian operator Δ acting on a smooth function f on M^n is defined by

$$\Delta f = -\operatorname{div}(\nabla f). \tag{2.9}$$

Let M^n be an n-dimensional biharmonic hypersurface in $(n+1)$-dimensional space form $\mathbb{R}^{n+1}(c)$. Suppose the mean curvature function H is not constant. From (2.8), we have that $\operatorname{grad} H$ is an eigenvector of the Weingarten operator A with the corresponding principal curvature $-nH/2$. Without loss of generality, we can choose e_1 such that e_1 is parallel to $\operatorname{grad} H$, and we can choose suitable orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ such that

$$Ae_i = \lambda_i e_i, \tag{2.10}$$

where $\lambda_1 = -nH/2$.

Denote the connection coefficients ω_{ij}^k by

$$\nabla_{e_i} e_j = \sum_{k=1}^{n} \omega_{ij}^k e_k, \quad \omega_{ij}^k + \omega_{ik}^j = 0, \quad i,j = 1, \cdots, n. \tag{2.11}$$

Lemma 2.2. *([FH]) Let M^n be an biharmonic hypersurface in a space form $\mathbb{R}^{n+1}(c)$ and assume the mean curvature H is non-constant. Then the multiplicity of the principal curvature $\lambda_1 (= -nH/2)$ is one, that is, $\lambda_j \neq \lambda_1$ for $2 \leq j \leq n$.*

Lemma 2.3. *([FH]) Let M^n be an biharmonic hypersurface in a space form $\mathbb{R}^{n+1}(c)$ and assume the mean curvature H is non-constant. Then the principal curvatures λ_i and the connection coefficients ω_{ii}^1 satisfy*

$$e_1 e_1(\lambda_1) = e_1(\lambda_1)\left(\sum_{j=2}^{n} \omega_{jj}^1\right) + \lambda_1\left(n(n-2)c - R + 4\lambda_1^2\right), \tag{2.12}$$

$$e_1(\lambda_i) = (\lambda_i - \lambda_1)\omega_{ii}^1, \quad 2 \le i \le n, \tag{2.13}$$

$$e_1\left(\omega_{ii}^1\right) = \left(\omega_{ii}^1\right)^2 + \lambda_1\lambda_i + c, \quad 2 \le i \le n, \tag{2.14}$$

where R is the scalar curvature.

In 2015, Fu [F2] proved the following theorem.

Theorem 2.4. *([F2]) Let M^n be a biharmonic hypersurface with at most three distinct principal curvatures in $\mathbb{R}^{n+1}(c)$. Then M^n has constant mean curvature.*

3. Four dimensional biharmonic hypersurfaces in $\mathbb{R}^5(c)$

From now on, we study the biharmonicity of a hypersurface M^4 in a space form $\mathbb{R}^5(c)$. By Theorem 2.4, we only need to work on the case that M^4 has four distinct principal curvatures, and we assume that the mean curvature H is non-constant. Then there exists a neighborhood of p such that $\operatorname{grad} H \neq 0$. The squared length of the second fundamental form of M is

$$S = \sum_{i=1}^{4} \lambda_i^2 = 4H^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2. \tag{3.1}$$

By using $\lambda_1 = -2H$, Gauss equation is

$$R = 12c + 16H^2 - S = 12c + 12H^2 - \lambda_2^2 - \lambda_3^2 - \lambda_4^2. \tag{3.2}$$

Since e_1 is parallel to $\operatorname{grad} H$, it follows that

$$e_1(H) \neq 0, \quad e_2(H) = e_3(H) = e_4(H) = 0. \tag{3.3}$$

The following result can be found in [F3].

Lemma 3.1. *([F3]) Let M^4 be a biharmonic hypersurface with four distinct principal curvatures in space forms $\mathbb{R}^5(c)$, then we have*

$$\nabla_{e_1} e_i = 0, \quad i = 1, 2, 3, 4, \tag{3.4}$$

$$\nabla_{e_i} e_1 = -\omega_{ii}^1 e_i, \quad i = 2, 3, 4, \tag{3.5}$$

$$\nabla_{e_i} e_i = \sum_{k=1, k\neq i}^{4} \omega_{ii}^k e_k, \quad i = 2, 3, 4, \tag{3.6}$$

$$\nabla_{e_i} e_j = -\omega_{ii}^j e_i + \omega_{ij}^k e_k \quad \textit{for distinct } i, j, k = 2, 3, 4, \tag{3.7}$$

where

$$\omega_{ii}^j = -\frac{e_j(\lambda_i)}{\lambda_j - \lambda_i}. \tag{3.8}$$

Note that we have

$$\lambda_2 + \lambda_3 + \lambda_4 = -3\lambda_1, \tag{3.9}$$

$$S = \sum_{i=1}^{4} \lambda_i^2 = \lambda_1^2 + \sum_{i=2}^{4} \lambda_i^2, \tag{3.10}$$

$$e_1(\lambda_1) \neq 0, \quad e_i(\lambda_1) = 0, \quad 2 \leq i \leq 4. \tag{3.11}$$

Following [FH] and [FHZ], the functions f_k are defined by

$$f_k = (\omega_{22}^1)^k + (\omega_{33}^1)^k + (\omega_{44}^1)^k, \quad k = 1, \cdots, 5. \tag{3.12}$$

The following lemma was proved by Fu-Hong-Zhan in [FHZ].

Lemma 3.2. *([FHZ]) With the notations f_k, the following two relations hold*

$$f_1^4 - 6f_1^2 f_2 + 3f_2^2 + 8f_1 f_3 - 6f_4 = 0, \tag{3.13}$$

$$f_1^5 - 5f_1^3 f_2 + 5f_1^2 f_3 + 5f_2 f_3 - 6f_5 = 0. \tag{3.14}$$

For simplicity, given a function g on M^4, we write $g' = e_1(g), g'' = e_1e_1(g), g''' = e_1e_1e_1(g)$ and $g'''' = e_1e_1e_1e_1(g)$. Also, we write $\lambda = \lambda_1$ and $f_1 = T$. Note that for $i = 2, 3, 4$, using Lemma 3.1, we have

$$\begin{aligned} e_i(\lambda') =& e_i e_1(\lambda) \\ =& e_1 e_i(\lambda) + [e_i, e_1](\lambda) \\ =& (\nabla_{e_i} e_1 - \nabla_{e_1} e_i)(\lambda) = 0, \end{aligned} \tag{3.15}$$

similarly we have

$$e_i(\lambda'') = e_i(\lambda''') = e_i(\lambda'''') = e_i(\lambda''''') = 0. \tag{3.16}$$

Following the argument of Fu-Hong-Zhan in [FHZ], we can prove

Lemma 3.3. *f_k can be written as*

$$\begin{cases} f_1 = T, \\ f_2 = T' + 3\lambda^2 - 3c, \\ f_3 = \dfrac{1}{2}T'' - \left(\lambda^2 + c\right) T + 6\lambda\lambda', \\ f_4 = \dfrac{1}{6}T''' - \dfrac{4}{3}\left(\lambda^2 + c\right) T' - \dfrac{5}{3}\lambda\lambda' T + 2\lambda'^2 + 4\lambda\lambda'' - 2\lambda^4 - c\lambda^2 + 3c^2, \\ f_5 = \dfrac{1}{24}T'''' - \dfrac{5}{6}\left(\lambda^2 + c\right) T'' + 2\lambda\lambda''' + \dfrac{5}{3}\lambda'\lambda'' - \dfrac{26}{3}\lambda^3\lambda' - \dfrac{47}{6}c\lambda\lambda' \\ \qquad -\dfrac{25}{12}\lambda\lambda' T' - \dfrac{1}{12}\left(13\lambda\lambda'' + \lambda'^2 - 12\lambda^4 - 24c\lambda^2 - 12c^2\right) T. \end{cases} \tag{3.17}$$

Proof. Since $e_1(\lambda) \neq 0$, from (2.12), we have

$$f_1 = \frac{e_1 e_1(\lambda) - \lambda(S-4c)}{e_1(\lambda)} = \frac{\lambda''}{\lambda'} - \frac{\lambda}{\lambda'}(S-4c) =: T. \tag{3.18}$$

