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ON PRODUCT AFFINE HYPERSPHERES IN $\mathbb{R}^{n+1}$

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Abstract. In this paper, we study locally strongly convex affine hyperspheres in the unimodular affine space $\mathbb{R}^{n+1}$ which, as Riemannian manifolds, are locally isometric to the Riemannian product of two Riemannian manifolds both possessing constant sectional curvatures. As the main result, a complete classification of such affine hyperspheres is established. Moreover, as direct consequences, affine hyperspheres of dimensions 3 and 4 with parallel Ricci tensor are also classified.

1. Introduction

In this paper, we study locally strongly convex affine hypersurfaces in the unimodular affine space $\mathbb{R}^{n+1}$. It is well known that on a nondegenerate affine hypersurface M^n in $\mathbb{R}^{n+1}$ there exists a canonical transversal vector field ξ which is called the affine normal vector field. If all the affine normal lines of M^n pass through a fixed point (resp. if all the affine normals are parallel), M^n is called a proper (resp. improper) affine hypersphere. The second fundamental form h associated with the affine normal vector field is called the (Blaschke) affine metric. As we consider only locally strongly convex affine hypersurfaces, the affine metric h is assumed to be positive definite, and in such situation, the proper affine hyperspheres are divided into two classes, i.e., the elliptic affine hyperspheres and the hyperbolic ones.

The affine hyperspheres form a very important class of affine hypersurfaces. From the global point of view that the affine metric h is complete, the improper (also called parabolic) affine hypersphere has to be the elliptic paraboloid, whereas the elliptic affine hypersphere has to be the ellipsoid. However, the class of locally strongly convex hyperbolic affine hyperspheres is very large and have been widely studied, see amongst others the works of [3, 6, 8, 13, 14, 15, 23] and also the recent monograph [17], or the survey paper [19]. Indeed, even assuming global conditions, the class of hyperbolic affine hyperspheres is surprisingly large, and one is still far from having a complete geometric understanding of them for all dimensions.

On the other hand, affine hyperspheres with constant sectional curvature are classified in [16] and [26] (see also [24, 25] for the general non-degenerate case), whereas in [12] it was further shown that all locally strongly convex Einstein affine hyperspheres in $\mathbb{R}^5$ are of constant sectional curvature. Contrary to the result of [12], the cases for locally strongly convex Einstein affine hyperspheres in $\mathbb{R}^{n+1}$ with $n \geq 5$ are different, and there exist Einstein affine hyperspheres which are not of

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constant sectional curvatures; actually, such examples occur for the standard embeddings of the noncompact symmetric spaces $\mathrm{E}_{6(-26)}/\mathrm{F}_4$, and $\mathrm{SL}(m,\mathbb{R})/\mathrm{SO}(m)$, $\mathrm{SL}(m,\mathbb{C})/\mathrm{SU}(m)$, $\mathrm{SU}^*(2m)/\mathrm{Sp}(m)$ for each $m \geq 3$ (cf. [2, 11] and [4, 5]). However, at present the complete classification of locally strongly convex Einstein affine hyperspheres in $\mathbb{R}^{n+1}$ is still an interesting and open problem.

In order to get further knowledge of the affine hyperspheres, the above mentioned facts motivate us to consider the following natural and interesting problem:

Classify all locally strongly convex affine hyperspheres which are locally isometric to the product $M_1^{n_1}(c_1) \times M_2^{n_2}(c_2)$, such that $n_1 + n_2 = n$ and $M_i^{n_i}(c_i)$ is an n_i-dimensional Riemannian manifold with constant sectional curvature c_i for $i = 1, 2$.

To consider this problem, we are sufficient to assume that $n \geq 3$. As the results of this paper, we have solved the above problem. More precisely, we have proved the following theorems.

Theorem 1.1. *Let $x : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex affine hypersphere. If (M^n, h) is locally isometric to the Riemannian product $M_1^{n_1}(c_1) \times M_2^{n_2}(c_2)$ for $n_1 \geq 2$ and $n_2 \geq 2$, such that $n_1 + n_2 = n$ and $M_i^{n_i}(c_i)$ is an n_i-dimensional Riemannian manifold with constant sectional curvature c_i for $i = 1, 2$. Then we have $c_1 c_2 = 0$, and one of the following cases occurs:*

- (i) *$c_1 = c_2 = 0$ and $x : M^n \to \mathbb{R}^{n+1}$ is locally affinely equivalent to either the paraboloid $x_{n+1} = \frac{1}{2}[(x_1)^2 + \cdots + (x_n)^2]$ or $Q(1, n)$: $x_1 x_2 \cdots x_{n+1} = 1$;*
- (ii) *$c_1 c_2 = 0$ and $c_1^2 + c_2^2 \neq 0$, assuming that $c_1 = 0$ and $c_2 \neq 0$, then $c_2 < 0$, $x : M^n \to \mathbb{R}^{n+1}$ is locally affinely equivalent to the Calabi composition*

$$(x_1 \cdots x_{n_1})^2 (x_{n+1}^2 - x_{n_1+1}^2 - \cdots - x_n^2)^{n_2+1} = 1,$$

where $(x_1, \ldots, x_{n+1})$ are the standard coordinates of $\mathbb{R}^{n+1}$.

Theorem 1.2. *Let $x : M^n \to \mathbb{R}^{n+1}$ $(n \geq 3)$ be a locally strongly convex affine hypersphere. If (M^n, h) is locally isometric to a Riemannian product $I \times \tilde{M}^{n-1}(c)$, with $I \subset \mathbb{R}$ and $\tilde{M}^{n-1}(c)$ an $(n-1)$-dimensional Riemannian manifold with constant sectional curvature $c \neq 0$. Then we have $c < 0$, and $x : M^n \to \mathbb{R}^{n+1}$ is locally affinely equivalent to the Calabi composition*

$$x_1^2 (x_{n+1}^2 - x_2^2 - \cdots - x_n^2)^n = 1,$$

where $(x_1, \ldots, x_{n+1})$ are the standard coordinates of $\mathbb{R}^{n+1}$.

As direct consequences of these theorems, we further have the following results.

Corollary 1.1. *Let $x : M^3 \to \mathbb{R}^4$ be a locally strongly convex affine hypersphere with parallel Ricci tensor. Then either M^3 is an open part of a locally strongly convex hyperquadric, or $x : M^3 \to \mathbb{R}^4$ is locally affinely equivalent to one of the following two hypersurfaces:*

- (i) $x_1 x_2 x_3 x_4 = 1$,
- (ii) $x_1^2 (x_4^2 - x_2^2 - x_3^2)^3 = 1$,

where (x_1, x_2, x_3, x_4) are the standard coordinates of $\mathbb{R}^4$.

Corollary 1.2. *Let $x : M^4 \to \mathbb{R}^5$ be a locally strongly convex affine hypersphere with parallel Ricci tensor. Then either M^4 is an open part of a locally strongly convex hyperquadric, or $x : M^4 \to \mathbb{R}^5$ is locally affinely equivalent to one of the following hypersurfaces:*

(i) $x_1x_2x_3x_4x_5 = 1$,
(ii) $(x_1x_2)^2(x_5^2 - x_3^2 - x_4^2)^3 = 1$,
(ii) $x_1^2(x_5^2 - x_2^2 - x_3^2 - x_4^2)^4 = 1$,

where $(x_1, x_2, x_3, x_4, x_5)$ *are the standard coordinates of* $\mathbb{R}^5$.

Remark 1.1. The above corollaries and the main results of [7] and [9] imply that for locally strongly convex affine hyperspheres in both $\mathbb{R}^4$ and $\mathbb{R}^5$, the parallelism of the intrinsic invariant *Ricci tensor* and that of the extrinsic invariant *cubic form* are actually equivalent.

The paper is arranged as follows: In section 2, we fix notations and briefly recall the local theory of equiaffine hypersurfaces. In section 3, the most technical parts of this paper are given and we prove the crucial lemmas which imply the existence of canonical local frame so that the difference tensor can be sufficiently determined. Finally, in section 4 we complete the proof of the preceding theorems and corollaries.

2. Preliminaries

In this section, we briefly recall the local theory of equiaffine hypersurfaces. For more details, we refer to the monographs [17, 21].

Let $\mathbb{R}^{n+1}$ be the standard $(n+1)$-dimensional real unimodular affine space that is equipped with its usual flat connection D and a parallel volume form given by the determinant. Let $x : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex hypersurface with affine normal ξ. Then, for any vector fields X and Y on M^n, we have

$$D_X x_*(Y) = x_*(\nabla_X Y) + h(X,Y)\xi, \tag{2.1}$$

$$D_X \xi = -x_*(SX), \tag{2.2}$$

where ∇, S and h are the induced affine connection, the affine shape operator and the affine metric, respectively. It is well known that M^n is an affine hypersphere if and only if $S = H\,\mathrm{id}$ with H being a constant; moreover, $x : M^n \to \mathbb{R}^{n+1}$ is a proper (resp. improper) affine hypersphere if and only if $H \neq 0$ (resp. $H = 0$).

Let $\hat\nabla$ denote the Levi-Civita connection of the affine metric h. The difference tensor K is defined by $K(X,Y) := K_XY := \nabla_XY - \hat\nabla_XY$; it is symmetric as both connections are torsion free. Moreover, $h(K(X,Y),Z)$ is a totally symmetric cubic form. For affine hyperspheres with affine shape operator $S = H\,\mathrm{id}$, the Riemannian curvature tensor $\hat R$ of the affine metric and the difference tensor K satisfy the following fundamental equations of Gauss and Codazzi:

$$\hat R(X,Y)Z = H\big[h(Y,Z)X - h(X,Z)Y\big] - [K_X, K_Y]Z, \tag{2.3}$$

$$(\hat\nabla_X K)(Y,Z) = (\hat\nabla_Y K)(X,Z). \tag{2.4}$$

As usual, we denote $(\hat\nabla K)(Z,X,Y) := (\hat\nabla_Z K)(X,Y)$, and define the second covariant differentiation $\hat\nabla^2 K$ of K by

$$\begin{aligned}(\hat\nabla^2 K)(W,Z,X,Y) :=& \hat\nabla_W((\hat\nabla K)(Z,X,Y)) - (\hat\nabla K)(\hat\nabla_W Z,X,Y)\\ &- (\hat\nabla K)(Z,\hat\nabla_W X,Y) - (\hat\nabla K)(Z,X,\hat\nabla_W Y).\end{aligned} \tag{2.5}$$

Then we have the following Ricci identity:

$$\begin{aligned}&(\hat\nabla^2 K)(W,X,Y,Z) - (\hat\nabla^2 K)(X,W,Y,Z)\\ &= \hat R(W,X)K(Y,Z) - K(\hat R(W,X)Y,Z) - K(Y,\hat R(W,X)Z).\end{aligned} \tag{2.6}$$

Moreover, for unimodular affine hypersurfaces of $\mathbb{R}^{n+1}$, K satisfies the so-called apolarity condition

$$\operatorname{trace} K_X = 0, \quad \forall X \in TM. \tag{2.7}$$

In the following, we will prove an additional relation that is very useful in our computations. To do so, we will make use of the technique introduced in [1], as the *Tsinghua Principle*. First, take the covariant derivative of (2.4) with respect to W, and use (2.4) and (2.5), to obtain straightforwardly that

$$(\hat{\nabla}^2 K)(W, X, Y, Z) - (\hat{\nabla}^2 K)(W, Y, X, Z) = 0. \tag{2.8}$$

Then we sum over cyclic permutations of the first three vector fields in the above equation and use the Ricci identity (2.6). It follows that

$$\begin{aligned} 0 =& \hat{R}(W, X)K(Y, Z) - K(\hat{R}(W, X)Z, Y) + \hat{R}(X, Y)K(W, Z) \\ &- K(\hat{R}(X, Y)Z, W) + \hat{R}(Y, W)K(X, Z) - K(\hat{R}(Y, W)Z, X). \end{aligned} \tag{2.9}$$

Additionally, if $(M^n, h) = M_1^{n_1}(c_1) \times M_2^{n_2}(c_2)$ and applying Corollary 58 on page 89 in [22], we know that

$$\begin{aligned} \hat{R}(X, Y)Z =& c_1 \big[h(Y_1, Z_1)X_1 - h(X_1, Z_1)Y_1\big] \\ &+ c_2 \big[h(Y_2, Z_2)X_2 - h(X_2, Z_2)Y_2\big], \end{aligned} \tag{2.10}$$

where, for $p \in M^n$ and $i = 1, 2$, X_i, Y_i, Z_i are the $T_pM_i^{n_i}$-component of $X, Y, Z \in T_pM^n$, respectively.

3. Lemmas on the Calculations of the Difference Tensor

In this section, we consider the n-dimensional locally strongly convex affine hypersphere $x : M^n \to \mathbb{R}^{n+1}$, such that (M^n, h) is locally isometric to a Riemannian product $M_1^{n_1}(c_1) \times M_2^{n_2}(c_2)$ for $n_1 \geq 2$ and $n_2 \geq 2$, $n_1 + n_2 = n$. Here, for $i = 1, 2$, $M_i^{n_i}(c_i)$ denotes an n_i-dimensional Riemannian manifold with constant sectional curvature c_i. We first assume that $c_1^2 + c_2^2 \neq 0$ in this section.

Now, we would emphasize that when we dealing with the product manifold $M_1^{n_1} \times M_2^{n_2}$, one should be aware that *throughout the paper* we will work with tangent vectors on M^n denoted by X and Y. In general, the X notation (as well as X_i, $1 \leq i \leq n_1$) will denote a tangent vector at $p = (p_1, p_2) \in M^n$, with zero component on $M_2^{n_2}$. Notice that, a priori, it means that X depends on p_2 as well, not only on p_1. A corresponding meaning is given to Y (or Y_j, $1 \leq j \leq n_2$), having zero components on $M_1^{n_1}$ and depending a priory on both p_1 and p_2. One should have in mind this meaning when reading $X \in T_pM_1^{n_1}$, respectively, $Y \in T_pM_2^{n_2}$. Nonetheless, a complete understanding will be acquired with the proofs of Lemmas 4.1 and 4.2.

We begin with the following result.

Lemma 3.1. *If $c_1^2 + c_2^2 \neq 0$, then the difference tensor K vanishes nowhere.*

Proof. Suppose on the contrary that the difference tensor K vanishes at the point $p = (p_1, p_2) \in M^n = M_1^{n_1} \times M_2^{n_2}$. Then, from (2.3) we know that

$$\hat{R}(X, Y)Z = H\big[h(Y, Z)X - h(X, Z)Y\big] \quad \text{at } p. \tag{3.1}$$

Thus (M^n, h) has constant sectional curvature H at p.