Taking the sum of i from 2 to 4 in (2.14) and (2.13) and using (3.9), we have

$$f_2 = 3\lambda^2 + e_1(f_1) - 3c = T' + 3\lambda^2 - 3c, \tag{3.19}$$

$$\begin{aligned} g_1 :&= \sum_{i=2}^{4} \lambda_i \omega_{ii}^1 \\ &= \lambda T - 3e_1(\lambda) = \lambda T - 3\lambda'. \end{aligned} \tag{3.20}$$

Multiplying ω_{ii}^1 on both sides of (2.14), we have

$$\frac{1}{2} e_1\left(\left(\omega_{ii}^1\right)^2\right) = \left(\omega_{ii}^1\right)^3 + \lambda\lambda_i\omega_{ii}^1 + c\omega_{ii}^1.$$

Taking the sum of i from 2 to 4, we have

$$\begin{aligned} f_3 &= \frac{1}{2} e_1(f_2) - \lambda g_1 - cT \\ &= \frac{1}{2} T'' - \left(\lambda^2 + c\right) T + 6\lambda\lambda'. \end{aligned} \tag{3.21}$$

Differentiating (3.20) with respect to e_1 and using (2.13) and (2.14), we have

$$e_1(g_1) = 2\sum_{i=2}^{4} \lambda_i \left(\omega_{ii}^1\right)^2 + \lambda \sum_{i=2}^{4} \lambda_i^2 - \lambda \sum_{i=2}^{4} \left(\omega_{ii}^1\right)^2 - 3c\lambda. \tag{3.22}$$

From (3.9), (3.10), (3.18), we have

$$\begin{aligned} g_2 := \sum_{i=2}^{4} \lambda_i \left(\omega_{ii}^1\right)^2 &= \frac{1}{2}\left\{e_1(g_1) - \lambda\left(S - \lambda^2\right) + \lambda f_2 + 3c\lambda\right\} \\ &= \frac{1}{2}\left\{e_1(g_1) - \lambda'' + \lambda' T + \lambda^3 + \lambda f_2 - c\lambda\right\}. \end{aligned}$$

Using (3.19) and (3.20), we have

$$g_2 = \lambda T' + \lambda' T - 2\lambda'' + 2\lambda^3 - 2c\lambda. \tag{3.23}$$

Multiplying $(\omega_{ii}^1)^2$ on both sides of (2.14), we have

$$\frac{1}{3} e_1\left(\left(\omega_{ii}^1\right)^3\right) = \left(\omega_{ii}^1\right)^4 + \lambda\lambda_i\left(\omega_{ii}^1\right)^2 + c\left(\omega_{ii}^1\right)^2.$$

Taking the sum of i from 2 to 4, we have

$$\begin{aligned} f_4 &= \frac{1}{3} e_1(f_3) - \lambda g_2 - c f_2 \\ &= \frac{1}{6} T''' - \frac{4}{3}\left(\lambda^2 + c\right) T' - \frac{5}{3}\lambda\lambda' T + 2\lambda'^2 + 4\lambda\lambda'' - 2\lambda^4 - c\lambda^2 + 3c^2. \end{aligned} \tag{3.24}$$

Multiplying λ_i on both sides of (2.13), we have

$$\lambda_i^2 \omega_{ii}^1 = \frac{1}{2} e_1\left(\lambda_i^2\right) + \lambda\lambda_i\omega_{ii}^1.$$

Using (3.10), we have

$$\begin{aligned} g_3 := \sum_{i=2}^{4} \lambda_i^2 \omega_{ii}^1 &= \frac{1}{2} e_1 \left(S - \lambda^2\right) + \lambda g_1 \\ &= \frac{1}{2} \left(\frac{\lambda'' - \lambda' T}{\lambda} - \lambda^2\right)' + \lambda g_1 \\ &= -\frac{\lambda'}{2\lambda} T' + \left(\lambda^2 - \frac{\lambda'' \lambda - \lambda'^2}{2\lambda^2}\right) T - 4\lambda\lambda' + \frac{\lambda''' \lambda - \lambda'' \lambda'}{2\lambda^2}. \end{aligned} \tag{3.25}$$

Differentiating (3.23) with respect to e_1 and using (2.13) and (2.14), we have

$$e_1 (g_2) = 3 \sum_{i=2}^{4} \lambda_i \left(\omega_{ii}^1\right)^3 - \lambda \sum_{i=2}^{4} \left(\omega_{ii}^1\right)^3 + 2\lambda \sum_{i=2}^{4} \lambda_i^2 \omega_{ii}^1 + 2c \sum_{i=2}^{4} \lambda_i \omega_{ii}^1,$$

that is

$$\begin{aligned} g_4 :=& \sum_{i=2}^{4} \lambda_i \left(\omega_{ii}^1\right)^3 \\ =& \frac{1}{3} \left(e_1 (g_2) + \lambda f_3 - 2\lambda g_3 - 2c g_1\right) \\ =& \frac{1}{2} \lambda T'' + \lambda' T' + \frac{1}{3} \left(2\lambda'' - 3\lambda^3 - \frac{\lambda'^2}{\lambda} - 3c\lambda\right) T \\ & - \lambda''' + \frac{20}{3} \lambda^2 \lambda' + \frac{\lambda'' \lambda'}{3\lambda} + \frac{4}{3} c\lambda'. \end{aligned} \tag{3.26}$$

Multiplying $(\omega_{ii}^1)^3$ on both sides of (2.14), we have

$$\frac{1}{4} e_1 \left(\left(\omega_{ii}^1\right)^4\right) = \left(\omega_{ii}^1\right)^5 + \lambda\lambda_i \left(\omega_{ii}^1\right)^3 + c \left(\omega_{ii}^1\right)^3.$$

Taking the sum of i from 2 to 4, we have

$$\begin{aligned} f_5 =& \frac{1}{4} e_1 (f_4) - \lambda g_4 - c f_3 \\ =& \frac{1}{24} T'''' - \frac{5}{6} \left(\lambda^2 + c\right) T'' + 2\lambda\lambda''' + \frac{5}{3} \lambda' \lambda'' - \frac{26}{3} \lambda^3 \lambda' - \frac{47}{6} c\lambda\lambda' \\ & - \frac{25}{12} \lambda\lambda' T' - \frac{1}{12} \left(13\lambda\lambda'' + \lambda'^2 - 12\lambda^4 - 24c\lambda^2 - 12c^2\right) T. \end{aligned} \tag{3.27}$$

□

Following the argument of Fu-Hong-Zhan in [FHZ], we can prove

Lemma 3.4. *The function T satisfies $e_i(T) = 0$ for $i = 2, 3, 4$.*

Proof. Assume that $T \neq 0$. Substituting (3.17) into (3.13) and (3.14), we have

$$\begin{aligned} & 9c^2 + 10cT^2 + T^4 - 48c\lambda^2 - 26T^2\lambda^2 + 39\lambda^4 \\ & -10cT' - 6T^2 T' + 26\lambda^2 T' + 3T'^2 + 58T\lambda\lambda' - 12\lambda'^2 \\ & +4TT'' - 24\lambda\lambda'' - T''' = 0, \end{aligned} \tag{3.28}$$

$$\begin{aligned}&36c^2T+40cT^3+4T^5-48cT\lambda^2-80T^3\lambda^2-84T\lambda^4\\&-20cTT'-20T^3T'-20T\lambda^2T'-172c\lambda\lambda'+120T^2\lambda\lambda'+568\lambda^3\lambda'\\&+170\lambda T'\lambda'+2T\lambda'^2-10cT''+10T^2T''+50\lambda^2T''+10T'T''\\&+26T\lambda\lambda''-40\lambda'\lambda''-48\lambda\lambda'''-T''''=0.\end{aligned}\tag{3.29}$$

Differentiating (3.28) with respect to e_1, we have

$$\begin{aligned}&20cTT'+4T^3T'-52T\lambda^2T'-12TT'^2-96c\lambda\lambda'-52T^2\lambda\lambda'\\&+156\lambda^3\lambda'+110\lambda T'\lambda'+58T\lambda'^2-10cT''-6T^2T''+26\lambda^2T''\\&+10T'T''+58T\lambda\lambda''-48\lambda'\lambda''+4TT'''-24\lambda\lambda'''-T''''=0.\end{aligned}\tag{3.30}$$

From (3.29) and (3.30), eliminating T'''' we have

$$\begin{aligned}&-36c^2T-40cT^3-4T^5+48cT\lambda^2+80T^3\lambda^2+84T\lambda^4+40cTT'\\&+24T^3T'-32T\lambda^2T'-12TT'^2+76c\lambda\lambda'-172T^2\lambda\lambda'-412\lambda^3\lambda'\\&-60\lambda T'\lambda'+56T\lambda'^2-16T^2T''-24\lambda^2T''+32T\lambda\lambda''-8\lambda'\lambda''\\&+4TT'''+24\lambda\lambda'''=0.\end{aligned}\tag{3.31}$$

From (3.28) and (3.31), eliminating T''' we have

$$\begin{aligned}&-36cT\lambda^2-6T^3\lambda^2+60T\lambda^4+18T\lambda^2T'+19c\lambda\lambda'+15T^2\lambda\lambda'\\&-103\lambda^3\lambda'-15\lambda T'\lambda'+2T\lambda'^2-6\lambda^2T''-16T\lambda\lambda''-2\lambda'\lambda''+6\lambda\lambda'''=0.\end{aligned}\tag{3.32}$$

Differentiating (3.32) with respect to e_1, we have

$$\begin{aligned}&-36c\lambda^2T'-18T^2\lambda^2T'+60\lambda^4T'+18\lambda^2T'^2-72cT\lambda\lambda'-12T^3\lambda\lambda'\\&+240T\lambda^3\lambda'+66T\lambda T'\lambda'+19c\lambda'^2+15T^2\lambda'^2-309\lambda^2\lambda'^2-13T'\lambda'^2\\&+18T\lambda^2T''-27\lambda\lambda'T''+19c\lambda\lambda''+15T^2\lambda\lambda''-103\lambda^3\lambda''-31\lambda T'\lambda''\\&-12T\lambda'\lambda''-2\lambda''^2-6\lambda^2T'''-16T\lambda\lambda'''+4\lambda'\lambda'''+6\lambda\lambda''''=0.\end{aligned}\tag{3.33}$$

From (3.28) and (3.33), eliminating T''' we have

$$\begin{aligned}&54c^2\lambda^2+60cT^2\lambda^2+6T^4\lambda^2-288c\lambda^4-156T^2\lambda^4+234\lambda^6\\&-24c\lambda^2T'-18T^2\lambda^2T'+96\lambda^4T'+72cT\lambda\lambda'+12T^3\lambda\lambda'+108T\lambda^3\lambda'\\&-66T\lambda T'\lambda'-19c\lambda'^2-15T^2\lambda'^2+237\lambda^2\lambda'^2+13T'\lambda'^2+6T\lambda^2T''\\&+27\lambda\lambda'T''-19c\lambda\lambda''-15T^2\lambda\lambda''-41\lambda^3\lambda''+31\lambda T'\lambda''+12T\lambda'\lambda''\\&+2\lambda''^2+16T\lambda\lambda'''-4\lambda'\lambda'''-6\lambda\lambda''''=0.\end{aligned}\tag{3.34}$$

From (3.32) and (3.34), eliminating T'' we have

$$a_1T'-a_1T^2+a_2T+a_3=0,\tag{3.35}$$

where

$$\begin{aligned}a_1=&62\lambda^2\lambda''-109\lambda\lambda'^2+192\lambda^5-48c\lambda^3,\\a_2=&44\lambda^2\lambda'''-124\lambda\lambda'\lambda''+550\lambda^4\lambda'+18\lambda'^3-142c\lambda^2\lambda',\\a_3=&-12\lambda^2\lambda''''+46\lambda\lambda'\lambda'''-82\lambda^4\lambda''+4\lambda\lambda''^2\\&-18\lambda'^2\lambda''-453\lambda^3\lambda'^2+468\lambda^7+108c^2\lambda^3-576c\lambda^5+133c\lambda\lambda'^2-38c\lambda^2\lambda''.\end{aligned}$$

Case (i): $a_1=0, a_2\neq 0$. (3.35) becomes

$$a_2T + a_3 = 0.$$

so we have $T = -\dfrac{a_3}{a_2}$, then $e_i(T) = 0$ for $i = 2, 3, 4$.

Case (ii): $a_1 = 0, a_2 = 0$. We have

$$62\lambda\lambda'' - 109\lambda'^2 + 192\lambda^4 - 48c\lambda^2 = 0, \tag{3.36}$$

$$44\lambda^2\lambda''' - 124\lambda\lambda'\lambda'' + 550\lambda^4\lambda' + 18\lambda'^3 - 142c\lambda^2\lambda' = 0. \tag{3.37}$$

Differentiating (3.36) with respect to e_1, from (3.37) we have

$$-1145c\lambda^2 + 77\lambda^4 + 279\lambda'^2 - 206\lambda\lambda'' = 0. \tag{3.38}$$

From (3.36) and (3.38), eliminating λ'' we have

$$-40439c\lambda^2 + 22163\lambda^4 - 2578\lambda'^2 = 0. \tag{3.39}$$

Differentiating (3.39) with respect to e_1, we have

$$-40439c\lambda + 44326\lambda^3 - 2578\lambda'' = 0. \tag{3.40}$$

Substituting (3.39) and (3.40) into (3.36), we have

$$1776889c + 827421\lambda^2 = 0,$$

which is a contradiction.

Case (iii): $a_1 \neq 0$. Differentiating (3.35) with respect to e_1, we have

$$\begin{aligned}
&96cT\lambda^3T' - 384T\lambda^5T' + 324c^2\lambda^2\lambda' + 144cT^2\lambda^2\lambda' - 2880c\lambda^4\lambda'\\
&-960T^2\lambda^4\lambda' + 3276\lambda^6\lambda' - 286c\lambda^2T'\lambda' + 1510\lambda^4T'\lambda' - 284cT\lambda\lambda'^2\\
&+2200T\lambda^3\lambda'^2 + 218T\lambda T'\lambda'^2 + 133c\lambda'^3 + 109T^2\lambda'^3 - 1359\lambda^2\lambda'^3\\
&-91T'\lambda'^3 - 48c\lambda^3T'' + 192\lambda^5T'' - 109\lambda\lambda'^2T'' - 142cT\lambda^2\lambda''\\
&+550T\lambda^4\lambda'' - 124T\lambda^2T'\lambda'' + 190c\lambda\lambda'\lambda'' + 94T^2\lambda\lambda'\lambda'' - 1234\lambda^3\lambda'\lambda''\\
&-218\lambda T'\lambda'\lambda'' - 70T\lambda'^2\lambda'' + 62\lambda^2T''\lambda'' - 124T\lambda\lambda''^2 - 32\lambda'\lambda''^2\\
&-38c\lambda^2\lambda''' - 62T^2\lambda^2\lambda''' - 82\lambda^4\lambda''' + 106\lambda^2T'\lambda''' - 36T\lambda\lambda'\lambda'''\\
&+28\lambda'^2\lambda''' + 54\lambda\lambda''\lambda''' + 44T\lambda^2\lambda'''' + 22\lambda\lambda'\lambda'''' - 12\lambda^2\lambda''''' = 0.
\end{aligned} \tag{3.41}$$

From (3.32) and (3.41), eliminating T'' we have

$$\begin{aligned}
&1728c^2T\lambda^4 + 288cT^3\lambda^4 - 9792cT\lambda^6 - 1152T^3\lambda^6\\
&+11520T\lambda^8 - 288cT\lambda^4T' + 1152T\lambda^6T' + 1032c^2\lambda^3\lambda'\\
&+144cT^2\lambda^3\lambda' - 8688c\lambda^5\lambda' - 2880T^2\lambda^5\lambda' - 120\lambda^7\lambda'\\
&-996c\lambda^3T'\lambda' + 6180\lambda^5T'\lambda' + 2124cT\lambda^2\lambda'^2 + 654T^3\lambda^2\lambda'^2\\
&+7044T\lambda^4\lambda'^2 - 654T\lambda^2T'\lambda'^2 - 1273c\lambda\lambda'^3 - 981T^2\lambda\lambda'^3\\
&+3073\lambda^3\lambda'^3 + 1089\lambda T'\lambda'^3 - 218T\lambda'^4 - 2316cT\lambda^3\lambda''\\
&-372T^3\lambda^3\lambda'' + 3948T\lambda^5\lambda'' + 372T\lambda^3T'\lambda'' + 2414c\lambda^2\lambda'\lambda''\\
&+1494T^2\lambda^2\lambda'\lambda'' - 14174\lambda^4\lambda'\lambda'' - 2238\lambda^2T'\lambda'\lambda'' + 1448T\lambda\lambda'^2\lambda''\\
&+218\lambda'^3\lambda'' - 1736T\lambda^2\lambda''^2 - 316\lambda\lambda'\lambda''^2 - 516c\lambda^3\lambda'''\\
&-372T^2\lambda^3\lambda''' + 660\lambda^5\lambda''' + 636\lambda^3T'\lambda''' - 216T\lambda^2\lambda'\lambda'''\\
&-486\lambda\lambda'^2\lambda''' + 696\lambda^2\lambda''\lambda''' + 264T\lambda^3\lambda'''' + 132\lambda^2\lambda'\lambda'''' - 72\lambda^3\lambda''''' = 0.
\end{aligned} \tag{3.42}$$