Now, taking unit vectors $X \in T_pM_1^{n_1}$ and $Y = Z \in T_pM_2^{n_2}$ in both (2.10) and (3.1), we get $H = 0$.

Next, taking unit vectors $X, Y = Z \in T_pM_1^{n_1}$ with $X \perp Y$ in both (2.10) and (3.1), we get $c_1 = 0$. Similarly, taking unit vectors $X, Y = Z \in T_pM_2^{n_2}$ with $X \perp Y$ in both (2.10) and (3.1), we get $c_2 = 0$.

Hence, $c_1 = c_2 = 0$. This is a contradiction to that $c_1^2 + c_2^2 \neq 0$. □

Notice that if $c_1c_2 = 0$, then without loss of generality we can assume that $c_1 = 0$ and $c_2 \neq 0$. Thus, in sequel we are sufficient to consider the following two cases:

Case $\mathfrak{C}_1$: $c_1 = 0$ and $c_2 \neq 0$; **Case** $\mathfrak{C}_2$: $c_1 \neq 0$ and $c_2 \neq 0$.

In the remaining of this section, we consider only **Case** $\mathfrak{C}_1$. In order to decide the difference tensor, first of all we have the following lemma.

Lemma 3.2. *For $p \in M_1^{n_1} \times M_2^{n_2}$, let $\{X_i\}_{1\le i\le n_1}$ and $\{Y_j\}_{1\le j\le n_2}$ be orthonormal bases of $T_pM_1^{n_1}$ and $T_pM_2^{n_2}$, respectively. Then, in **Case** $\mathfrak{C}_1$, we have*

$$K_{X_i}Y_\alpha = \mu(X_i)Y_\alpha, \quad 1 \le i \le n_1,\ 1 \le \alpha \le n_2, \tag{3.2}$$

where $\mu(X_i) =: \mu_i$ depends only on X_i for $i = 1, \dots, n_1$. Moreover, it holds that

$$\mu(X_1)^2 + \cdots + \mu(X_{n_1})^2 = -\tfrac{n_1}{n_2+1}H. \tag{3.3}$$

Proof. Let $\{X_1, \dots, X_{n_1}\}$ (resp. $\{Y_1, \dots, Y_{n_2}\}$) be an orthonormal basis of $T_pM_1^{n_1}$ (resp. $T_pM_2^{n_2}$). Taking $X = X_i$, $Y = Y_\alpha$ and $Z = W = Y_\beta$ ($\alpha \neq \beta$) in (2.9), then using (2.10) we obtain

$$0 = c_2\sum_{m=1}^{n_2}(\delta_{\beta m}Y_\alpha - \delta_{\alpha m}Y_\beta)h(K_{X_i}Y_\beta, Y_m) - c_2K_{X_i}Y_\alpha. \tag{3.4}$$

Taking the component of (3.4) on Y_β, we have that

$$h(K_{X_i}Y_\alpha, Y_\beta) = 0, \quad 1 \le i \le n_1,\ 1 \le \alpha \neq \beta \le n_2. \tag{3.5}$$

Taking the component of (3.4) on Y_α, we have

$$h(K_{X_i}Y_\alpha, Y_\alpha) = h(K_{X_i}Y_\beta, Y_\beta), \quad 1 \le i \le n_1,\ 1 \le \alpha, \beta \le n_2. \tag{3.6}$$

Similarly, taking $X = Y_\alpha$, $Y = X_i$, $Z = X_j$ and $W = Y_\beta$ in (2.9), then using (2.10) we obtain

$$0 = c_2\sum_{m=1}^{n_2}(\delta_{m\alpha}Y_\beta - \delta_{\beta m}Y_\alpha)h(K_{X_i}X_j, Y_m). \tag{3.7}$$

Let $\alpha \neq \beta$, then (3.7) implies that

$$h(K_{X_i}X_j, Y_\alpha) = 0, \quad 1 \le i, j \le n_1,\ 1 \le \alpha \le n_2. \tag{3.8}$$

Combining (3.5), (3.6) and (3.8), the assertion (3.2) immediately follows.

Next, we compute the sectional curvature $K(\pi(X_i, Y_j))$ of the plane π spanned by X_i and Y_j, for some fixed $i \in \{1, \dots, n_1\}$ and $j \in \{1, \dots, n_2\}$. For that purpose, using (2.10) on the one hand, and (2.3) on the other hand, together with applying (3.2), we obtain

$$\begin{aligned} 0 =& H - h(K_{Y_j}Y_j, K_{X_i}X_i) + h(K_{X_i}Y_j, K_{Y_j}X_i) \\ =& H + \mu(X_i)^2 - h(K_{Y_j}Y_j, K_{X_i}X_i), \quad 1 \le i \le n_1,\ 1 \le j \le n_2. \end{aligned}$$

Then, taking summation over $i = 1, \dots, n_1$, and using (3.2), we get

$$0 = n_1 H + \sum_{i=1}^{n_1} \mu(X_i)^2 - h(K_{Y_j} Y_j, \sum_{i=1}^{n_1} K_{X_i} X_i) \tag{3.9}$$
$$= n_1 H + \sum_{i=1}^{n_1} \mu(X_i)^2 - \sum_{k=1}^{n_1} \sum_{i=1}^{n_1} h(K_{X_k} X_i, X_i) \mu(X_k).$$

On the other hand, the apolarity condition implies that, for each $k = 1, \dots, n_1$,

$$0 = \sum_{i=1}^{n_1} h(K_{X_k} X_i, X_i) + \sum_{j=1}^{n_2} h(K_{X_k} Y_j, Y_j) = \sum_{i=1}^{n_1} h(K_{X_k} X_i, X_i) + n_2 \mu(X_k). \tag{3.10}$$

Therefore, from (3.9) and (3.10), we obtain

$$\mu(X_1)^2 + \cdots + \mu(X_{n_1})^2 = -\tfrac{n_1}{n_2+1} H. \tag{3.11}$$

This completes the proof of Lemma 3.2. □

Now, before going to show the next lemma, we will describe the construction of a typical orthonormal basis, which was introduced by N. Ejiri and has been widely applied, and proved to be very useful for various situations, see e.g. [10] and [18, 20]. The idea is to construct a basis from a self-adjoint operator at a point; then one extends the basis to local orthonormal vector fields. In this paper, we have the general principle as below:

For an arbitrary $p \in M^n = M_1^{n_1} \times M_2^{n_2}$, let $U_p M_1^{n_1} = \{u \in T_p M_1^{n_1} \mid h(u,u) = 1\}$ and $E_p \subset T_{p_1} M_1^{n_1} \times \{0\}$ a vector subspace. Since M^n is locally strongly convex, $U_p M_1^{n_1} \cap E_p$ is compact. We define on this set the function

$$f_1(u) = h(K_u u, u), \quad u \in U_p M_1^{n_1} \cap E_p.$$

Then there is an element $e_1 \in U_p M_1^{n_1} \cap E_p$ at which the function $f_1(u)$ attains the absolute maximum. Let $u \in U_p M_1^{n_1} \cap E_p$ such that $h(u, e_1) = 0$, and define a function g by $g(t) := f_1\big(\cos t\, e_1 + \sin t\, u\big)$. Then we have

$$g'(0) = 3\, h(K_{e_1} e_1, u), \quad g''(0) = 6\, h(K_{e_1} u, u) - 3\, f_1(e_1). \tag{3.12}$$

Since g attains an absolute maximum at $t = 0$, we have $g'(0) = 0, g''(0) \leq 0$, i.e.,

$$h(K_{e_1} e_1, u) = 0,\ h(K_{e_1} e_1, e_1) \geq 2h(K_{e_1} u, u),\ h(u,u) = 1,\ u \perp e_1. \tag{3.13}$$

Analogously, we can define a function f_2 on $U_p M_2^{n_2} \cap \tilde{E}_p$, where $U_p M_2^{n_2} = \{u \in T_p M_2^{n_2} \mid h(u,u) = 1\}$ and $\tilde{E}_p \subset \{0\} \times T_{p_2} M_2^{n_2}$ a vector subspace. We can choose $e_1 \in U_p M_2^{n_2} \cap \tilde{E}_p$ such that (3.13) holds for $u \in U_p M_2^{n_2} \cap \tilde{E}_p$ with $u \perp e_1$.

In the following, we will apply the above principle of choosing the unit vector e_1 many times.

Now, as a supplement to Lemma 3.2, we can prove the following lemma.

Lemma 3.3. *Given $p = (p_1, p_2) \in M_1^{n_1} \times M_2^{n_2}$. Let $\{X_i\}_{1 \leq i \leq n_1}$ and $\{Y_j\}_{1 \leq j \leq n_2}$ be the orthonormal bases of $T_p M_1^{n_1}$ and $T_p M_2^{n_2}$, respectively. Then, in **Case** $\mathfrak{C}_1$, we have*

$$K_{Y_\alpha} Y_\beta = \delta_{\alpha\beta}(\mu_1 X_1 + \cdots + \mu_{n_1} X_{n_1}), \quad 1 \leq \alpha, \beta \leq n_2, \tag{3.14}$$

Moreover, we have $c_2 = \frac{n+1}{n_2+1} H < 0$.

Proof. Let $\{X_i\}_{1\le i\le n_1}$ and $\{Y_j\}_{1\le j\le n_2}$ be orthonormal bases of $T_pM_1^{n_1}$ and $T_pM_2^{n_2}$, respectively. Then, according to Lemma 3.2, there are constants $\{\theta_{\alpha\beta}^{\gamma}\}$ such that

$$K_{Y_\alpha}Y_\beta=\delta_{\alpha\beta}(\mu_1X_1+\cdots+\mu_{n_1}X_{n_1})+\sum_{\gamma=1}^{n_2}\theta_{\alpha\beta}^{\gamma}Y_l,\quad 1\le\alpha,\beta\le n_2.$$

We will show that $\theta_{\alpha\beta}^{\gamma}=0$ for $1\le\alpha,\beta,\gamma\le n_2$, or equivalently,

$$h(K_{Y_\alpha}Y_\beta,Y_\gamma)=0,\quad 1\le\alpha,\beta,\gamma\le n_2. \tag{3.15}$$

We will prove (3.15) by contradiction.

Suppose on the contrary that (3.15) does not hold. Then, following the preceding stated procedure, we can choose a unit vector in $U_pM_2^{n_2}$, denoted by $\bar Y_1$, such that $\theta_1:=h(K_{\bar Y_1}\bar Y_1,\bar Y_1)>0$ is the maximum of the function f_2 defined on $U_{p_2}M_2^{n_2}$.

Define an operator $\mathcal{A}:T_pM_2^{n_2}\to T_pM_2^{n_2}$ by

$$\mathcal{A}(Y):=K_{\bar Y_1}Y-h(K_{\bar Y_1}Y,X_1)X_1-\cdots-h(K_{\bar Y_1}Y,X_{n_1})X_{n_1}.$$

Then, it is easy to show that $\mathcal{A}$ is self-adjoint and satisfies $\mathcal{A}(\bar Y_1)=\theta_1\bar Y_1$. We can choose orthonormal vectors in $U_pM_2^{n_2}$ orthogonal to $\bar Y_1$, denoted by $\bar Y_2,\ \ldots,\bar Y_{n_2}$, which are the remaining eigenvectors of the operator $\mathcal{A}$, with associated eigenvalues $\theta_2,\ldots,\theta_{n_2}$, respectively. Thus, by Lemma 3.2, we get the conclusion that

$$K_{\bar Y_1}\bar Y_1=\mu_1X_1+\cdots+\mu_{n_1}X_{n_1}+\theta_1\bar Y_1,\quad K_{\bar Y_1}\bar Y_i=\theta_i\bar Y_i,\quad 2\le i\le n_2. \tag{3.16}$$

In order to solve $\{\theta_i\}$ in (3.16), taking $X=Z=\bar Y_1$ and $Y=\bar Y_i$, $2\le i\le n_2$, in (2.3), using (2.10), (3.16) and Lemma 3.2, we can obtain

$$\theta_i^2-\theta_1\theta_i+\tfrac{n_1+1}{n_2+1}H-c_2=0,\quad 2\le i\le n. \tag{3.17}$$

From (3.17) and the statement of (3.13), we obtain that

$$\theta_2=\cdots=\theta_{n_2}=\tfrac12\big(\theta_1-\sqrt{\theta_1^2-4(\tfrac{n+1}{n_2+1}H-c_2)}\,\big). \tag{3.18}$$

Using (3.2), (3.16), (3.18) and $\operatorname{trace}K_{Y_1}=0$, we get

$$(n_2+1)\theta_1=(n_2-1)\sqrt{\theta_1^2-4(\tfrac{n+1}{n_2+1}H-c_2)}. \tag{3.19}$$

Then, we have

$$4\left(c_2-\tfrac{n+1}{n_2+1}H\right)=\left[\left(\tfrac{n_2+1}{n_2-1}\right)^2-1\right]\theta_1^2>0.$$

It follows that $c_2>\frac{n+1}{n_2+1}H$ and

$$\theta_1=(n_2-1)\sqrt{\tfrac{(n_2+1)c_2-(n+1)H}{n_2(n_2+1)}}. \tag{3.20}$$

Next, we intend to extend $\bar Y_1\in U_pM_2^{n_2}$, that satisfying (3.16), to be a local unit vector field around $p\in M^n$. For that purpose, we first make the following Claim.

Claim 1. *For every $p=(p_1,p_2)\in M^n$, the set*

$$\Omega_p:=\Big\{\lambda\in\mathbb{R}\mid V\in U_pM_2^{n_2}\ \text{s.t.}\ K_VV=\lambda V+\sum_{i=1}^{n_1}\mu_iX_i\Big\}$$

consists of finite numbers, which are independent of the point $p\in M^n$.

To verify the claim, we notice that, for any fixed $p\in M^n$, the above discussion implies that we have $\theta_1\in\Omega_p$ with $V=\bar Y_1$. Thus, the set Ω_p is non-empty.

Next, assume an arbitrary $\lambda \in \Omega_p$ associated with $V \in U_pM_2^{n_2}$ such that

$$K_V V = \lambda V + \mu_1 X_1 + \cdots + \mu_{n_1} X_{n_1}.$$

Then we put $\tilde{Y}_1 = V$, $\tilde{\theta}_1 = \lambda$ and define an operator $\mathcal{B} : T_pM_2^{n_2} \to T_pM_2^{n_2}$ by

$$\mathcal{B}(Y) = K_{\tilde{Y}_1} Y - h(K_{\tilde{Y}_1} Y, X_1)X_1 - \cdots - h(K_{\tilde{Y}_1} Y, X_{n_1})X_{n_1}.$$

It is easily seen that $\mathcal{B}$ is self-adjoint and $\mathcal{B}(\tilde{Y}_1) = \tilde{\theta}_1 \tilde{Y}_1$. Then, we may complete $\tilde{Y}_1$ to get an orthonormal basis $\{\tilde{Y}_i\}_{1\le i\le n_2}$ of $T_pM_2^{n_2}$ by letting $\tilde{Y}_2, \ldots, \tilde{Y}_{n_2}$ to be the eigenvectors of $\mathcal{B}$, with eigenvalues $\tilde{\theta}_2, \ldots, \tilde{\theta}_{n_2}$, respectively.

Similar to the proof of (3.17), we have the existence of an integer $n_{2,1}$ with $0 \le n_{2,1} \le n_2 - 1$ such that, if necessary, after renumbering the basis, it holds

$$\begin{cases} \tilde{\theta}_2 = \cdots = \tilde{\theta}_{n_{2,1}+1} = \frac{1}{2}\left(\tilde{\theta}_1 + \sqrt{\tilde{\theta}_1^2 - 4(\frac{n+1}{n_2+1}H - c_2)}\right), \\ \tilde{\theta}_{n_{2,1}+2} = \cdots = \tilde{\theta}_{n_2} = \frac{1}{2}\left(\tilde{\theta}_1 - \sqrt{\tilde{\theta}_1^2 - 4(\frac{n+1}{n_2+1}H - c_2)}\right). \end{cases} \tag{3.21}$$

Then, by $\operatorname{trace} K_{\tilde{Y}_1} = 0$, we find that

$$(n_2 + 1)\tilde{\theta}_1 - (n_2 - 2n_{2,1} - 1)\sqrt{\tilde{\theta}_1^2 - 4(\tfrac{n+1}{n_2+1}H - c_2)} = 0. \tag{3.22}$$

This implies that $\tilde{\theta}_1 = \lambda$ is independent of the point p and takes value of only finite possibilities. The assertion of Claim 1 immediately follows.

To extend $\bar{Y}_1$ differentiably to a unit vector field on a neighbourhood $U \subset M^n$ around p, which is still denoted by $\bar{Y}_1$, such that, at every point $q \in U$, f_2 attains an absolute maximum at $\bar{Y}_1(q)$, we first take differentiable h-orthonormal vector fields $\{E_1, \ldots, E_{n_2}\}$ defined on a neighbourhood U' of p and satisfying $E_i(q) \in T_qM_2^{n_2}, q \in U', 1 \le i \le n_2$, such that $E_i(p) = \bar{Y}_i$ for $1 \le i \le n_2$. Then, we define a function γ by

$$\gamma : \mathbb{R}^{n_2} \times U' \to \mathbb{R}^{n_2} \text{ by } (a_1, \ldots, a_{n_2}, q) \mapsto (b_1, \ldots, b_{n_2}),$$

where

$$b_k = \sum_{i,j=1}^{n_2} a_i a_j h(K_{E_i} E_j, E_k) - \theta_1 a_k, \quad 1 \le k \le n_2, \tag{3.23}$$

are regarded as functions on $\mathbb{R}^{n_2} \times U'$: $b_k = b_k(a_1, \ldots, a_{n_2}, q)$.

Using (3.16) and the fact that f_2 attains an absolute maximum at $E_1(p)$, we then obtain that

$$\begin{aligned} \tfrac{\partial b_k}{\partial a_m}(1, 0, \ldots, 0, p) &= 2h(K_{E_1(p)} E_m(p), E_k(p)) - \theta_1 \delta_{km} \\ &= \begin{cases} 0, & \text{if } k \ne m, \\ \theta_1, & \text{if } k = m = 1, \\ 2\theta_k - \theta_1, & \text{if } k = m \ge 2. \end{cases} \end{aligned}$$

Notice that, by assumption, (3.18) and (3.19), we have $\theta_1 > 0$ and $2\theta_k - \theta_1 \ne 0$ for $2 \le k \le n_2$. Then, the implicit function theorem shows that there exist differentiable functions $\{a_i(q)\}_{1\le i\le n_2}$ defined on a neighbourhood $U'' \subset U'$ of p, such that

$$\begin{cases} a_1(p) = 1, \ a_2(p) = \cdots = a_{n_2}(p) = 0, \\ b_i(a_1(q), \ldots, a_{n_2}(q), q) \equiv 0, \quad 1 \le i \le n_2. \end{cases} \tag{3.24}$$

Define the local vector field V on U'' by

$$V(q)=a_1(q)E_1(q)+\cdots+a_{n_2}(q)E_{n_2}(q),\quad q\in U''.$$

Then, for local basis of $TM_1^{n_1}$ around U'', still denoted by $\{X_i\}_{1\le i\le n_1}$, from (3.23), (3.24) and Lemma 3.2, we have $K_{X_i}Y=\mu_iY$ for any $Y\in TM_2^{n_2}$, and that

$$K_VV=\theta_1V+\mu_1h(V,V)X_1+\cdots+\mu_{n_1}h(V,V)X_{n_1}.$$

Let us define $\|V\|=\sqrt{h(V,V)}$. Since $\|V\|(p)=1$, there exists a neighbourhood $U\subset U''$ of p such that $V\neq 0$ on U. Then, $W=\frac{V}{\|V\|}$ is a unit vector field on U that satisfies

$$K_WW=\tfrac{\theta_1}{\sqrt{h(V,V)}}W+\mu_1X_1+\cdots+\mu_{n_1}X_{n_1}.$$

Denote $\tilde\theta_1=\theta_1/\sqrt{h(V,V)}$. Then, the proof of Claim 1 implies that, as a function on U, $\tilde\theta_1$ takes values of finite number, which satisfy (3.22) for some $0\le n_{2,1}\le n_2-1$. This further implies from the fact $h(V,V)(p)=1$ and the continuity of the function $\theta_1/\sqrt{h(V,V)}$ that $h(V,V)\equiv 1$ on U.

Let $\bar Y_1=W$ and take orthonormal vector fields $\bar Y_2,\dots,\bar Y_{n_2}$ orthogonal to $\bar Y_1$ so that $\{\bar Y_1,\dots,\bar Y_{n_1}\}$ forms a local orthonormal basis of $TM_2^{n_2}$ on U. Then, according to (3.16), (3.18) and (3.20), we have a constant $\theta_2=\cdots=\theta_{n_2}$ such that the difference tensor satisfies

$$K_{\bar Y_1}\bar Y_1=\mu_1X_1+\cdots+\mu_{n_1}X_{n_1}+\theta_1\bar Y_1,\quad K_{\bar Y_1}\bar Y_i=\theta_i\bar Y_i,\quad 2\le i\le n_2. \tag{3.25}$$

Now, we can apply the Codazzi equation (2.4) to the basis $\{\bar Y_i\}_{1\le i\le n_2}$.

By the property $h(\hat\nabla_{\bar Y_i}\bar Y_j,X_k)=0$ of product manifold and (3.25), we have the following calculations:

$$\begin{aligned}(\hat\nabla_{\bar Y_i}K)(\bar Y_1,\bar Y_1)&=\hat\nabla_{\bar Y_i}K(\bar Y_1,\bar Y_1)-2K(\hat\nabla_{\bar Y_i}\bar Y_1,\bar Y_1)\\&=(\theta_1-2\theta_2)\hat\nabla_{\bar Y_i}\bar Y_1+\sum_{k=1}^{n_1}\Big(\mu_k\hat\nabla_{\bar Y_i}X_k+\bar Y_i(\mu_k)X_k\Big),\\&=(\theta_1-2\theta_2)\sum_{j=1}^{n_2}h(\hat\nabla_{\bar Y_i}\bar Y_1,\bar Y_j)\bar Y_j\\&\qquad+\sum_{k=1}^{n_1}\Big(\mu_k\hat\nabla_{\bar Y_i}X_k+\bar Y_i(\mu_k)X_k\Big),\end{aligned}\tag{3.26}$$

$$\begin{aligned}(\hat\nabla_{\bar Y_1}K)(\bar Y_i,\bar Y_1)&=\hat\nabla_{\bar Y_1}K(\bar Y_i,\bar Y_1)-K(\hat\nabla_{\bar Y_1}\bar Y_i,\bar Y_1)-K(\hat\nabla_{\bar Y_1}\bar Y_1,\bar Y_i)\\&=\theta_2\hat\nabla_{\bar Y_1}\bar Y_i-K(\hat\nabla_{\bar Y_1}\bar Y_i,\bar Y_1)-K(\hat\nabla_{\bar Y_1}\bar Y_1,\bar Y_i)\\&=\theta_2h(\hat\nabla_{\bar Y_1}\bar Y_i,\bar Y_1)\bar Y_1-h(\hat\nabla_{\bar Y_1}\bar Y_i,\bar Y_1)K(\bar Y_1,\bar Y_1)\\&\qquad-\sum_{j=2}^{n_2}h(\hat\nabla_{\bar Y_1}\bar Y_1,\bar Y_j)K(\bar Y_j,\bar Y_i).\end{aligned}\tag{3.27}$$

Then, using $h((\hat\nabla_{\bar Y_i}K)(\bar Y_1,\bar Y_1),\bar Y_1)=h((\hat\nabla_{\bar Y_1}K)(\bar Y_i,\bar Y_1),\bar Y_1)$ for $i\ge 2$ we get $\hat\nabla_{\bar Y_1}\bar Y_1=0$. This and (3.27) give that $(\hat\nabla_{\bar Y_1}K)(\bar Y_i,\bar Y_1)=0$ for $1\le i\le n_2$. Thus, using (2.4) and (3.26), we can finally get

$$\hat\nabla_{\bar Y_i}\bar Y_1=0,\quad 1\le i\le n_2. \tag{3.28}$$

It follows that $c_2=h(\hat R(\bar Y_2,\bar Y_1)\bar Y_1,\bar Y_2)=0$ and as desired we get a contradiction. Therefore, (3.15) does hold.

Finally, taking $X = \bar{Y}_2$ and $Y = Z = \bar{Y}_1$ in (2.3), with using (2.10), (3.2) and (3.14), we easily get the relation $c_2 = \frac{n+1}{n_2+1}H$. This together with (3.3) further implies that $H < 0$.

We have completed the proof of Lemma 3.3. □

For the difference tensor, besides the conclusions as stated in Lemmas 3.2 and 3.3, we shall construct in the following Lemma 3.4 a typical local orthonormal frame on M^n so that more information of the difference tensor can be derived for **Case** $\mathfrak{C}_1$. However, the proof of Lemma 3.4 becomes more complicated when we compare it with that of Lemma 3.3.

Lemma 3.4. *In* ***Case*** $\mathfrak{C}_1$*, given* $p \in M^n$*, there exist local orthonormal vector fields* $\{X_i\}_{1\le i\le n_1}$ *defined on a neighbourhood* U *of* p*, and satisfying* $X_i(q) \in T_qM_1^{n_1}$ *for* $q \in U$ *and* $1 \le i \le n_1$*, such that the difference tensor* K *takes the following form:*

$$\begin{cases} K_{X_1}X_1 = \lambda_{1,1}X_1, \\ K_{X_i}X_i = \mu_1X_1 + \cdots + \mu_{i-1}X_{i-1} + \lambda_{i,i}X_i, \quad 2 \le i \le n_1, \\ K_{X_i}X_j = \mu_iX_j, \quad 1 \le i < j \le n_1, \\ K_{X_i}Y = \mu_iY, \ Y(q) \in T_qM_2^{n_2}, \quad 1 \le i \le n_1, \end{cases} \tag{3.29}$$

where $\lambda_{i,i}$ *and* μ_i $(1 \le i \le n_1)$ *are constants, and they satisfy the relations*

$$\begin{cases} \lambda_{i,i} + (n-i)\mu_i = 0, \quad 1 \le i \le n_1, \\ \lambda_{i,i} > 0, \ 1 \le i \le n_1 - 1; \quad \lambda_{n_1,n_1} \ge 0. \end{cases} \tag{3.30}$$

Proof. We give the proof by induction on the subscript i of K_{X_i}. According to the general principle of induction method, this consists of two steps as below.

The first step of induction.

In this step, we should verify the assertion for $i = 1$. To do so, we have to show that, around any given $p \in M_1^{n_1} \times M_2^{n_2}$, there exist orthonormal vector fields $\{X_i\}_{1\le i\le n_1}$ defined on a neighbourhood U of p and satisfying $X_i(q) \in T_qM_1^{n_1}$ for $q \in U$ and $1 \le i \le n_1$, and real numbers $\lambda_{1,1} > 0$ and μ_1, so that we have

$$\begin{cases} K_{X_1}X_1 = \lambda_{1,1}X_1, \quad K_{X_1}X_i = \mu_1X_i, \quad 2 \le i \le n_1, \\ K_{X_1}Y = \mu_1Y, \quad Y(q) \in T_qM_2^{n_2}, \\ \lambda_{1,1} + (n-1)\mu_1 = 0. \end{cases}$$

The proof of the above assertion will be divided into four claims as below.

Claim I-(1). *Given* $p \in M_1^{n_1} \times M_2^{n_2}$*, there exists an orthonormal basis* $\{X_i\}_{1\le i\le n_1}$ *of* $T_pM_1^{n_1}$*, real numbers* $\lambda_{1,1} > 0$*,* $\lambda_{1,2} = \cdots = \lambda_{1,n_1}$ *and* μ_1*, such that* $\lambda_{1,1}$ *is the maximum of* f_1 *defined on* $U_pM_1^{n_1}$*, and the following relations hold:*

$$\begin{cases} K_{X_1}X_1 = \lambda_{1,1}X_1, \ K_{X_1}X_i = \lambda_{1,i}X_i, \quad 2 \le i \le n_1, \\ K_{X_1}Y = \mu_1Y, \quad Y \in T_pM_2^{n_2}. \end{cases} \tag{3.31}$$

Proof of ***Claim I-(1)***. First, if for an orthonormal vectors $\{X_i\}_{1\le i\le n_1}$ and for any $i, j, k = 1, \ldots, n_1$, it holds $h(K_{X_i}X_j, X_k) = 0$. Then in (2.3) taking $X = X_1$ and $Y = Z = X_2$, using (2.10) and (3.2), we obtain $H = 0$. This is a contradiction to Lemma 3.3.