From (3.35) and (3.42), eliminating T' we have

$$b_1 T + b_2 = 0, \tag{3.43}$$

where

$$\begin{aligned}
b_1 = &-6480c^3\lambda^6 + 63936c^2\lambda^8 - 204336c\lambda^{10} + 209088\lambda^{12} \\
&-40350c^2\lambda^4\lambda'^2 + 237750c\lambda^6\lambda'^2 - 309288\lambda^8\lambda'^2 + 4812c\lambda^2\lambda'^4 \\
&-227013\lambda^4\lambda'^4 + 520\lambda'^6 + 20898c^2\lambda^5\lambda'' - 125856c\lambda^7\lambda'' \\
&+174078\lambda^9\lambda'' - 25773c\lambda^3\lambda'^2\lambda'' + 302157\lambda^5\lambda'^2\lambda'' - 975\lambda\lambda'^4\lambda'' \\
&-5622c\lambda^4\lambda''^2 - 7830\lambda^6\lambda''^2 + 1350\lambda^2\lambda'^2\lambda''^2 - 13640\lambda^3\lambda''^3 \\
&+19719c\lambda^4\lambda'\lambda''' - 89523\lambda^6\lambda'\lambda''' - 717\lambda^2\lambda'^3\lambda''' + 18354\lambda^3\lambda'\lambda''\lambda''' \\
&-3498\lambda^4\lambda'''^2 - 2016c\lambda^5\lambda'''' + 8064\lambda^7\lambda'''' - 4578\lambda^3\lambda'^2\lambda'''' \\
&+2604\lambda^4\lambda''\lambda'''', \\
b_2 = \ & 7254c^3\lambda^5\lambda' - 78246c^2\lambda^7\lambda' + 295434c\lambda^9\lambda' - 364410\lambda^{11}\lambda' \\
&-4566c^2\lambda^3\lambda'^3 - 11349c\lambda^5\lambda'^3 + 361623\lambda^7\lambda'^3 - 760c\lambda\lambda'^5 \\
&+19795\lambda^3\lambda'^5 + 18996c^2\lambda^4\lambda'\lambda'' - 66342c\lambda^6\lambda'\lambda'' - 146838\lambda^8\lambda'\lambda'' \\
&-3926c\lambda^2\lambda'^3\lambda'' + 120509\lambda^4\lambda'^3\lambda'' - 520\lambda'^5\lambda'' + 10472c\lambda^3\lambda'\lambda''^2 \\
&-143462\lambda^5\lambda'\lambda''^2 + 415\lambda\lambda'^3\lambda''^2 - 1330\lambda^2\lambda'\lambda''^3 - 5490c^2\lambda^5\lambda''' \\
&+29448c\lambda^7\lambda''' - 21366\lambda^9\lambda''' + 5100c\lambda^3\lambda'^2\lambda''' - 20178\lambda^5\lambda'^2\lambda''' \\
&+360\lambda\lambda'^4\lambda''' - 5154c\lambda^4\lambda''\lambda''' + 28338\lambda^6\lambda''\lambda''' + 1050\lambda^2\lambda'^2\lambda''\lambda''' \\
&+5076\lambda^3\lambda''^2\lambda''' - 3657\lambda^3\lambda'\lambda'''^2 - 2286c\lambda^4\lambda'\lambda'''' + 12438\lambda^6\lambda'\lambda'''' \\
&-165\lambda^2\lambda'^3\lambda'''' - 2334\lambda^3\lambda'\lambda''\lambda'''' + 954\lambda^4\lambda'''\lambda'''' + 432c\lambda^5\lambda''''' \\
&-1728\lambda^7\lambda''''' + 981\lambda^3\lambda'^2\lambda''''' - 558\lambda^4\lambda''\lambda'''''.
\end{aligned}$$

If $b_1 \neq 0$, then $T = -\dfrac{b_2}{b_1}$, and $e_i(T) = 0$ for $i = 2,3,4$. If $b_1 = 0$, then $b_2 = 0$. Using the similar technique in [FHZ] and (3.15), (3.16), we can eliminate $\lambda''''', \lambda'''', \lambda''', \lambda'', \lambda'$ and get a nontrivial polynomial of λ with constant coefficients. Thus λ is a constant, which is a contradiction. □

Following the argument of Fu-Hong-Zhan in [FHZ], we can get

Lemma 3.5. *Suppose M^4 has four distinct principal curvatures, then $e_j(\lambda_i) = 0$ for $2 \le i, j \le 4$.*

Moreover, from (3.8) we have $\omega_{ii}^j = 0$ for $2 \le i, j \le 4$.

4. Proof of Theorem 1.1

We need the following two lemmas proved by Fu and Hong in [FH].

Lemma 4.1. *([FH])*

$$\omega_{23}^4 (\lambda_3 - \lambda_4) = \omega_{32}^4 (\lambda_2 - \lambda_4) = \omega_{43}^2 (\lambda_3 - \lambda_2), \tag{4.1}$$

$$\omega_{23}^4\omega_{32}^4 + \omega_{34}^2\omega_{43}^2 + \omega_{24}^3\omega_{42}^3 = 0, \tag{4.2}$$

$$\omega_{23}^4\left(\omega_{33}^1-\omega_{44}^1\right)=\omega_{32}^4\left(\omega_{22}^1-\omega_{44}^1\right)=\omega_{43}^2\left(\omega_{33}^1-\omega_{22}^1\right). \tag{4.3}$$

Lemma 4.2. *([FH])*

$$\omega_{22}^1\omega_{33}^1-2\omega_{23}^4\omega_{32}^4=-\lambda_2\lambda_3-c, \tag{4.4}$$

$$\omega_{22}^1\omega_{44}^1-2\omega_{24}^3\omega_{42}^3=-\lambda_2\lambda_4-c, \tag{4.5}$$

$$\omega_{33}^1\omega_{44}^1-2\omega_{34}^2\omega_{43}^2=-\lambda_3\lambda_4-c. \tag{4.6}$$

The proof of Theorem 1.1:

Let M^4 be a biharmonic hypersurface in $\mathbb{R}^5(c)(c\neq 0)$. There exists a smooth function a such that

$$\omega_{23}^4=a(\lambda_2-\lambda_3)(\lambda_2-\lambda_4). \tag{4.7}$$

From (4.1) and (4.7) we have

$$\omega_{34}^2=a(\lambda_3-\lambda_4)(\lambda_3-\lambda_2), \tag{4.8}$$

$$\omega_{42}^3=a(\lambda_4-\lambda_2)(\lambda_4-\lambda_3). \tag{4.9}$$

Taking $X=e_1,Y=e_2,Z=e_3$ in Gauss equation (2.4), we have

(4.10)
$$\begin{aligned}e_1(a)=&-\frac{a}{3(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}\{(5\lambda_2^2+\lambda_2\lambda_3-\lambda_3^2+\lambda_2\lambda_4-5\lambda_3\lambda_4-\lambda_4^2)\omega_{22}^1\\&+(-\lambda_2^2-4\lambda_2\lambda_3+4\lambda_3\lambda_4+\lambda_4^2)\omega_{33}^1\\&+(-\lambda_2^2+\lambda_3^2-4\lambda_2\lambda_4+4\lambda_3\lambda_4)\omega_{44}^1\}.\end{aligned}$$