Next, let $p \in M^n = M_1^{n_1}(c_1) \times M_2^{n_2}(c_2)$. We choose $X_1 \in U_pM_1^{n_1}$ such that $\lambda_{1,1} = h(K_{X_1}X_1, X_1)$ is the maximum of $f_1(u)$ on $U_pM_1^{n_1}$ and it must be the case

$\lambda_{1,1} > 0$. Then, according to (3.2) and the statement of (3.13), we know that X_1 is an eigenvector of K_{X_1} and we can choose orthonormal vectors $X_2, \ldots, X_{n_1} \in T_pM_1^{n_1}$ orthogonal to X_1 such that $K_{X_1}X_i = \lambda_{1,i}X_i$ for $1 \le i \le n_1$, and $K_{X_1}Y = \mu_1 Y$ for any $Y \in T_pM_2^{n_2}$.

Taking in (2.3) $X = Z = X_1$ and $Y = X_k$, and using (2.10), we can obtain

$$\lambda_{1,k}^2 - \lambda_{1,1}\lambda_{1,k} + H = 0, \quad 2 \le k \le n_1. \tag{3.32}$$

Similar to the proof of (3.13), we have $\lambda_{1,1} \ge 2\lambda_{1,k}$ for $2 \le k \le n_1$. Thus, solving (3.32) we obtain $\lambda_{1,2} = \cdots = \lambda_{1,n_1}$ with

$$\lambda_{1,k} = \tfrac{1}{2}\Big(\lambda_{1,1} - \sqrt{\lambda_{1,1}^2 - 4H}\Big), \quad 2 \le k \le n_1. \tag{3.33}$$

Furthermore, taking in (2.3) $X = Z = X_1$ and $Y \in T_pM_2^{n_2}$ be a unit vector, using (2.10) and (3.2), we get

$$\mu_1^2 - \mu_1\lambda_{1,1} + H = 0. \tag{3.34}$$

Hence we have

$$\mu_1 = \tfrac{1}{2}\Big(\lambda_{1,1} + \varepsilon_1\sqrt{\lambda_{1,1}^2 - 4H}\Big), \quad \varepsilon_1 = \pm 1. \tag{3.35}$$

Finally, by (3.2), (3.33), (3.35) and $\operatorname{trace} K_{X_1} = 0$, we obtain

$$(n+1)\lambda_{1,1} + (-n_1 + 1 + \varepsilon_1 n_2)\sqrt{\lambda_{1,1}^2 - 4H} = 0, \tag{3.36}$$

and therefore, we have

$$\lambda_{1,1} = 2\sqrt{\frac{-H}{(\frac{n+1}{n_1 - \varepsilon_1 n_2 - 1})^2 - 1}}. \tag{3.37}$$

From (3.33), (3.35) and (3.37), we have completed the proof of **Claim I-(1)**. □

Claim I-(2). *The real numbers described in Claim I-(1) satisfy the relations:*

$$\lambda_{1,2} = \cdots = \lambda_{1,n_1} = \mu_1 \quad \text{and} \quad \lambda_{1,1} + (n-1)\mu_1 = 0.$$

*Proof of **Claim I-(2)**.* From (3.33), (3.35) and $\operatorname{trace} K_{X_1} = 0$, the assertions are equivalent to that $\varepsilon_1 = -1$. Suppose on the contrary that $\varepsilon_1 = 1$. Then we have

$$\mu_1\lambda_{1,2} = H, \tag{3.38}$$

and (3.36) implies that

$$n_1 > n_2 + 1 \ge 3. \tag{3.39}$$

Put $V_1 = \{u \in T_pM_1^{n_1} |\, u \perp X_1\}$. Then, by arguments as in the beginning of the proof for Claim I-(1) shows that the function $f_1 \ne 0$ restricting on $V_1 \cap U_pM_1^{n_1}$. We rechoose a unit vector $X_2 \in V_1$ such that $\lambda_{2,2} = h(K_{X_2}X_2, X_2) > 0$ is the maximum of $f_1(u)$ restricted on $\{u \in U_pM_1^{n_1} |\, u \perp X_1\}$.

Then, according to Lemma 3.2, we can define a linear mapping $\mathcal{A}: V_1 \to V_1$ by $\mathcal{A}(X) := K_{X_2}X - h(K_{X_2}X, X_1)X_1$. It is easily seen that $\mathcal{A}$ is self-adjoint and X_2 is one of its eigenvector. We can choose orthonormal vectors $X_3, \ldots, X_{n_1} \in T_pM_1^{n_1}$ orthogonal to X_2, which are the remaining eigenvectors of the operator $\mathcal{A}$, associated to the eigenvalues $\lambda_{2,3}, \ldots, \lambda_{2,n_1}$, respectively. Therefore, we have

$$K_{X_2}X_2 = \lambda_{1,2}X_1 + \lambda_{2,2}X_2, \ K_{X_2}X_i = \lambda_{2,i}X_i, \quad 3 \le i \le n_1. \tag{3.40}$$

Now, we can make use of (3.40) to derive the expected contradiction.

Taking in (2.3) $X = Z = X_2$ and $Y = X_k$, using (2.10) and (3.40), we can obtain

$$\lambda_{2,k}^2 - \lambda_{2,2}\lambda_{2,k} + H - \lambda_{1,2}^2 = 0, \quad 3 \le k \le n_1. \tag{3.41}$$

Similar to the proof of (3.13), we have $\lambda_{2,2} \ge 2\lambda_{2,k}$ for $3 \le k \le n_1$. Then, solving (3.13), we get $\lambda_{2,3} = \cdots = \lambda_{2,n_1}$ with

$$\lambda_{2,k} = \tfrac{1}{2}\Big(\lambda_{2,2} - \sqrt{\lambda_{2,2}^2 - 4(H - \lambda_{1,2}^2)}\Big), \quad 3 \le k \le n_1. \tag{3.42}$$

Similarly, taking in (2.3) $X = Z = X_2$ and $Y \in T_pM_2^{n_2}$ a unit vector, using (2.10), (3.2) and (3.40), we get

$$\mu_2^2 - \mu_2\lambda_{2,2} + H - \lambda_{1,2}\mu_1 = 0. \tag{3.43}$$

By using (3.38), we can reduce (3.43) to be

$$\mu_2^2 - \mu_2\lambda_{2,2} = 0. \tag{3.44}$$

It follows that

$$\mu_2 = \tfrac{1}{2}(\lambda_{2,2} + \varepsilon_2\lambda_{2,2}), \quad \varepsilon_2 = \pm 1. \tag{3.45}$$

Then, by trace $K_{X_2} = 0$, and using (3.2), (3.40), (3.42) and (3.45), we have

$$(n_1 + n_2 + \varepsilon_2 n_2)\lambda_{2,2} = (n_1 - 2)\sqrt{\lambda_{2,2}^2 + 4(\lambda_{1,2}^2 - H)}, \tag{3.46}$$

which implies that

$$\lambda_{2,2} = \sqrt{\frac{4(\lambda_{1,2}^2 - H)}{\big(\frac{n_1+n_2+\varepsilon_2 n_2}{n_1-2}\big)^2 - 1}}. \tag{3.47}$$

Note that $\varepsilon_1 = 1$, from (3.37) we have

$$\lambda_{1,1} = \sqrt{\frac{-4H}{\big(\frac{n_1+n_2+1}{n_1-n_2-1}\big)^2 - 1}}. \tag{3.48}$$

Noticing that $n_2 \ge 2$ and, by (3.39), $n_1 \ge n_2 + 2$, we have

$$\tfrac{n_1+n_2+1}{n_1-n_2-1} - \tfrac{n_1+n_2+\varepsilon_2 n_2}{n_1-2} > \tfrac{n_1+n_2+1}{n_1-n_2-1} - \tfrac{n_1+2n_2}{n_1-2} = \tfrac{2(n_2+1)(n_2-1)}{(n_1-n_2-1)(n_1-2)} > 0.$$

This, together with $H < 0$, implies that $\lambda_{2,2} > \lambda_{1,1}$. This is a contradiction.

Hence, we have $\varepsilon_1 = -1$ and $\lambda_{1,2} = \cdots = \lambda_{1,n_1} = \mu_1$.

Then, by trace $K_{X_1} = 0$ we get the second assertion. □

Claim I-(3). *For every point $p = (p_1, p_2) \in M^n$, the set*

$$\Omega_p := \Big\{\lambda \in \mathbb{R} \mid V \in U_pM_1^{n_1} \text{ s.t. } K_V V = \lambda V\Big\}$$

consists of finite numbers, which are independent of $p \in M^n$.

Proof of ***Claim I-(3)***. Claim I-(1) implies that Ω_p is non-empty. Assume that there exists a unit vector $V \in T_pM_1^{n_1}$ such that $K_V V = \lambda V$. Let $X_1 := V$ and $\lambda_{1,1} = \lambda$. Then, according to Lemma 3.2, we may complete X_1 to obtain an orthonormal basis $\{X_i\}_{1 \le i \le n_1}$ of $T_pM_1^{n_1}$ such that, for each $2 \le k \le n_1$, X_k is the eigenvector of K_{X_1} with eigenvalue $\lambda_{1,k}$.

Then we have (3.32), from which we have an integer $n_{1,1}$, $0 \le n_{1,1} \le n_1 - 1$, such that, if necessary after renumbering the basis, we have

$$\begin{cases} \lambda_{1,2} = \cdots = \lambda_{1,n_{1,1}+1} = \tfrac{1}{2}\big(\lambda_{1,1} + \sqrt{\lambda_{1,1}^2 - 4H}\,\big), \\ \lambda_{1,n_{1,1}+2} = \lambda_{1,n_1} = \tfrac{1}{2}\big(\lambda_{1,1} - \sqrt{\lambda_{1,1}^2 - 4H}\,\big). \end{cases} \tag{3.49}$$

Similarly, we have (3.35). Then, by trace $K_{X_1} = 0$, we have

$$(n+1)\lambda_{1,1} + (2n_{1,1} - n_1 + 1 + \varepsilon_1 n_2)\sqrt{\lambda_{1,1}^2 - 4H} = 0. \tag{3.50}$$

If $2n_{1,1} - n_1 + 1 + \varepsilon_1 n_2 = 0$, then $\lambda_{1,1} = 0$.
If $2n_{1,1} - n_1 + 1 + \varepsilon_1 n_2 < 0$, then we have

$$\lambda_{1,1} = \sqrt{\frac{4H}{1 - (\frac{n_1+n_2+1}{2n_{1,1}-n_1+\varepsilon_1 n_2+1})^2}}. \tag{3.51}$$

It follows that $\lambda_{1,1}$ has finite possibilities, and Claim I-(3) is verified. □

Claim I-(4). *The unit vector $X_1 \in U_pM_1^{n_1}$ given in Claim-I-(1) can be extended differentiably to a unit vector field, still denoted by X_1, in a neighbourhood $U \subset M^n$ of p, such that, for each $q \in U$, the function f_1 defined on $U_qM_1^{n_1}$ attains its absolute maximum at $X_1(q)$.*

*Proof of **Claim I-(4)***. Let $\{E_1, \ldots, E_{n_1}\}$ be differentiable orthonormal vector fields defined on a neighbourhood U' of p and satisfying $E_i(q) \in T_qM_1^{n_1}, q \in U', 1 \le i \le n_1$, such that $E_i(p) = X_i$ for $1 \le i \le n_1$. Then, from the fact $K_{X_1}X_1 = \lambda_{1,1}X_1$ at p, we define a function γ by

$$\gamma : \mathbb{R}^{n_1} \times U' \to \mathbb{R}^{n_1} \quad \text{by} \quad (a_1, \ldots, a_{n_1}, q) \mapsto (b_1, \ldots, b_{n_1}),$$

where

$$b_k = \sum_{i,j=1}^{n_1} a_i a_j h(K_{E_i}E_j, E_k) - \lambda_{1,1}a_k, \quad k = 1, 2, \ldots, n_1, \tag{3.52}$$

are regarded as functions on $\mathbb{R}^{n_1} \times U'$: $b_k = b_k(a_1, \ldots, a_{n_1}, q)$. Here, according to (3.37) and the proof of Claim I-(2), the maximum of f_1 defined on $U_qM_1^{n_1}$ is independent of $q \in U'$, and it is equal to $\lambda_{1,1} = (n-1)\sqrt{-H/n}$.

Using (3.31) and the fact that f_1 attains the absolute maximum $\lambda_{1,1}$ at $E_1(p)$, we obtain that

$$\begin{aligned} \tfrac{\partial b_k}{\partial a_m}(1, 0, \ldots, 0, p) &= 2h(K_{E_1(p)}E_m(p), E_k(p)) - \lambda_{1,1}\delta_{km} \\ &= \begin{cases} 0, & \text{if } k \ne m, \\ \lambda_{1,1}, & \text{if } k = m = 1, \\ 2\lambda_{1,k} - \lambda_{1,1}, & \text{if } k = m \ge 2. \end{cases} \end{aligned}$$

From the proof of Claim-I-(1) we have $\lambda_{1,1} > 0$ and $\lambda_{1,1} > 2\lambda_{1,k}$ for $2 \le k \le n_1$. Then, the implicit function theorem shows that there exist differentiable functions $\{a_i(q)\}_{1 \le i \le n_1}$, defined on a neighbourhood $U'' \subset U'$ of p, such that

$$\begin{cases} a_1(p) = 1, \ a_2(p) = \cdots = a_{n_1}(p) = 0, \\ b_i(a_1(q), \ldots, a_{n_1}(q), q) \equiv 0, \quad 1 \le i \le n_1. \end{cases} \tag{3.53}$$

Define the local vector field V on U'' by

$$V(q) = a_1(q)E_1(q) + \cdots + a_{n_1}(q)E_{n_1}(q), \quad q \in U''.$$

Then, from (3.52), (3.53) and (3.2), we get

$$K_V V = \lambda_{1,1} V. \tag{3.54}$$

Let us define $\|V\| = \sqrt{h(V,V)}$. Since $\|V\|(p) = 1$, there exists a neighbourhood $U \subset U''$ of p, such that $V \neq 0$ on U, and it holds that

$$K_{\frac{V}{\sqrt{h(V,V)}}} \frac{V}{\sqrt{h(V,V)}} = \frac{\lambda_{1,1}}{\sqrt{h(V,V)}} \frac{V}{\sqrt{h(V,V)}}.$$

From Claim I-(3), we know that $\frac{\lambda_{1,1}}{\sqrt{h(V,V)}}$ takes values of finite number. On the other hand, $\frac{\lambda_{1,1}}{\sqrt{h(V,V)}}$ is continuous and $h(V,V)(p) = 1$. Thus $h(V,V) \equiv 1$. It follows from (3.54) that, for any point $q \in U$, the function f_1 attains its absolute maximum in $V(q)$.