Taking $X=e_1,Y=e_3,Z=e_2$ in Gauss equation (2.4), we have

$$\begin{aligned}e_1(a)=&-\frac{a}{3(\lambda_2-\lambda_3)(\lambda_3-\lambda_4)}\{(4\lambda_2\lambda_3+\lambda_3^2-4\lambda_2\lambda_4-\lambda_4^2)\omega_{22}^1\\&+(\lambda_2^2-\lambda_2\lambda_3-5\lambda_3^2+5\lambda_2\lambda_4-\lambda_3\lambda_4+\lambda_4^2)\omega_{33}^1\\&+(-\lambda_2^2+\lambda_3^2-4\lambda_2\lambda_4+4\lambda_3\lambda_4)\omega_{44}^1\}.\end{aligned} \tag{4.11}$$

Taking $X=e_1,Y=e_4,Z=e_2$ in Gauss equation (2.4), we have

$$\begin{aligned}e_1(a)=&-\frac{a}{3(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}\{(4\lambda_2\lambda_3+\lambda_3^2-4\lambda_2\lambda_4-\lambda_4^2)\omega_{22}^1\\&+(\lambda_2^2+4\lambda_2\lambda_3-4\lambda_3\lambda_4-\lambda_4^2)\omega_{33}^1\\&+(-\lambda_2^2-5\lambda_2\lambda_3-\lambda_3^2+\lambda_2\lambda_4+\lambda_3\lambda_4+5\lambda_4^2)\omega_{44}^1\}.\end{aligned} \tag{4.12}$$

(4.10)+(4.11)+(4.12) implies

$$e_1(a)=\frac{a}{9(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)(\lambda_3-\lambda_4)}(k_2\omega_{22}^1-k_3\omega_{33}^1+k_4\omega_{44}^1), \tag{4.13}$$

where

$$\begin{aligned}k_2=&(\lambda_3-\lambda_4)(-13\lambda_2^2+2\lambda_3^2+7\lambda_3\lambda_4+2\lambda_4^2+\lambda_2\lambda_3+\lambda_2\lambda_4),\\k_3=&(\lambda_2-\lambda_4)(2\lambda_2^2-13\lambda_3^2+\lambda_3\lambda_4+2\lambda_4^2+\lambda_2\lambda_3+7\lambda_2\lambda_4),\\k_4=&(\lambda_2-\lambda_3)(2\lambda_2^2+7\lambda_2\lambda_3+2\lambda_3^2+\lambda_2\lambda_4+\lambda_3\lambda_4-13\lambda_4^2).\end{aligned}$$

Taking $X = e_2, Y = e_4, Z = e_1$ in Gauss equation (2.4), we have

$$a\{(\lambda_3 - \lambda_4)\omega_{22}^1 - (\lambda_2 - \lambda_4)\omega_{33}^1 + (\lambda_2 - \lambda_3)\omega_{44}^1\} = 0. \tag{4.14}$$

We can rewrite the biharmonic equation (2.7) as

$$-e_1e_1(\lambda_1) + e_1(\lambda_1)(\omega_{22}^1 + \omega_{33}^1 + \omega_{44}^1) + \lambda_1(8c + 4\lambda_1^2 - R) = 0. \tag{4.15}$$

Taking $X = e_2, Y = e_4, Z = e_2$ in Gauss equation (2.4), we have $e_2(a) = 0$ and

$$\omega_{22}^1\omega_{44}^1 + 2a^2(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)^2(\lambda_3 - \lambda_4) + \lambda_2\lambda_4 + c = 0. \tag{4.16}$$

By symmetry, we have $e_3(a) = e_4(a) = 0$ and

$$\omega_{33}^1\omega_{44}^1 - 2a^2(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4)^2 + \lambda_3\lambda_4 + c = 0, \tag{4.17}$$

$$\omega_{22}^1\omega_{33}^1 + \lambda_2\lambda_3 - 2a^2(\lambda_2 - \lambda_3)^2(\lambda_2 - \lambda_4)(\lambda_3 - \lambda_4) + c = 0. \tag{4.18}$$

Here we introduce the new variables y_1, y_2, y_3 by

$$y_1 = \lambda_2 + \lambda_3 + \lambda_4 = -3\lambda_1, \tag{4.19}$$

$$y_2 = \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4, \tag{4.20}$$

$$y_3 = \lambda_2\lambda_3\lambda_4. \tag{4.21}$$

Next we consider two cases.

Case A: $a \neq 0$. From (4.14) we have

$$(\lambda_3 - \lambda_4)\omega_{22}^1 - (\lambda_2 - \lambda_4)\omega_{33}^1 + (\lambda_2 - \lambda_3)\omega_{44}^1 = 0. \tag{4.22}$$

From (4.1) and (4.3), there exists smooth functions κ and τ such that

$$\omega_{ii}^1 = \kappa\lambda_i + \tau, \qquad i = 2, 3, 4. \tag{4.23}$$

From (2.13) and (2.14), we have

$$e_1(\kappa) = -\frac{1}{3}(1 + \kappa^2)(\lambda_2 + \lambda_3 + \lambda_4) + \kappa\tau, \tag{4.24}$$

$$e_1(\tau) = c - \frac{1}{3}\kappa\tau(\lambda_2 + \lambda_3 + \lambda_4) + \tau^2. \tag{4.25}$$

(4.16)+(4.17)+(4.18) implies

$$3c + (1 + \kappa^2)y_2 + 2\kappa\tau y_1 + 3\tau^2 = 0. \tag{4.26}$$

From (4.26) we can solve for y_2

$$y_2 = -\frac{1}{1 + \kappa^2}(3c + 2\kappa\tau y_1 + 3\tau^2). \tag{4.27}$$

From (2.13), (2.14), (4.23), we have

$$e_1(y_1) = \frac{4}{3}\kappa y_1^2 - 2\kappa y_2 + 2\tau y_1. \tag{4.28}$$

From (4.24), (4.25), (4.27), (4.28), we have

$$\begin{aligned} e_1(y_1^2 - 2y_2) =& \frac{4}{3(1 + \kappa^2)^2}(2y_1^3\kappa(1 + \kappa^2)^2 + 15cy_1(\kappa + \kappa^3) + 9c(1 + 2\kappa^2)\tau \\ &+ y_1^2(1 + \kappa^2)(2 + 13\kappa^2)\tau + 3y_1\kappa(7 + 9\kappa^2)\tau^2 + 9(1 + 2\kappa^2)\tau^3). \end{aligned} \tag{4.29}$$

From (2.13), (2.14), (4.23), we have

$$e_1(\lambda_2^2 + \lambda_3^2 + \lambda_4^2) = \frac{2}{3}(4y_1^3\kappa - 11y_1y_2\kappa + 9y_3\kappa + 4y_1^2\tau - 6y_2\tau). \tag{4.30}$$

From (4.29) and (4.30), we have

$$-cy_1-3y_3-cy_1\kappa^2-6y_3\kappa^2-3y_3\kappa^4+6c\kappa\tau-y_1\tau^2+3y_1\kappa^2\tau^2+6\kappa\tau^3=0. \tag{4.31}$$

From (4.31), we can solve for y_3

$$y_3=\frac{1}{3(1+\kappa^2)^2}(-c(1+\kappa^2)y_1+6c\kappa\tau+(3\kappa^2-1)\tau^2y_1+6\kappa\tau^3). \tag{4.32}$$

Using (4.23), (4.15) can be rewritten as

$$27cy_1-7y_1^3+12y_1y_2+8y_1^3\kappa^2-27y_1y_2\kappa^2+27y_3\kappa^2+6y_1^2\kappa\tau-18y_2\kappa\tau=0. \tag{4.33}$$

Differentiating (4.33) with respect to e_1, we have

$$\begin{aligned}&-684cy_1^2\kappa-100y_1^4\kappa+2106c^2\kappa^3+324cy_1^2\kappa^3\\&-120y_1^4\kappa^3+1008cy_1^2\kappa^5+60y_1^4\kappa^5+80y_1^4\kappa^7\\&-324cy_1\tau-108y_1^3\tau+648cy_1\kappa^2\tau-618y_1^3\kappa^2\tau\\&+4806cy_1\kappa^4\tau+330y_1^3\kappa^4\tau+840y_1^3\kappa^6\tau-1062y_1^2\kappa\tau^2\\&+5346c\kappa^3\tau^2+558y_1^2\kappa^3\tau^2+3240y_1^2\kappa^5\tau^2-486y_1\tau^3\\&+324y_1\kappa^2\tau^3+5400y_1\kappa^4\tau^3+3240\kappa^3\tau^4=0.\end{aligned} \tag{4.34}$$