Define $X_1 := V$ on U. Then we have completed the proof of Claim I-(4). □

Finally, having determined the unit vector field X_1 as in Claim I-(4), we can further choose orthonormal vectors $X_2, \ldots, X_{n_1}$ orthogonal to X_1, defined on U and satisfying $X_i(q) \in T_q M_1^{n_1}, q \in U, 2 \leq i \leq n_1$. Then, it is easily seen that, combining with Lemma 3.2, Claim I-(1), Claim I-(2) and their proofs, $\{X_1, \ldots, X_{n_1}\}$ turns into the desired local orthonormal vector fields so that we have completed the proof for the first step of induction.

The second step of induction

In this step, we first assume the assertion of Lemma 3.4 for all $1 \leq i \leq k$, where $k \in \{1, 2, \ldots, n_1 - 2\}$ is a fixed integer. Thus, we have:

Around any given $p \in M_1^{n_1} \times M_2^{n_2}$, there exist local orthonormal vector fields $\{X_i\}_{1 \leq i \leq n_1}$ defined on a neighborhood U of p and satisfying $X_i(q) \in T_q M_1^{n_1}, q \in U, 1 \leq i \leq n_1$, such that the difference tensor K takes the form:

$$\begin{cases} K_{X_1} X_1 = \lambda_{1,1} X_1, \\ K_{X_i} X_i = \mu_1 X_1 + \cdots + \mu_{i-1} X_{i-1} + \lambda_{i,i} X_i, \quad 2 \leq i \leq k, \\ K_{X_i} X_j = \mu_i X_j, \quad 1 \leq i \leq k, \ i < j \leq n_1, \\ K_{X_i} Y = \mu_i Y, \ Y(q) \in T_q M_2^{n_2}, \quad 1 \leq i \leq k, \end{cases} \tag{3.55}$$

where, μ_i and $\lambda_{i,i}$ for $1 \leq i \leq k$ are real numbers, and they satisfy the relations:

$$\lambda_{i,i} + (n - i)\mu_i = 0, \ \lambda_{i,i} > 0, \quad 1 \leq i \leq k. \tag{3.56}$$

Moreover, at any $q \in U$, the number $\lambda_{i,i}$ is the maximum of the function f_1 defined on

$$\{u \in U_q M_1^{n_1} \,|\, u \perp X_1(q), \ldots, u \perp X_{i-1}(q)\},$$

for each $1 \leq i \leq k$.

Then, as purpose of the second step, we should verify the assertion of Lemma 3.4 for $i = k + 1$. To do so, we are sufficient to show that:

There exist an orthonormal frame $\{\tilde{X}_i\}_{1 \leq i \leq n_1}$ on $TM_1^{n_1}$ around p, given by

$$\tilde{X}_1 = X_1, \ldots, \tilde{X}_k = X_k; \ \tilde{X}_l = \sum_{t=k+1}^{n_1} T_l^t X_t, \quad k + 1 \leq l \leq n_1,$$

such that $T=(T_l^t)_{k+1\le l,t\le n_1}$ is an orthogonal matrix, and the difference tensor K takes the following form:

$$\begin{cases} K_{\tilde X_1}\tilde X_1=\lambda_{1,1}\tilde X_1,\\ K_{\tilde X_i}\tilde X_i=\mu_1\tilde X_1+\cdots+\mu_{i-1}\tilde X_{i-1}+\lambda_{i,i}\tilde X_i,\quad 2\le i\le k+1,\\ K_{\tilde X_i}\tilde X_j=\mu_i\tilde X_j,\quad 1\le i\le k+1,\ i+1\le j\le n_1,\\ K_{\tilde X_i}Y=\mu_iY,\quad Y(q)\in T_qM_2^{n_2},\quad 1\le i\le k+1,\end{cases}\tag{3.57}$$

where, μ_i and $\lambda_{i,i}$, for $1\le i\le k+1$, are real numbers, and they satisfy the relations

$$\lambda_{i,i}+(n-i)\mu_i=0,\quad 1\le i\le k+1.\tag{3.58}$$

Moreover, at any q around p, the number $\lambda_{i,i}$ is the maximum of the function f_1 defined on

$$\{u\in U_qM_1^{n_1}\mid u\perp \operatorname{span}\{\tilde X_1(q),\dots,\tilde X_{i-1}(q)\}\},$$

for each $1\le i\le k+1$.

In order to prove the above conclusions, similar to the proof in the first step, we also divide it into the verification of the following four claims.

Claim II-(1). *For any $p\in M_1^{n_1}\times M_2^{n_2}$, there exist an orthonormal basis $\{\bar X_i\}_{1\le i\le n_1}$ of $T_pM_1^{n_1}$ and, real numbers $\lambda_{k+1,k+1}>0$, $\lambda_{k+1,k+2}=\cdots=\lambda_{k+1,n_1}$ and μ_{k+1}, such that the following relations hold:*

$$\begin{cases} K_{\bar X_1}\bar X_1=\lambda_{1,1}\bar X_1,\\ K_{\bar X_i}\bar X_i=\mu_1\bar X_1+\cdots+\mu_{i-1}\bar X_{i-1}+\lambda_{i,i}X_i,\quad 2\le i\le k+1,\\ K_{\bar X_i}\bar X_j=\mu_i\bar X_j,\quad 1\le i\le k,\quad i+1\le j\le n_1,\\ K_{\bar X_{k+1}}\bar X_i=\lambda_{k+1,i}\bar X_i,\quad k+2\le i\le n_1,\\ K_{\bar X_{k+1}}Y=\mu_{k+1}Y,\quad Y\in T_pM_2^{n_2}.\end{cases}\tag{3.59}$$

*Proof of **Claim II-(1)***. By the assumption of induction, we have local orthonormal vector fields $\{X_i\}_{1\le i\le n_1}$ defined on a neighborhood U of p and satisfying $X_i(q)\in T_qM_1^{n_1}$ for $q\in U$ and $1\le i\le n_1$, such that (3.55) and (3.56) hold. We first take $\bar X_1=X_1(p),\dots,\bar X_k=X_k(p)$ and put

$$V_k=\{u\in T_pM_1^{n_1}\mid u\perp\bar X_1,\dots,u\perp\bar X_k\}.$$

Then, similar argument as in the proof of Claim I-(1) shows that when restricting on $V_k\cap U_pM_1^{n_1}$ the function $f_1\not\equiv 0$. Thus, we can choose a unit vector $\bar X_{k+1}\in V_k$ such that $\lambda_{k+1,k+1}=h(K_{\bar X_{k+1}}\bar X_{k+1},\bar X_{k+1})$ is the maximum of f_1 on $V_k\cap U_pM_1^{n_1}$ with $\lambda_{k+1,k+1}>0$.

Define a linear transformation $\mathfrak{A}:V_k\to V_k$ by

$$\mathfrak{A}(X)=K_{\bar X_{k+1}}X-\sum_{i=1}^k h(K_{\bar X_{k+1}}X,\bar X_i)\bar X_i,\quad \forall\, X\in V_k.$$

It is easily seen that $\mathfrak{A}$ is self-adjoint and $\mathfrak{A}(\bar X_{k+1})=\lambda_{k+1,k+1}\bar X_{k+1}$. We can choose orthonormal vectors $\bar X_{k+2},\dots,\bar X_{n_1}\in V_k$ orthogonal to $\bar X_{k+1}$, which are the remaining eigenvectors of $\mathfrak{A}$ with associated eigenvalues $\lambda_{k+1,k+2},\dots,\lambda_{k+1,n_1}$, respectively. Then, by the assumption (3.55) of induction, we can show that

$$\begin{cases} K_{\bar X_{k+1}}\bar X_{k+1}=\mu_1\bar X_1+\cdots+\mu_{k-1}\bar X_{k-1}+\mu_k\bar X_k+\lambda_{k+1,k+1}\bar X_{k+1},\\ K_{\bar X_{k+1}}\bar X_i=\lambda_{k+1,i}\bar X_i,\quad k+2\le i\le n_1.\end{cases}\tag{3.60}$$

Taking $X = Z = \bar{X}_{k+1}$ and $Y = \bar{X}_i$ in (2.3) for $k+2 \le i \le n_1$, using (2.10) and (3.60), we can obtain

$$\lambda_{k+1,i}^2 - \lambda_{k+1,k+1}\lambda_{k+1,i} + H - \sum_{l=1}^{k} \mu_l^2 = 0, \quad k+2 \le i \le n_1. \tag{3.61}$$

Similar to the proof of (3.13), we have $\lambda_{k+1,k+1} \ge 2\lambda_{k+1,i}$ for $k+2 \le i \le n_1$. Then, solving (3.61), we get $\lambda_{k+1,k+2} = \cdots = \lambda_{k+1,n_1}$ with

$$\lambda_{k+1,i} = \tfrac{1}{2}\Big(\lambda_{k+1,k+1} - \Big[\lambda_{k+1,k+1}^2 - 4(H - \sum_{l=1}^{k}\mu_l^2)\Big]^{1/2}\Big), \quad k+2 \le i \le n_1. \tag{3.62}$$

Similarly, taking in (2.3) $X = Z = X_{k+1}$ and $Y \in T_pM_2^{n_2}$ a unit vector, then using (2.10) and (3.2), we get

$$\mu_{k+1}^2 - \mu_{k+1}\lambda_{k+1,k+1} + H - \sum_{l=1}^{k} \mu_l^2 = 0. \tag{3.63}$$

Hence, we have

$$\mu_{k+1} = \tfrac{1}{2}\Big(\lambda_{k+1,k+1} + \varepsilon_{k+1}\Big[\lambda_{k+1,k+1}^2 - 4(H - \sum_{l=1}^{k}\mu_l^2)\Big]^{1/2}\Big), \quad \varepsilon_{k+1} = \pm 1. \tag{3.64}$$

On the other hand, by applying $\operatorname{trace} K_{\bar{X}_{k+1}} = 0$, we get $n_1 - n_2\varepsilon_{k+1} - k - 1 > 0$ and that

$$\lambda_{k+1,k+1} = 2(n_1 - n_2\varepsilon_{k+1} - k - 1)\sqrt{\tfrac{\sum_{l=1}^{k}\mu_l^2 - H}{(n_1+n_2-k+1)^2 - (n_1 - n_2\varepsilon_{k+1} - k - 1)^2}}. \tag{3.65}$$

From (3.62), (3.64), (3.65) and the assumption that $\mu_1, \ldots, \mu_k$ are real numbers, we see that, as claimed, $\lambda_{k+1,k+2} = \cdots = \lambda_{k+1,n_1}$ and μ_{k+1} are also constants.

Moreover, by (3.60) and the assumption (3.55) of induction, we get the assertion that (3.59) holds. $\square$

Claim II-(2). *The real numbers described in Claim II-(1) satisfy the relations:*

$$\lambda_{k+1,k+2} = \cdots = \lambda_{k+1,n_1} = \mu_{k+1} \quad \text{and} \quad \lambda_{k+1,k+1} + (n-k-1)\mu_{k+1} = 0.$$

Proof of ***Claim II-(2)***. From (3.62) and (3.64), the first assertion is equivalent to showing that $\varepsilon_{k+1} = -1$. Suppose on the contrary that $\varepsilon_{k+1} = 1$. Then we have

$$\mu_{k+1}\lambda_{k+1,i} = H - \sum_{l=1}^{k}\mu_l^2, \quad k+2 \le i \le n_1. \tag{3.66}$$

Now from $\operatorname{trace} K_{\bar{X}_{k+1}} = 0$ and $\lambda_{k+1,k+1} > 0$ we obtain

$$n_1 - n_2 - k - 1 > 0, \tag{3.67}$$

and that

$$\lambda_{k+1,k+1} = 2(n_1 - n_2 - k - 1)\sqrt{\tfrac{\sum_{l=1}^{k}\mu_l^2 - H}{(n_1+n_2-k+1)^2 - (n_1 - n_2 - k - 1)^2}}. \tag{3.68}$$

Put $V_{k+1} = \{u \in T_pM_1^{n_1} \mid u \perp \bar{X}_1, \ldots, u \perp \bar{X}_{k+1}\}$. Then (3.67) shows that $\dim V_{k+1} = n_1 - k - 1 \ge n_2 + 1 \ge 3$. Again, similar argument as in the proof of Claim I-(1) shows that, restricting on $V_{k+1} \cap U_pM_1^{n_1}$, the function $f_1 \ne 0$.

Now, by a totally similar argument as in the proof of Claim II-(1), we can choose a new orthonormal basis $\{\tilde{X}_i\}_{1\le i\le n_1}$ of $T_pM_1^{n_1}$ with $\tilde{X}_j = \bar{X}_j$ for $1 \le j \le k+1$, such that f_1, restricting on $V_{k+1}\cap U_pM_1^{n_1}$, attains its maximum $\lambda_{k+2,k+2}>0$ at $\tilde X_{k+2}$ so that $\lambda_{k+2,k+2}=h(K_{\tilde X_{k+2}}\tilde X_{k+2},\tilde X_{k+2})$.

Similar as before, we define a self-adjoint operator $\mathfrak{B}:\ V_{k+1}\to V_{k+1}$ by

$$\mathfrak{B}(X)=K_{\tilde X_{k+2}}X-\sum_{i=1}^{k+1}h(K_{\tilde X_{k+2}}X,\tilde X_i)\tilde X_i.$$

Then $\mathfrak{B}(\tilde X_{k+2})=\lambda_{k+2,k+2}\tilde X_{k+2}$. As before we can choose orthonormal vectors $\tilde X_{k+3},\dots,\tilde X_{n_1}\in V_{k+1}$, orthogonal to $\tilde X_{k+2}$, which are the remaining eigenvectors of $\mathfrak{B}:\ V_{k+1}\to V_{k+1}$, with associated eigenvalues $\lambda_{k+2,k+3},\dots,\lambda_{k+2,n_1}$, respectively.

In this way, by using (3.59), we can show that

$$\begin{cases} K_{\tilde X_{k+2}}\tilde X_{k+2}=\mu_1\tilde X_1+\cdots+\mu_k\tilde X_k+\lambda_{k+1,k+2}\tilde X_{k+1}+\lambda_{k+2,k+2}\tilde X_{k+2},\\ K_{\tilde X_{k+2}}\tilde X_i=\lambda_{k+2,i}\tilde X_i,\quad k+3\le i\le n_1.\end{cases}\tag{3.69}$$

Taking in (2.3) that $X=Z=\tilde X_{k+2}$ and $Y=\tilde X_i$ for $k+3\le i\le n_1$, and using (2.10), we can obtain

$$\lambda_{k+2,i}^2-\lambda_{k+2,k+2}\lambda_{k+2,i}+H-\sum_{l=1}^{k}\mu_l^2-\lambda_{k+1,i}^2=0,\quad k+3\le i\le n_1.\tag{3.70}$$

Notice that $\lambda_{k+2,k+2}\ge 2\lambda_{k+2,i}$ for $k+3\le i\le n_1$. Then, solving (3.70), we get

$$\lambda_{k+2,i}=\tfrac12\Big(\lambda_{k+2,k+2}-\Big[\lambda_{k+2,k+2}^2-4(H-\sum_{l=1}^{k}\mu_l^2-\lambda_{k+1,i}^2)\Big]^{1/2}\Big)\tag{3.71}$$

for $k+3\le i\le n_1$. Thus, $\lambda_{k+2,k+3}=\cdots=\lambda_{k+2,n_1}$.

On the other hand, taking in (2.3) $X=Z=\tilde X_{k+2}$ and $Y\in T_pM_2^{n_2}$ a unit vector, then using (2.10) and (3.2), we get

$$\mu_{k+2}^2-\mu_{k+2}\lambda_{k+2,k+2}+H-\lambda_{k+1,i}\mu_{k+1}-\sum_{l=1}^{k}\mu_l^2=0,\quad k+2\le i\le n_1.\tag{3.72}$$

From (3.66) and (3.72), we get

$$\mu_{k+2}^2-\mu_{k+2}\lambda_{k+2,k+2}=0,\tag{3.73}$$

and, equivalently,

$$\mu_{k+2}=\tfrac12(\lambda_{k+2,k+2}+\varepsilon_{k+2}\lambda_{k+2,k+2}),\quad \varepsilon_{k+2}=\pm1.\tag{3.74}$$

Then, from $\operatorname{trace}K_{\tilde X_{k+2}}=0$ and $\lambda_{k+2,k+2}>0$, we get $n_1-k-2>0$ and that

$$\lambda_{k+2,k+2}=2(n_1-k-2)\sqrt{\tfrac{\lambda_{k+1,k+2}^2+\sum_{l=1}^{k}\mu_l^2-H}{(n_1+n_2-k+\varepsilon_{k+2}n_2)^2-(n_1-k-2)^2}}\,.\tag{3.75}$$

From (3.67) and that $n_2\ge2$, we have the following calculations

$$\begin{aligned}\tfrac{n_1+n_2-k+1}{n_1-n_2-k-1}-\tfrac{n_1+n_2+\varepsilon_{k+2}n_2-k}{n_1-k-2}&>\tfrac{n_1+n_2-k+1}{n_1-n_2-k-1}-\tfrac{n_1+2n_2-k}{n_1-k-2}\\&=\tfrac{2(n_2+1)(n_2-1)}{(n_1-n_2-k-1)(n_1-k-2)}>0.\end{aligned}\tag{3.76}$$

Then, by (3.68) and (3.75), we get $\lambda_{k+2,k+2}>\lambda_{k+1,k+1}$, which is a contradiction. Hence, as claimed we have $\varepsilon_{k+1}=-1$ and $\lambda_{k+1,k+2}=\cdots=\lambda_{k+1,n_1}=\mu_{k+1}$. Finally, by $\operatorname{trace}K_{\bar X_{k+1}}=0$ and (3.59), we get the second assertion that

$$\lambda_{k+1,k+1}+(n-k-1)\mu_{k+1}=0.$$

This completes the proof of Claim II-(2). □

Claim II-(3). *Under the assumptions of induction, the set*

$$\Omega_{p,k} := \Big\{\lambda \in \mathbb{R}\,|V \in U_pM_1^{n_1} \setminus \mathrm{span}\{X_i(p)\}_{i=1}^k \text{ s. t. } K_VV = \lambda V + \sum_{i=1}^k \mu_i X_i\Big\}$$

consists of finite numbers, which are independent of $p \in M^n$.

*Proof of **Claim II-(3)***. We first notice that, for any fixed $p \in M^n$, Claim II-(1) shows that $\lambda_{k+1,k+1} \in \Omega_{p,k}$ with $V = \bar{X}_{k+1}$. Thus, the set $\Omega_{p,k}$ is non-empty.

Next, with the local orthonormal vector fields $\{X_i\}_{1\le i\le n_1}$ around $p \in M_1^{n_1} \times M_2^{n_2}$, given by the assumption of induction, we assume an arbitrary $\lambda \in \Omega_{p,k}$ associated with $V \in U_pM_1^{n_1} \setminus \mathrm{span}\{X_i(p)\}_{i=1}^k$ such that

$$K_VV = \lambda V + \mu_1X_1 + \cdots + \mu_kX_k.$$

Then, at p, we put $\tilde{X}_{k+1} := V$, $\tilde{X}_i = X_i(p)$ for $1 \le i \le k$ and $\tilde{\lambda}_{k+1,k+1} = \lambda$.

Put $W_k = \{u \in T_pM_1^{n_1} \mid u \perp \tilde{X}_1, \ldots, u \perp \tilde{X}_k\}$ and define $\mathfrak{F} : W_k \to W_k$ by

$$\mathfrak{F}(X) = K_{\tilde{X}_{k+1}}X - \sum_{i=1}^k h(K_{\tilde{X}_{k+1}}X, \tilde{X}_i)\tilde{X}_i, \quad X \in W_k.$$

Then $\mathfrak{F}$ is a self-adjoint linear transformation and that $\mathfrak{F}(\tilde{X}_{k+1}) = \tilde{\lambda}_{k+1,k+1}\tilde{X}_{k+1}$. Thus, we can choose an orthonormal basis $\{\tilde{X}_i\}_{k+1\le i\le n_1}$ of W_k, such that

$$\mathfrak{F}(\tilde{X}_i) = \tilde{\lambda}_{k+1,i}\tilde{X}_i, \quad k+2 \le i \le n_1.$$

Then, just like having did with equation (3.61), we have an integer $n_{1,k+1}$ with $0 \le n_{1,k+1} \le n_1 - (k+1)$ such that, if necessary after renumbering the basis of W_k, it holds

$$\begin{cases} \tilde{\lambda}_{k+1,k+2} = \cdots = \tilde{\lambda}_{k+1,n_{1,k+1}+k+1} \\ \qquad\qquad = \dfrac{1}{2}\Big(\tilde{\lambda}_{k+1,k+1} + \Big[\tilde{\lambda}^2_{k+1,k+1} - 4(H - \sum_{l=1}^k \mu_l^2)\Big]^{1/2}\Big), \\ \tilde{\lambda}_{k+1,n_{1,k+1}+k+2} = \cdots = \tilde{\lambda}_{k+1,n_1} \\ \qquad\qquad = \dfrac{1}{2}\Big(\tilde{\lambda}_{k+1,k+1} - \Big[\tilde{\lambda}^2_{k+1,k+1} - 4(H - \sum_{l=1}^k \mu_l^2)\Big]^{1/2}\Big). \end{cases} \tag{3.77}$$

Similar as deriving (3.64), now we also have

$$\mu_{k+1} = \tfrac{1}{2}\Big(\tilde{\lambda}_{k+1,k+1} + \varepsilon_{k+1}\Big[\tilde{\lambda}^2_{k+1,k+1} - 4(H - \sum_{l=1}^k \mu_l^2)\Big]^{1/2}\Big), \quad \varepsilon_{k+1} = \pm 1. \tag{3.78}$$

.

Then, computing trace $K_{\tilde{X}_{k+1}} = 0$, gives that

$$\begin{aligned} &(n_1 + n_2 - k + 1)\tilde{\lambda}_{k+1,k+1} \\ &+ (2n_{1,k+1} - n_1 + n_2\varepsilon_{k+1} + k + 1)\Big[\tilde{\lambda}^2_{k+1,k+1} - 4(H - \sum_{l=1}^k \mu_l^2)\Big]^{1/2} = 0. \end{aligned} \tag{3.79}$$

From (3.79) we have proved the assertion that $\lambda = \tilde{\lambda}_{k+1,k+1}$ takes values of only finite possibilities and they are independent of the point p. □

Claim II-(4). *Under the assumptions of induction, the unit vector $\bar{X}_{k+1} \in T_pM_1^{n_1}$, determined by Claim II-(1), can be extended differentiably to a local unit vector field in a neighbourhood U of p, denoted by $\tilde{X}_{k+1}$, such that for each $q \in U$ the function f_1, defined on*

$$\mathcal{U}_k(q) = \{u \in U_qM_1^{n_1} \mid u \perp X_1(q), \ldots, u \perp X_k(q)\},$$

attains its absolute maximum at $\tilde{X}_{k+1}(q)$.

Proof of ***Claim II-(4)***. First of all, according to (3.65) and the proof of Claim II-(2), we notice that for any q around p, the maximum of f_1 defined on $\mathcal{U}_k(q)$ is independent of q, and it equals to $\lambda_{k+1,k+1} = (n-k-1)\sqrt{(\sum_{l=1}^{k}\mu_l^2 - H)/(n-k)}$.

Now, we choose arbitrary differentiable orthonormal vector fields $\{E_{k+1}, \ldots, E_{n_1}\}$, defined on a neighbourhood U' of p such that, for $k+1 \le i \le n_1$ and $q \in U'$, we have $E_i(p) = \bar{X}_i$ and $E_i(q) \in \mathcal{U}_k(q)$.

Next, we define a function γ by

$$\begin{aligned} \gamma:\ & \mathbb{R}^{n_1-k} \times U' \to \mathbb{R}^{n_1-k}, \\ & (a_{k+1}, \ldots, a_{n_1}, q) \mapsto (b_{k+1}, \ldots, b_{n_1}), \end{aligned}$$

where

$$b_l = \sum_{i,j=k+1}^{n_1} a_i a_j h(K_{E_i}E_j, E_l) - \lambda_{k+1,k+1}a_l, \quad k+1 \le l \le n_1, \tag{3.80}$$

are regarded as functions on $\mathbb{R}^{n_1-k} \times U'$: $b_l = b_l(a_{k+1}, \ldots, a_{n_1}, q)$.

Using Claim II-(1), the fact that f_1 attains its absolute maximum $\lambda_{k+1,k+1}$ at $E_{k+1}(p)$, and that

$$h(K_{E_{k+1}}E_i, E_j)|_p = \lambda_{k+1,i}\delta_{ij}, \quad k+1 \le i, j \le n_1,$$

where $\lambda_{k+1,i}$ is given by (3.62), we then obtain that

$$\begin{aligned} \tfrac{\partial b_l}{\partial a_m}(1, 0, \ldots, 0, p) &= 2h(K_{E_{k+1}(p)}E_m(p), E_l(p)) - \lambda_{k+1,k+1}\delta_{lm} \\ &= \begin{cases} 0, & \text{if } l \ne m, \\ \lambda_{k+1,k+1}, & \text{if } l = m = k+1, \\ 2\lambda_{k+1,l} - \lambda_{k+1,k+1}, & \text{if } k+2 \le l = m \le n_1. \end{cases} \end{aligned}$$

Given that $\lambda_{k+1,k+1} > 0$ and $\lambda_{k+1,k+1} - 2\lambda_{k+1,l} > 0$ for $k+2 \le l \le n_1$, the implicit function theorem shows that in a neighbourhood $U'' \subset U'$ of p there exist differentiable functions $\{a_{k+1}, \ldots, a_{n_1}\}$ satisfying

$$\begin{cases} a_{k+1}(p) = 1,\ a_{k+2}(p) = \cdots = a_{n_1}(p) = 0, \\ b_l(a_{k+1}(q), \ldots, a_{n_1}(q), q) \equiv 0,\ q \in U'', \quad k+1 \le l \le n_1. \end{cases} \tag{3.81}$$

Define a local vector field V on U'' by

$$V(q) = a_{k+1}(q)E_{k+1}(q) + \cdots + a_{n_1}(q)E_{n_1}(q), \quad q \in U''.$$

Then $V(p) = \bar{X}_{k+1}$, there exists a neighbourhood $U \subset U''$ of p, such that $V \ne 0$ on U. Using (3.80), (3.81) and (3.2), we easily see that

$$K_VV = \lambda_{k+1,k+1}V + \mu_1 h(V,V)X_1 + \cdots + \mu_k h(V,V)X_k,$$

or, equivalently,

$$K_{\frac{V}{\sqrt{h(V,V)}}}\frac{V}{\sqrt{h(V,V)}} = \frac{\lambda_{k+1,k+1}}{\sqrt{h(V,V)}}\frac{V}{\sqrt{h(V,V)}} + \sum_{i=1}^{k}\mu_i X_i, \quad \text{in } U. \tag{3.82}$$

Now, according to Claim II-(3), the function $\frac{\lambda_{k+1,k+1}}{\sqrt{h(V,V)}}$ takes values of only finite possibilities. On the other hand, $\frac{\lambda_{k+1,k+1}}{\sqrt{h(V,V)}}$ is continuous and $h(V,V)(p) = 1$. Thus $h(V,V)|_U \equiv 1$. Let $\tilde{X}_{k+1} := V$. Then, (3.82) with $h(V,V) = 1$ implies that

$$K_{\tilde{X}_{k+1}}\tilde{X}_{k+1} = \lambda_{k+1,k+1}\tilde{X}_{k+1} + \mu_1 X_1 + \cdots + \mu_k X_k,$$

and for any $q \in U$, f_1 attains its absolute maximum $\lambda_{k+1,k+1}$ at $\tilde{X}_{k+1}(q)$. □

Let $\tilde{X}_1 = X_1, \ldots, \tilde{X}_k = X_k$ and choose vector fields $\tilde{X}_{k+2}, \ldots, \tilde{X}_{n_1}$ such that, with $\tilde{X}_{k+1}$ obtained as in Claim II-(4), $\{\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_{n_1}\}$ is a local orthonormal frame of $TM_1^{n_1}$ defined on a neighborhood U of p and satisfies $X_i(q) \in T_qM_1^{n_1}$ for $q \in U$ and $1 \le i \le n_1$. Then, with respect to $\{\tilde{X}_i\}_{1\le i\le n_1}$ and combining with Lemma 3.2, we immediately fulfil the second step of induction.