Differentiating (4.34) with respect to e_1, we have

$$\begin{aligned}&-972c^2y_1+360cy_1^3+100y_1^5-29970c^2y_1\kappa^2\\&-14454cy_1^3\kappa^2-1040y_1^5\kappa^2+15390c^2y_1\kappa^4-18684cy_1^3\kappa^4\\&-3000y_1^5\kappa^4+44388c^2y_1\kappa^6+7434cy_1^3\kappa^6-1760y_1^5\kappa^6\\&+11304cy_1^3\kappa^8+820y_1^5\kappa^8+720y_1^5\kappa^{10}-5832c^2\kappa\tau\\&-24732cy_1^2\kappa\tau-2652y_1^4\kappa\tau+62694c^2\kappa^3\tau-77112cy_1^2\kappa^3\tau\\&-18294y_1^4\kappa^3\tau+137538c^2\kappa^5\tau+59022cy_1^2\kappa^5\tau-16992y_1^4\kappa^5\tau\\&+111402cy_1^2\kappa^7\tau+10290y_1^4\kappa^7\tau+11640y_1^4\kappa^9\tau-7290cy_1\tau^2\\&-1206y_1^3\tau^2-77436cy_1\kappa^2\tau^2-35964y_1^3\kappa^2\tau^2+158922cy_1\kappa^4\tau^2\\&-58194y_1^3\kappa^4\tau^2+333396cy_1\kappa^6\tau^2+48564y_1^3\kappa^6\tau^2+72000y_1^3\kappa^8\tau^2\\&-14580c\kappa\tau^3-29268y_1^2\kappa\tau^3+136566c\kappa^3\tau^3-89154y_1^2\kappa^3\tau^3\\&+302778c\kappa^5\tau^3+115074y_1^2\kappa^5\tau^3+213840y_1^2\kappa^7\tau^3-7290y_1\tau^4\\&-54270y_1\kappa^2\tau^4+144180y_1\kappa^4\tau^4+304560y_1\kappa^6\tau^4-8748\kappa\tau^5\\&+73872\kappa^3\tau^5+165240\kappa^5\tau^5=0.\end{aligned} \tag{4.35}$$

From (4.33) and (4.34), eliminating τ we have

$$\sum_{m=0}^{8}P_{2m}\kappa^{2m}=0, \tag{4.36}$$

where

$$
\begin{aligned}
P_0 =& -164025c^3y_1^4 - 149445c^2y_1^6 - 17739cy_1^8 - 567y_1^{10},\\
P_2 =& -157464c^4y_1^2 + 988524c^3y_1^4 - 1879848c^2y_1^6 + 449388cy_1^8 - 18072y_1^{10},\\
P_4 =& \ \ 6114852c^4y_1^2 + 8627715c^3y_1^4 - 4223745c^2y_1^6 + 79353cy_1^8 - 8223y_1^{10},\\
P_6 =& \ \ 6141096c^5 - 46189440c^4y_1^2 + 84187836c^3y_1^4 - 36738684c^2y_1^6\\
& +4146300cy_1^8 - 172964y_1^{10},\\
P_8 =& \ \ 6141096c^5 - 223047756c^4y_1^2 + 137387340c^3y_1^4 - 83992140c^2y_1^6\\
& +12535884cy_1^8 - 488104y_1^{10},\\
P_{10} =& -35311302c^5 + 54154494c^4y_1^2 - 49487436c^3y_1^4 - 69538500c^2y_1^6\\
& +11334546cy_1^8 - 413882y_1^{10},\\
P_{12} =& -18423288c^5 + 1192107456c^4y_1^2 - 355203792c^3y_1^4 - 6360768c^2y_1^6\\
& +3415752cy_1^8 - 121088y_1^{10},\\
P_{14} =& \ \ 55269864c^5 + 1533226968c^4y_1^2 - 345382704c^3y_1^4 + 18635184c^2y_1^6\\
& +46152cy_1^8 - 15496y_1^{10},\\
P_{16} =& \ \ 587865600c^4y_1^2 - 105629184c^3y_1^4 + 5505408c^2y_1^6 - 36864cy_1^8 - 2432y_1^{10}.
\end{aligned}
$$

From (4.33) and (4.35), eliminating τ we have

$$
\sum_{m=0}^{13} Q_{2m}\kappa^{2m} = 0, \tag{4.37}
$$

where

$$
\begin{aligned}
Q_0 =& \ \ 89813529c^4y_1^4 + 623321244c^3y_1^6 + 1108270998c^2y_1^8\\
& +92935836cy_1^{10} + 1996569y_1^{12},\\
Q_2 =& \ \ 7620155352c^5y_1^2 + 18958390038c^4y_1^4 - 2051878392c^3y_1^6\\
& +18386833140c^2y_1^8 - 3132184464cy_1^{10} + 199117734y_1^{12},\\
Q_4 =& \ \ 2550916800c^6 - 3713237316c^5y_1^2 + 396683144775c^4y_1^4\\
& -281478327804c^3y_1^6 + 162873940914c^2y_1^8 - 23218046136cy_1^{10}\\
& +1170436927y_1^{12},\\
Q_6 =& \ \ 21533989320c^6 - 253030869900c^5y_1^2 + 2983350761604c^4y_1^4\\
& -2233514241576c^3y_1^6 + 897578511552c^2y_1^8 - 110846416140cy_1^{10}\\
& +5032123316y_1^{12},\\
Q_8 =& \ \ 169801776792c^6 - 4722853102668c^5y_1^2 + 19054644126723c^4y_1^4\\
& -11292218972532c^3y_1^6 + 3695744541258c^2y_1^8 - 455003760600cy_1^{10}\\
& +19351326643y_1^{12},\\
Q_{10} =& \ \ 554978521890c^6 - 30549126042468c^5y_1^2 + 83834398712052c^4y_1^4\\
& -42196006098576c^3y_1^6 + 11288209375170c^2y_1^8 - 1281849786108cy_1^{10}\\
& +48941467416y_1^{12},
\end{aligned}
$$

$$\begin{aligned}
Q_{12} =& -310363669764c^6 - 57933464951484c^5y_1^2 + 211026987943857c^4y_1^4 \\
&-105270974689716c^3y_1^6 + 23637191018346c^2y_1^8 - 2258485653816cy_1^{10} \\
&+72974080657y_1^{12}, \\
Q_{14} =& -4681635955884c^6 + 135848247734196c^5y_1^2 + 275115121235820c^4y_1^4 \\
&-165192051148632c^3y_1^6 + 32239821679308c^2y_1^8 - 2470452821820cy_1^{10} \\
&+62750560596y_1^{12}, \\
Q_{16} =& -5901463112004c^6 + 844079357467764c^5y_1^2 + 106282529065260c^4y_1^4 \\
&-157860516846024c^3y_1^6 + 27762314926644c^2y_1^8 - 1657847338092cy_1^{10} \\
&+29261950884y_1^{12}, \\
Q_{18} =& \ 6145346701314c^6 + 1705659405062004c^5y_1^2 - 177158182581522c^4y_1^4 \\
&-87108572161368c^3y_1^6 + 14676006828366c^2y_1^8 - 656114701212cy_1^{10} \\
&+5566963010y_1^{12}, \\
Q_{20} =& \ 17073860098680c^6 + 1765021213832760c^5y_1^2 - 262032349848480c^4y_1^4 \\
&-25663081979952c^3y_1^6 + 4778135352216c^2y_1^8 - 147573429768cy_1^{10} \\
&-751835760y_1^{12}, \\
Q_{22} =& \ 9338536521864c^6 + 941182952949648c^5y_1^2 - 130448854754568c^4y_1^4 \\
&-5532982699104c^3y_1^6 + 1153495908792c^2y_1^8 - 23152890672cy_1^{10} \\
&-611376440y_1^{12}, \\
Q_{24} =& \ 205652969940096c^5y_1^2 - 16668899694720c^4y_1^4 - 2570855428224c^3y_1^6 \\
&+271365697152c^2y_1^8 - 3749437440cy_1^{10} - 144665088y_1^{12}, \\
Q_{26} =& \ 3762339840000c^4y_1^4 - 676026777600c^3y_1^6 + 35234611200c^2y_1^8 \\
&-235929600cy_1^{10} - 15564800y_1^{12}.
\end{aligned}$$

From (4.36), (4.37), eliminating κ we get a polynomial of y_1 with constant coefficients of degree 428 and then y_1 is a constant, which is a contradiction.

Case B: $a = 0$. (4.4)-(4.6) becomes

$$\omega_{22}^1\omega_{33}^1 = -\lambda_2\lambda_3 - c, \tag{4.38}$$

$$\omega_{22}^1\omega_{44}^1 = -\lambda_2\lambda_4 - c, \tag{4.39}$$

$$\omega_{33}^1\omega_{44}^1 = -\lambda_3\lambda_4 - c. \tag{4.40}$$

From the assumption $c \neq 0$ and (4.38)-(4.40), we can get $\omega_{ii}^1 \neq 0$ and $\lambda_i\lambda_j + c \neq 0$ for $2 \leq i, j \leq 4, i \neq j$, then

$$\omega_{33}^1 = -\frac{\lambda_2\lambda_3 + c}{\omega_{22}^1}, \tag{4.41}$$

$$\omega_{44}^1 = -\frac{\lambda_2\lambda_4 + c}{\omega_{22}^1}, \tag{4.42}$$

$$(\omega_{22}^1)^2 = -\frac{(\lambda_2\lambda_3 + c)(\lambda_2\lambda_4 + c)}{\lambda_3\lambda_4 + c}. \tag{4.43}$$

From (2.13), (2.14), (4.41)-(4.43), we have

$$e_1(y_1) = -\frac{1}{3\omega_{22}^1(\lambda_3\lambda_4 + c)}(6c^2y_1 + 5cy_1y_2 - 9cy_3 + 4y_1^2y_3 - 6y_2y_3), \tag{4.44}$$

$$e_1(y_2) = -\frac{1}{3\omega_{22}^1(\lambda_3\lambda_4 + c)}(2c^2y_1^2 + 6c^2y_2 + cy_1^2y_2 + 6cy_2^2 - 3cy_1y_3 + 5y_1y_2y_3 - 9y_3^2), \tag{4.45}$$

$$e_1(y_3) = -\frac{1}{3\omega_{22}^1(\lambda_3\lambda_4 + c)}(c^2y_1y_2 + 9c^2y_3 + 2cy_1^2y_3 + 6cy_2y_3 + 6y_1y_3^2). \tag{4.46}$$

Using (4.41)-(4.43), (4.15) can be rewritten as

$$\begin{aligned} &27c^4y_1 - 7c^3y_1^3 + 36c^3y_1y_2 - 7c^2y_1^3y_2 + 7c^2y_1y_2^2 + 27c^3y_3 \\ &+27c^2y_1^2y_3 - 7cy_1^4y_3 + 45c^2y_2y_3 + cy_1^2y_2y_3 + 24cy_2^2y_3 + 54cy_1y_3^2 \\ &-15y_1^3y_3^2 + 39y_1y_2y_3^2 - 27y_3^3 = 0. \end{aligned} \tag{4.47}$$

Differentiating (4.47) with respect to e_1, we have

$$\begin{aligned} &162c^6y_1 - 54c^5y_1^3 - 14c^4y_1^5 + 594c^5y_1y_2 - 182c^4y_1^3y_2 - 14c^3y_1^5y_2 \\ &+567c^4y_1y_2^2 - 132c^3y_1^3y_2^2 + 143c^3y_1y_2^3 + 900c^4y_1^2y_3 - 238c^3y_1^4y_3 - 14c^2y_1^6y_3 \\ &+351c^4y_2y_3 + 1395c^3y_1^2y_2y_3 - 328c^2y_1^4y_2y_3 + 765c^3y_2^2y_3 + 420c^2y_1^2y_2^2y_3 + 390c^2y_2^3y_3 \\ &+513c^3y_1y_3^2 + 444c^2y_1^3y_3^2 - 214cy_1^5y_3^2 + 1890c^2y_1y_2y_3^2 - 23cy_1^3y_2y_3^2 + 1269cy_1y_2^2y_3^2 \\ &-1620c^2y_3^3 + 981cy_1^2y_3^3 - 360y_1^4y_3^3 - 1593cy_2y_3^3 + 1089y_1^2y_2y_3^3 - 234y_2^2y_3^3 - 837y_1y_3^4 = 0. \end{aligned} \tag{4.48}$$

Differentiating (4.48) with respect to e_1, we have

$$\begin{aligned} &972c^8y_1 + 216c^7y_1^3 - 784c^6y_1^5 - 28c^5y_1^7 \\ &+7938c^7y_1y_2 - 1416c^6y_1^3y_2 - 1802c^5y_1^5y_2 - 28c^4y_1^7y_2 \\ &+17091c^6y_1y_2^2 - 4395c^5y_1^3y_2^2 - 1026c^4y_1^5y_2^2 + 13836c^5y_1y_2^3 \\ &-2715c^4y_1^3y_2^3 + 3679c^4y_1y_2^4 - 1458c^7y_3 + 19926c^6y_1^2y_3 \\ &-2736c^5y_1^4y_3 - 2000c^4y_1^6y_3 - 28c^3y_1^8y_3 - 1053c^6y_2y_3 \\ &+65034c^5y_1^2y_2y_3 - 13479c^4y_1^4y_2y_3 - 2266c^3y_1^6y_2y_3 + 11610c^5y_2^2y_3 \\ &+67221c^4y_1^2y_2^2y_3 - 11346c^3y_1^4y_2^2y_3 + 19611c^4y_2^3y_3 + 21341c^3y_1^2y_2^3y_3 \\ &+8502c^3y_2^4y_3 - 10287c^5y_1y_3^2 + 40437c^4y_1^3y_3^2 - 11912c^3y_1^5y_3^2 \\ &-1276c^2y_1^7y_3^2 + 13716c^4y_1y_2y_3^2 + 72372c^3y_1^3y_2y_3^2 - 17465c^2y_1^5y_2y_3^2 \\ &+64944c^3y_1y_2^2y_3^2 + 28074c^2y_1^3y_2^2y_3^2 + 39249c^2y_1y_2^3y_3^2 - 51516c^4y_3^3 \\ &+3348c^3y_1^2y_3^3 + 13011c^2y_1^4y_3^3 - 9008cy_1^6y_3^3 - 115587c^3y_2y_3^3 \\ &+78183c^2y_1^2y_2y_3^3 - 304cy_1^4y_2y_3^3 - 80649c^2y_2^2y_3^3 + 68562cy_1^2y_2^2y_3^3 \\ &-14634cy_2^3y_3^3 - 94203c^2y_1y_3^4 + 28710cy_1^3y_3^4 - 12240y_1^5y_3^4 \\ &-113724cy_1y_2y_3^4 + 42399y_1^3y_2y_3^4 - 19620y_1y_2^2y_3^4 + 21870cy_3^5 \\ &-33237y_1^2y_3^5 + 9234y_2y_3^5 = 0. \end{aligned} \tag{4.49}$$

From(4.47), (4.48), eliminating y_2 we have

$$
\begin{aligned}
&(cy_1-3y_3)(c^3-cy_1y_3-2y_3^2)(961551c^7y_1^4-550638c^6y_1^6\\
&+92295c^5y_1^8+1078c^4y_1^{10}+3241134c^6y_1^3y_3\\
&-3718062c^5y_1^5y_3+852336c^4y_1^7y_3-16548c^3y_1^9y_3\\
&+2071089c^5y_1^2y_3^2-10454265c^4y_1^4y_3^2+4078449c^3y_1^6y_3^2\\
&-228795c^2y_1^8y_3^2-4234032c^4y_1y_3^3-9157698c^3y_1^3y_3^3\\
&+9454374c^2y_1^5y_3^3-621684cy_1^7y_3^3-4185918c^3y_3^4\\
&+4435965c^2y_1^2y_3^4+10034766cy_1^4y_3^4-479115y_1^6y_3^4\\
&+7050888cy_1y_3^5+4106700y_1^3y_3^5+2217618y_3^6)=0.
\end{aligned}
\tag{4.50}
$$