In this way, the method of induction allows us to obtain the desired orthonormal vector fields $\{X_1, \ldots, X_{n_1-1}\}$ defined on a neighborhood U of p and satisfying $X_i(q) \in T_qM_1^{n_1}$ for $q \in U$ and $1 \le i \le n_1 - 1$. Finally, we choose a unit vector field X_{n_1} that is orthogonal to $\{X_1, \ldots, X_{n_1-1}\}$ and that satisfies $X_{n_1}(q) \in T_qM_1^{n_1}$, such that $\lambda_{n_1,n_1} \ge 0$ (if necessary we change X_{n_1} by $-X_{n_1}$). Then, it is easy to see that $\{X_1, \ldots, X_{n_1}\}$ are the desired orthonormal vector fields. Accordingly, we have completed the proof of Lemma 3.4. □

4. Proofs of the Theorems and Corollaries

First of all, continuing with the study of **Case** $\mathfrak{C}_1$ in last section, we show that the local orthonormal vector fields $\{X_i\}_{1\le i\le n_1}$, as determined in Lemma 3.4, consist of parallel vector fields such that $\hat{\nabla}X_i = 0$.

Lemma 4.1. *The local orthonormal vector fields $\{X_1, \ldots, X_{n_1}\}$, as described by Lemma 3.4, consist of parallel vector fields, i.e.,*

$$\hat{\nabla}X_i = 0, \quad 1 \le i \le n_1.$$

Proof. We shall give the proof by induction on i. First of all, we prove $\hat{\nabla}X_1 = 0$.

In fact, for $j \ge 2$, applying (3.29), we have the following calculations

$$(\hat{\nabla}_{X_j}K)(X_1, X_1) = (\lambda_{1,1} - 2\mu_1)\hat{\nabla}_{X_j}X_1, \tag{4.1}$$

$$(\hat{\nabla}_{X_1}K)(X_j, X_1) = \mu_1\hat{\nabla}_{X_1}X_j - K(\hat{\nabla}_{X_1}X_j, X_1) - K(\hat{\nabla}_{X_1}X_1, X_j). \tag{4.2}$$

Now, the Codazzi equation $(\hat{\nabla}_{X_j}K)(X_1, X_1) = (\hat{\nabla}_{X_1}K)(X_j, X_1)$ gives that

$$(\lambda_{1,1} - 2\mu_1)\hat{\nabla}_{X_j}X_1 = \mu_1\hat{\nabla}_{X_1}X_j - K(\hat{\nabla}_{X_1}X_j, X_1) - K(\hat{\nabla}_{X_1}X_1, X_j). \tag{4.3}$$

Then, taking the component of (4.3) in direction of X_1 for each $j \ge 2$ and using the fact that $h(\hat{\nabla}_{X_1}X_1, Y) = 0$ for $Y(q) \in T_qM_2^{n_2}$, and (3.29) again, we get $\hat{\nabla}_{X_1}X_1 = 0$. Substituting $\hat{\nabla}_{X_1}X_1 = 0$ into (4.3), and then taking its component in direction of X_j, we get $h(\hat{\nabla}_{X_j}X_1, X_k) = 0$ for $2 \le j, k \le n_1$. This, together with the fact that $h(\hat{\nabla}_{X_j}X_1, Y) = 0$ for $Y(q) \in T_qM_2^{n_2}$, implies that

$$\hat{\nabla}_{X_j}X_1 = 0, \quad 1 \le j \le n_1. \tag{4.4}$$

Take a unit vector field Y with $Y(q) \in T_qM_2^{n_2}$. By a direct calculation of $h((\hat{\nabla}_Y K)(X_1, X_i), X_1) = h((\hat{\nabla}_{X_1}K)(Y, X_i), X_1)$, we obtain $h(\hat{\nabla}_Y X_1, X_i) = 0$ for $2 \le i \le n_1$. This, together with $h(\hat{\nabla}_Y X_1, Y') = 0$ for $Y'(q) \in T_qM_2^{n_2}$, implies that

$$\hat{\nabla}_Y X_1 = 0. \tag{4.5}$$

Combining (4.4) and (4.5), we have proved the assertion $\hat{\nabla}X_1 = 0$.

Next, by induction we show that if for any fixed $2 \le i \le n_1 - 1$ *satisfying*

$$\hat{\nabla}X_k = 0, \quad k = 1, \dots, i-1, \tag{4.6}$$

then it holds $\hat{\nabla}X_i = 0$.

To state a proof of the above second step, we consider five cases below:

(i) By (4.6) and that $h(X_k, X_l) = \delta_{kl}$, we get

$$\begin{aligned} &h(\hat{\nabla}_{X_j}X_i, X_k) = -h(\hat{\nabla}_{X_j}X_k, X_i) = 0, \quad 1 \le j \le n_1, \ k \le i \\ &h(\hat{\nabla}_Y X_i, X_k) = -h(\hat{\nabla}_Y X_k, X_i) = 0, \quad 1 \le k \le i \le n_1, \ . \end{aligned} \tag{4.7}$$

(ii) For $j \le i-1$, by using (3.29), (4.6) and (4.7), we can show that

$$\begin{aligned} (\hat{\nabla}_{X_j}K)(X_i, X_i) &= \hat{\nabla}_{X_j}K(X_i, X_i) - 2K(\hat{\nabla}_{X_j}X_i, X_i) \\ &= (\lambda_{i,i} - 2\mu_i)\hat{\nabla}_{X_j}X_i, \end{aligned} \tag{4.8}$$

$$\begin{aligned} (\hat{\nabla}_{X_i}K)(X_j, X_i) &= \hat{\nabla}_{X_i}K(X_j, X_i) - K(\hat{\nabla}_{X_i}X_j, X_i) - K(\hat{\nabla}_{X_i}X_i, X_j) \\ &= \mu_j\hat{\nabla}_{X_i}X_i - K(\hat{\nabla}_{X_i}X_i, X_j). \end{aligned} \tag{4.9}$$

Then, by $(\hat{\nabla}_{X_j}K)(X_i, X_i) = (\hat{\nabla}_{X_i}K)(X_j, X_i)$, for $k \ge i+1$ we obtain

$$\begin{aligned} (\lambda_{i,i} - 2\mu_i)h(\hat{\nabla}_{X_j}X_i, X_k) &= \mu_j h(\hat{\nabla}_{X_i}X_i, X_k) - h(K(\hat{\nabla}_{X_i}X_i, X_j), X_k) \\ &= \mu_j h(\hat{\nabla}_{X_i}X_i, X_k) - h(\hat{\nabla}_{X_i}X_i, K_{X_j}X_k) = 0. \end{aligned}$$

It follows that

$$h(\hat{\nabla}_{X_j}X_i, X_k) = 0, \quad j \le i-1, \ k \ge i+1. \tag{4.10}$$

(iii) Similar to the above case (ii), for $j \ge i+1$, we have

$$\begin{aligned} (\hat{\nabla}_{X_j}K)(X_i, X_i) &= \hat{\nabla}_{X_j}K(X_i, X_i) - 2K(\hat{\nabla}_{X_j}X_i, X_i) \\ &= (\lambda_{i,i} - 2\mu_i)\hat{\nabla}_{X_j}X_i, \end{aligned} \tag{4.11}$$

$$\begin{aligned} (\hat{\nabla}_{X_i}K)(X_j, X_i) &= \hat{\nabla}_{X_i}K(X_j, X_i) - K(\hat{\nabla}_{X_i}X_j, X_i) - K(\hat{\nabla}_{X_i}X_i, X_j) \\ &= \mu_i\hat{\nabla}_{X_i}X_j - K(\hat{\nabla}_{X_i}X_j, X_i) - K(\hat{\nabla}_{X_i}X_i, X_j). \end{aligned} \tag{4.12}$$

Then, taking the X_i-components of $(\hat\nabla_{X_j}K)(X_i,X_i)=(\hat\nabla_{X_i}K)(X_j,X_i)$, with using (3.29) and (4.6), we obtain

$$\begin{aligned}0&=(\lambda_{i,i}-2\mu_i)h(\hat\nabla_{X_j}X_i,X_i)\\&=\mu_ih(\hat\nabla_{X_i}X_j,X_i)-h(K(\hat\nabla_{X_i}X_j,X_i),X_i)-h(K(\hat\nabla_{X_i}X_i,X_j),X_i)\\&=-\mu_ih(\hat\nabla_{X_i}X_i,X_j)-h(\hat\nabla_{X_i}X_j,K_{X_i}X_i)-h(\hat\nabla_{X_i}X_i,K_{X_i}X_j)\\&=(\lambda_{i,i}-2\mu_i)h(\hat\nabla_{X_i}X_i,X_j).\end{aligned}$$

Hence, we obtain

$$h(\hat\nabla_{X_i}X_i,X_j)=0,\ \ j\ge i+1. \tag{4.13}$$

(iv) By using $(\hat\nabla_{X_j}K)(X_i,X_i)=(\hat\nabla_{X_i}K)(X_j,X_i)$ and taking its X_k-components for $j,k\ge i+1$, then applying (4.13) we obtain

$$\begin{aligned}(\lambda_{i,i}-2\mu_i)h(\hat\nabla_{X_j}X_i,X_k)&=\mu_ih(\hat\nabla_{X_i}X_j,X_k)-h(K(\hat\nabla_{X_i}X_j,X_i),X_k)\\&=\mu_ih(\hat\nabla_{X_i}X_j,X_k)-h(\hat\nabla_{X_i}X_j,K_{X_i}X_k)\\&=0.\end{aligned}$$

$$h(\hat\nabla_{X_j}X_i,X_k)=0,\quad j,k\ge i+1. \tag{4.14}$$

(v) If Y is a unit vector field with $Y(q)\in T_qM_2^{n_2}$, by a direct calculation of $h((\hat\nabla_YK)(X_i,X_k),X_i)=h((\hat\nabla_{X_i}K)(Y,X_k),X_i)$ for $i+1\le k$, we obtain

$$h(\hat\nabla_YX_i,X_k)=0,\quad i+1\le k. \tag{4.15}$$

For Y,Y' with $Y(q),Y'(q)\in T_qM_2^{n_2}$, we have $h(\hat\nabla_{X_j}X_i,Y)=h(\hat\nabla_YX_i,Y')=0$. Hence, combining (4.7), (4.10) and (4.13)–(4.15), we finally get

$$\hat\nabla X_i=0. \tag{4.16}$$

Therefore, by induction we have proved that

$$\hat\nabla X_i=0,\quad 1\le i\le n_1-1. \tag{4.17}$$

Finally, for vector fields X,Y with $X(q)\in T_qM_1^{n_1}$, $Y(q)\in T_qM_2^{n_2}$ and $k\le n_1-1$, from (4.17) it is easily seen the following

$$\begin{aligned}&h(\hat\nabla_XX_{n_1},X_k)=-h(\hat\nabla_XX_k,X_{n_1})=0,\ h(\hat\nabla_XX_{n_1},X_{n_1})=0,\\&h(\hat\nabla_YX_{n_1},X_k)=-h(\hat\nabla_YX_k,X_{n_1})=0,\ h(\hat\nabla_YX_{n_1},X_{n_1})=0,\end{aligned}$$

so that it holds also $\hat\nabla X_{n_1}=0$.

We have completed the proof of Lemma 4.1. □

Moreover, we have the following further conclusion.

Lemma 4.2. *Let $x:M^n\to\mathbb{R}^{n+1}$ be an n-dimensional locally strongly convex affine hypersphere such that* ***Case*** *$\mathfrak{C}_1$ in section 3 occurs, then the difference tensor is parallel, i.e., $\hat\nabla K=0$.*

Proof. Let $\{X_1,\dots,X_{n_1}\}$ be the local orthonormal vector fields as described by Lemma 3.4. Then Lemma 4.1 shows that

$$\hat\nabla X_i=0,\quad 1\le i\le n_1. \tag{4.18}$$

On the other hand, as $(M^n,h)=M_1^{n_1}(c_1)\times M_2^{n_2}(c_2)$, by Proposition 56 in p.89 of [22], we can choose local orthonormal vector fields $\{Y_1,\dots,Y_{n_2}\}$ with $Y_i(q)\in T_qM_2^{n_2}$, such that

$$\hat\nabla_{X_i}Y_\alpha=0,\quad 1\le i\le n_1,\ 1\le\alpha\le n_2. \tag{4.19}$$

Then, using (4.18), (4.19) and properties of the difference tensor established by Lemmas 3.2, 3.3 and 3.4, direct calculations immediately give the assertion that $\hat\nabla K=0$. □

Theorem 4.1. *Let $x:M^n\to\mathbb{R}^{n+1}$ be an n-dimensional locally strongly convex affine hypersphere such that* ***Case*** *$\mathfrak{C}_1$ in section 3 occurs. Then $x:M^n\to\mathbb{R}^{n+1}$ is locally affinely equivalent to the Calabi composition*

$$(x_1\cdots x_{n_1})^2(x_{n+1}^2-x_{n_1+1}^2-\cdots-x_n^2)^{n_2+1}=1, \tag{4.20}$$

where $(x_1,\dots,x_{n+1})$ are the standard coordinates of $\mathbb{R}^{n+1}$.

Proof. By Lemma 3.4 and Lemma 4.2, we can apply Theorem 4.1 of [10] with X_1 being regarded as e_1 there. Then $x:M^n\to\mathbb{R}^{n+1}$ is a Calabi product of a point G_1 with a hyperbolic affinesphere $G_2':\tilde M_1^{n-1}\to\mathbb{R}^n$ with parallel cubic form and affine mean curvature H_2, so that we have the decomposition $M^n=I_1\times\tilde M_1^{n-1}$, $I\subset\mathbb{R}$, and the parametrization

$$x(s_1,\tilde p_1)=-\tfrac{\mu_1}{H^2+\mu_1^2}e^{-s_1}G_1+\tfrac{1}{H^2+\mu_1^2}e^{s_1/n}G_2'(\tilde p_1),\quad s_1\in I_1,\ \tilde p_1\in\tilde M_1^{n-1}.$$

Moreover, the affine metric of $G_2':\tilde M_1^{n-1}\to\mathbb{R}^n$ is $(\mu_1^2-H)h|_{T\tilde M_1^{n-1}}$ (cf. [10]).