From(4.47), (4.49), eliminating y_2 we have

$$
\begin{aligned}
&(cy_1-3y_3)(c^3-cy_1y_3-2y_3^2)(121234158c^{12}y_1^5+4502127582c^{11}y_1^7\\
&-4130694792c^{10}y_1^9+1312681986c^9y_1^{11}-138793970c^8y_1^{13}\\
&-1131900c^7y_1^{15}-6493539798c^{11}y_1^4y_3+17508226059c^{10}y_1^6y_3\\
&-26579271393c^9y_1^8y_3+12168575541c^8y_1^{10}y_3-1901077395c^7y_1^{12}y_3\\
&+41618150c^6y_1^{14}y_3-19265759766c^{10}y_1^3y_3^2+4495598658c^9y_1^5y_3^2\\
&-42049903410c^8y_1^7y_3^2+42284538216c^7y_1^9y_3^2-10550910060c^6y_1^{11}y_3^2\\
&+560212114c^5y_1^{13}y_3^2-6970891914c^9y_1^2y_3^3-13811443002c^8y_1^4y_3^3\\
&+83680475085c^7y_1^6y_3^3+21298486113c^6y_1^8y_3^3-21213196659c^5y_1^{10}y_3^3\\
&+1830863937c^4y_1^{12}y_3^3+30101526828c^8y_1y_3^4+45869158224c^7y_1^3y_3^4\\
&+279229797288c^6y_1^5y_3^4-232982629074c^5y_1^7y_3^4+23242316040c^4y_1^9y_3^4\\
&-602724402c^3y_1^{11}y_3^4+23531498892c^7y_3^5+95793204717c^6y_1^2y_3^5\\
&-35556908661c^5y_1^4y_3^5-505869269463c^4y_1^6y_3^5+173198646333c^3y_1^8y_3^5\\
&-12359609514c^2y_1^{10}y_3^5-28067406876c^5y_1y_3^6-471693791502c^4y_1^3y_3^6\\
&-188063792496c^3y_1^5y_3^6+305129824146c^2y_1^7y_3^6-20136670848cy_1^9y_3^6\\
&-99047493522c^4y_3^7-128261179269c^3y_1^2y_3^7+309915643131c^2y_1^4y_3^7\\
&+254417953509cy_1^6y_3^7-9828590625y_1^8y_3^7+92615758344c^2y_1y_3^8\\
&+252963396576cy_1^3y_3^8+90957709080y_1^5y_3^8\\
&+45154416006cy_3^9+25402919166y_1^2y_3^9)=0.
\end{aligned}
\tag{4.51}
$$

Next we check three subcases.

Case B.1: $cy_1-3y_3=0$. Substituting $y_3=\dfrac{cy_1}{3}$ into (4.47), (4.48) gives

$$(9c+4y_1^2+9y_2)(12c-3y_1^2+5y_2)=0, \tag{4.52}$$

$$(9c+4y_1^2+9y_2)(162c^2+171cy_1^2-94y_1^4+549cy_2+19y_1^2y_2+273y_2^2)=0. \tag{4.53}$$

From (4.52), (4.53), we consider the two following subcases:
(i) $9c+4y_1^2+9y_2\neq 0$, or
(ii) $9c+4y_1^2+9y_2=0$.

If (i) holds, by eliminating y_2 we can get that y_1 satisfies a polynomial with constant coefficients and y_1 is a constant. If (ii) holds, taking $9c+4y_1^2+9y_2=0$ into (4.44) we get $e_1(y_1)=0$ and y_1 is also a constant. So in both subcases we get that y_1 is a constant, which is a contradiction.

Case B.2: $c^3-cy_1y_3-2y_3^2=0$. Since $c\neq 0$, we solve for y_1 in terms of y_3. Substituting $y_1=\dfrac{c^3-2y_3^2}{cy_3}$ into (4.47) and (4.48), we have

$$(2c^3+c^2y_2-y_3^2)(7c^9-65c^6y_3^2-7c^5y_2y_3^2+89c^3y_3^4-10c^2y_2y_3^4+8y_3^6)=0, \tag{4.54}$$

$$\begin{aligned}&(2c^3+c^2y_2-y_3^2)(14c^{15}+106c^{12}y_3^2+132c^{11}y_2y_3^2-2004c^9y_3^4\\&-1493c^8y_2y_3^4-143c^7y_2^2y_3^4+5885c^6y_3^6+2429c^5y_2y_3^6\\&-104c^4y_2^2y_3^6-3842c^3y_3^8-68c^2y_2y_3^8+192y_3^{10})=0.\end{aligned} \tag{4.55}$$

From (4.54), (4.55), we consider the two following subcases:
(i) $2c^3+c^2y_2-y_3^2\neq 0$, or
(ii) $2c^3+c^2y_2-y_3^2=0$.
If (i) holds, by eliminating y_2 we can get that y_3 satisfies a polynomial with constant coefficients and y_3 is a constant. If (ii) holds, taking $2c^3+c^2y_2-y_3^2=0$ into (4.46) we get $e_1(y_3)=0$ and y_3 is also a constant. So in both subcases we get that y_3 is a constant, then y_1 is a constant, which is a contradiction.

Case B.3: $(cy_1-3y_3)(c^3-cy_1y_3-2y_3^2)\neq 0$. From (4.50), (4.51), we have

$$\begin{aligned}&961551c^7y_1^4-550638c^6y_1^6+92295c^5y_1^8+1078c^4y_1^{10}\\&+3241134c^6y_1^3y_3-3718062c^5y_1^5y_3+852336c^4y_1^7y_3-16548c^3y_1^9y_3\\&+2071089c^5y_1^2y_3^2-10454265c^4y_1^4y_3^2+4078449c^3y_1^6y_3^2-228795c^2y_1^8y_3^2\\&-4234032c^4y_1y_3^3-9157698c^3y_1^3y_3^3+9454374c^2y_1^5y_3^3-621684cy_1^7y_3^3\\&-4185918c^3y_3^4+4435965c^2y_1^2y_3^4+10034766cy_1^4y_3^4-479115y_1^6y_3^4\\&+7050888cy_1y_3^5+4106700y_1^3y_3^5+2217618y_3^6=0,\end{aligned} \tag{4.56}$$

$$\begin{aligned}&121234158c^{12}y_1^5+4502127582c^{11}y_1^7-4130694792c^{10}y_1^9\\&+1312681986c^9y_1^{11}-138793970c^8y_1^{13}-1131900c^7y_1^{15}\\&-6493539798c^{11}y_1^4y_3+17508226059c^{10}y_1^6y_3-26579271393c^9y_1^8y_3\\&+12168575541c^8y_1^{10}y_3-1901077395c^7y_1^{12}y_3+41618150c^6y_1^{14}y_3\\&-19265759766c^{10}y_1^3y_3^2+4495598658c^9y_1^5y_3^2-42049903410c^8y_1^7y_3^2\\&+42284538216c^7y_1^9y_3^2-10550910060c^6y_1^{11}y_3^2+560212114c^5y_1^{13}y_3^2\\&-6970891914c^9y_1^2y_3^3-13811443002c^8y_1^4y_3^3+83680475085c^7y_1^6y_3^3\\&+21298486113c^6y_1^8y_3^3-21213196659c^5y_1^{10}y_3^3+1830863937c^4y_1^{12}y_3^3\\&+30101526828c^8y_1y_3^4+45869158224c^7y_1^3y_3^4+279229797288c^6y_1^5y_3^4\\&-232982629074c^5y_1^7y_3^4+23242316040c^4y_1^9y_3^4-602724402c^3y_1^{11}y_3^4\\&+23531498892c^7y_3^5+95793204717c^6y_1^2y_3^5-35556908661c^5y_1^4y_3^5\\&-505869269463c^4y_1^6y_3^5+173198646333c^3y_1^8y_3^5-12359609514c^2y_1^{10}y_3^5\end{aligned} \tag{4.57}$$

$$
\begin{aligned}
&-28067406876c^5y_1y_3^6-471693791502c^4y_1^3y_3^6-188063792496c^3y_1^5y_3^6\\
&+305129824146c^2y_1^7y_3^6-20136670848cy_1^9y_3^6-99047493522c^4y_3^7\\
&-128261179269c^3y_1^2y_3^7+309915643131c^2y_1^4y_3^7+254417953509cy_1^6y_3^7\\
&-9828590625y_1^8y_3^7+92615758344c^2y_1y_3^8+252963396576cy_1^3y_3^8\\
&+90957709080y_1^5y_3^8+45154416006cy_3^9+25402919166y_1^2y_3^9=0.
\end{aligned}
$$

From (4.56) and (4.57), eliminating y_3 we get a polynomial of y_1 with constant coefficients of degree 118 and then y_1 is a constant, that is, λ_1 is constant, which is a contradiction. Therefore we complete the proof of Theorem 1.1.

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