Notice that $T\tilde M_1^{n-1}=\mathrm{span}\{X_2,\dots,X_{n_1},Y_1,\dots,Y_{n_2}\}$. Let us denote by K^1 the difference tensor of $G_2':\tilde M_1^{n-1}\to\mathbb{R}^n$, then from the proof of Theorem 4.1 of [10] and Lemmas 3.3 and 3.4, we can derive that K^1 has the expressions as follows:

$$\begin{cases} K^1_{X_i}X_i=\mu_2X_2+\cdots+\mu_{i-1}X_{i-1}+\lambda_{i,i}X_i, & 2\le i\le n_1,\\ K^1_{X_i}X_j=\mu_iX_j, & 2\le i<j\le n_1,\\ K^1_{X_i}Y_\alpha=\mu_iY_\alpha, & 2\le i\le n_1,\ 2\le\alpha\le n_1,\\ K^1_{Y_\alpha}Y_\beta=\delta_{\alpha\beta}(\mu_2X_2+\cdots+\mu_{n_1}X_{n_1}), & 1\le\alpha,\beta\le n_2.\end{cases} \tag{4.21}$$

Notice also that, up to scaling a constant multiple, $\{X_2,\dots,X_{n_1},Y_1,\dots,Y_{n_2}\}$ are the orthomormal basis of the affine metric of $G_2':\tilde M_1^{n-1}\to\mathbb{R}^n$. Applying Theorem 4.1 in [10] once again by regarding X_2 as e_1 there, then $G_2':\tilde M_1^{n-1}\to\mathbb{R}^n$ is a Calabi product of a point G_2 with a hyperbolic affinesphere $G_3':\tilde M_2^{n-2}\to\mathbb{R}^{n-1}$ with parallel cubic form, so that we have the decomposition $\tilde M_1^{n-1}=I_2\times\tilde M_2^{n-2}$, $I_2\subset\mathbb{R}$, and the further parametrization

$$\begin{aligned} x(s_1,s_2,\tilde p_2)=&-\tfrac{\mu_1}{H^2+\mu_1^2}e^{-s_1}G_1-\tfrac{1}{H^2+\mu_1^2}\tfrac{\mu_2}{H_2^2+\mu_2^2}e^{\frac{s_1}{n}-s_2}G_2\\ &+\tfrac{1}{H^2+\mu_1^2}\tfrac{1}{H_2^2+\mu_2^2}e^{\frac{s_1}{n}-\frac{s_2}{n-1}}G_3'(\tilde p_2),\quad (s_1,s_2)\in I_1\times I_2,\ \tilde p_2\in\tilde M_2^{n-2}.\end{aligned}$$

Continuing in this way n_1 times, we finally see that $M^n = M_1^{n_1} \times M_2^{n_2}$, with $M_1^{n_1} \cong I_1 \times I_2 \times \cdots \times I_{n_1}$, and $x : M^n \to \mathbb{R}^{n+1}$ has a parametrization

$$\begin{aligned} x(s_1, \ldots, s_{n_1}, p_2) &= -\tfrac{\mu_1}{H^2+\mu_1^2} e^{-s_1} G_1 - \tfrac{1}{H^2+\mu_1^2} \tfrac{\mu_2}{H_2^2+\mu_2^2} e^{\frac{s_1}{n_1+n_2} - s_2} G_2 - \cdots \\ &- \tfrac{1}{H^2+\mu_1^2} \tfrac{1}{H_2^2+\mu_2^2} \cdots \tfrac{1}{H_{n_1-1}^2+\mu_{n_1-1}^2} \tfrac{\mu_{n_1}}{H_{n_1}^2+\mu_{n_1}^2} e^{\frac{s_1}{n_1+n_2} + \cdots + \frac{s_{n_1-1}}{n_2+2} - s_{n_1}} G_{n_1} \\ &+ \tfrac{1}{H^2+\mu_1^2} \tfrac{1}{H_2^2+\mu_2^2} \cdots \tfrac{1}{H_{n_1}^2+\mu_{n_1}^2} e^{\frac{s_1}{n_1+n_2} + \cdots + \frac{s_{n_1}}{n_2+1}} G'_{n_1+1}(p_2), \quad p_2 \in M_2^{n_2}, \end{aligned} \tag{4.22}$$

where, $(s_1, \ldots, s_{n_1}) \in M_1^{n_1}$, $\{G_i\}_{1 \le i \le n_1}$ are constant vectors and $G'_{n_1+1} : M_2^{n_2} \to \mathbb{R}^{n_2+1}$ is a hyperbolic affine hypersphere with parallel cubic form.

Furthermore, from the above procedure of induction, it can be easily seen that $G'_{n_1+1} : M_2^{n_2} \to \mathbb{R}^{n_2+1}$ has vanishing difference tensor. This implies that $G'_{n_1+1} : M_2^{n_2} \to \mathbb{R}^{n_2+1}$ is a hyperboloid. Therefore, up to an affine transformation, there exist constant vectors $G_{n_1+1}, \ldots, G_{n+1}$ such that

$$G'_{n_1+1} = y_1 G_{n_1+1} + y_2 G_{n_1+2} + \cdots + y_{n_2+1} G_{n+1}, \tag{4.23}$$

where $y_1^2 + \cdots + y_{n_2}^2 - y_{n_2+1}^2 = -1$.

Combining (4.22) and (4.23), we finally see that, up to an affine transformation, $x : M^n = M_1^{n_1} \times M_2^{n_2} \to \mathbb{R}^{n+1}$ can be written as

$$\begin{aligned} x &= (x_1, \ldots, x_{n_1}, x_{n_1+1}, \ldots, x_{n+1}) \\ &= \Big(e^{-s_1}, e^{\frac{s_1}{n} - s_2}, \ldots, e^{\frac{s_1}{n_1+n_2} + \cdots + \frac{s_{n_1-1}}{n_2+2} - s_{n_1}}, e^{\frac{s_1}{n_1+n_2} + \cdots + \frac{s_{n_1}}{n_2+1}} (y_1, \ldots, y_{n_2+1}) \Big). \end{aligned}$$

Hence, $x : M^n \to \mathbb{R}^{n+1}$ is affinely equivalent to the affine hypersphere (4.20). □

Next, we consider **Case** $\mathfrak{C}_2$ as stated in section 3 such that $x : M^n \to \mathbb{R}^{n+1}$ is an n-dimensional locally strongly convex affine hypersphere with $(M^n, h) = M_1^{n_1}(c_1) \times M_2^{n_2}(c_2)$, $n_1 \ge 2,\ n_2 \ge 2$ and $c_1 c_2 \ne 0$. Then, similar to that in Lemma 3.2 for the proof of (3.8), we can obtain the following result.

Lemma 4.3. *For $p \in M_1^{n_1} \times M_2^{n_2}$, let $\{X_i\}_{1 \le i \le n_1}$ and $\{Y_j\}_{1 \le j \le n_2}$ be orthonormal bases of $T_p M_1^{n_1}$ and $T_p M_2^{n_2}$, respectively. Then, in **Case** $\mathfrak{C}_2$, the difference tensor satisfies*

$$\begin{cases} h(K_{X_i} X_j, Y_\gamma) = 0, & 1 \le i, j \le n_1, \quad 1 \le \gamma \le n_2, \\ h(K_{Y_\alpha} Y_\beta, X_k) = 0, & 1 \le \alpha, \beta \le n_2, \quad 1 \le k \le n_1. \end{cases} \tag{4.24}$$

Moreover, we have the following further conclusion.

Theorem 4.2. *Let $x : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex product affine hypersphere, then **Case** $\mathfrak{C}_2$ in section 3 does not occur.*

Proof. If otherwise, we assume that **Case** $\mathfrak{C}_2$ does occur. Then, as by Lemma 3.1 the difference tensor K vanishes nowhere, we may assume that for an arbitrary fixed $p \in M^n = M_1^{n_1} \times M_2^{n_2}$ there exists $X \in T_p M_1^{n_1}$ such that $K_X \ne 0$. Now, similar to the proof for the first step of induction in the proof of Lemma 3.4, we can show that around $p \in M^n$ there exist local orthonormal vector fields $\{X_1, \ldots, X_{n_1}\}$ with $X_i(q) \in T_q M_1^{n_1}, 1 \le i \le n_1$, such that the difference tensor takes the form

$$K_{X_1} X_1 = \lambda_1 X_1,\ K_{X_1} X_i = \lambda_2 X_i, \quad 2 \le i \le n_1, \tag{4.25}$$

where, λ_1 and λ_2 are real numbers with $\lambda_1 > 0$ and $\lambda_1 + (n-1)\lambda_2 = 0$. Then, similar to the proof of (3.28), we can show that $\hat{\nabla}_{X_i} X_1 = 0$ for $1 \le i \le n_1$. It follows that $\hat{R}(X_1, X_2)X_1 = 0$, which is a contradiction to that $c_1 c_2 \neq 0$. □

The Completion of Theorem 1.1's Proof.

If $c_1 = c_2 = 0$, it follows from (2.10) that (M^n, h) is flat. Then, according to the result of [26], we get the assertion (i) of Theorem 1.1.

If $c_1^2 + c_2^2 \neq 0$, we have two cases: **Case** $\mathfrak{C}_1$ and **Case** $\mathfrak{C}_2$, as preceding described.

If **Case** $\mathfrak{C}_1$ occurs, then by Theorem 4.1, we obtain the hypersphere as stated in (ii) of Theorem 1.1. Moreover, according to Theorem 4.2, **Case** $\mathfrak{C}_2$ does not occur.

We have completed the proof of Theorem 1.1. □

Next, we come to give the proof of Theorem 1.2. First of all, similar to the proof of Lemma 3.1, we can obtain the following result.

Lemma 4.4. *Let $x : M^n \to \mathbb{R}^{n+1}$ $(n \ge 3)$ be a locally strongly convex affine hypersphere such that (M^n, h) is locally isometric to a Riemannian product $\mathbb{R} \times M_2^{n-1}(c_2)$, where $M_2^{n-1}(c_2)$ is an $(n-1)$-dimensional Riemannian manifold with constant sectional curvature $c_2 \neq 0$. Then the difference tensor K of $x : M^n \to \mathbb{R}^{n+1}$ vanishes nowhere.*

Next, similar to the proofs of (3.5) and (3.6), we have

Lemma 4.5. *Let $x : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex affine hypersphere as described in Lemma 4.4. For $p \in M^n$, assume that $\{Y_\alpha\}_{1\le\alpha\le n-1}$ is an orthonormal basis of $T_pM_2^{n-1}$ and $X \in T_p\mathbb{R}$ is a unit vector, then we have*

$$K_X Y_\alpha = \mu(X) Y_\alpha, \quad 1 \le \alpha \le n_2, \tag{4.26}$$

where $\mu(X) =: \mu$ depends only on X.

Now, we will prove a lemma which plays the same important role as Lemma 3.4.

Lemma 4.6. *Let $x : M^n \to \mathbb{R}^{n+1}$ be a locally strongly convex affine hypersphere as described in Lemma 4.4 with $S = H\,\mathrm{id}$. Then, around any point $p \in M^n$, there exists a local orthonormal frame $\{X_1, Y_1, \ldots, Y_{n-1}\}$ on M^n with $X_1(q) \in T_q\mathbb{R}$ and $Y_\alpha(q) \in T_qM_2^{n-1}$, $1 \le \alpha \le n-1$, such that the difference tensor of $x : M^n \to \mathbb{R}^{n+1}$ takes the following form*

$$\begin{cases} K_{X_1}X_1 = (n-1)\sqrt{-\frac{H}{n}}\,X_1, \ K_{X_1}Y_\alpha = -\sqrt{-\frac{H}{n}}\,Y_\alpha, \quad 1 \le \alpha \le n-1, \\ K_{Y_\alpha}Y_\beta = -\sqrt{-\frac{H}{n}}\,\delta_{\alpha\beta}X_1, \quad 1 \le \alpha, \beta \le n-1. \end{cases} \tag{4.27}$$

Moreover, we have $c_2 = (n+1)H/n < 0$ and $\hat{\nabla}K = 0$.

Proof. Around any point $p \in M^n = I \times M_2^{n-1}$, we take local unit vector fields X and Y, with $X(q) \in T_q\mathbb{R}$ and $Y(q) \in T_qM_2^{n-1}$. Then, similar to the proof of (3.8), and applying Lemma 4.5, we obtain

$$K_X X = \lambda X, \quad K_X Y = \mu Y. \tag{4.28}$$

Moreover, by using (2.3) and the fact $\hat{R}(Y, X)X = 0$, we have

$$\mu^2 - \lambda\mu + H = 0. \tag{4.29}$$

On the other hand, by trace $K_X = 0$, we get $\lambda + (n-1)\mu = 0$. This together with (4.29) implies that, if necessary replacing X by $-X$,

$$H \le 0, \quad \lambda = (n-1)\sqrt{-\tfrac{H}{n}}, \quad \mu = -\sqrt{-\tfrac{H}{n}}. \tag{4.30}$$

Similar to the proof of Lemma 3.3, we can also show that

$$h(K_Y Y', Y'') = 0, \quad \forall\, Y, Y', Y'' \in T_q M_2^{n_2}.$$

Since $(M^n, h) = \mathbb{R} \times M_2^{n-1}(c_2)$, by Proposition 56 in p.89 of [22], we can take an orthonormal frame $\{X_1, Y_1, \ldots, Y_{n-1}\}$ on M^n with $X_1 = X$, such that

$$\hat\nabla_{X_1} Y_\alpha = 0, \quad 1 \le \alpha \le n-1. \tag{4.31}$$

Then, w.r.t $\{X_1, Y_\alpha\}$, (4.27) immediately follows from the preceding conclusions.

Next, using (4.27), we can apply (2.3) and (2.10), with $X = Y_2$ and $Y = Z = Y_1$, to obtain that $c_2 = (n+1)H/n < 0$.

Finally, similar to the proof of $\hat\nabla X_1 = 0$ in Lemma 4.1, by (2.4) and (4.27), we can show that $\hat\nabla X_1 = 0$. From this, together with (4.27) and (4.31), we can show by direct calculations that $\hat\nabla K = 0$. □

The Completion of Theorem 1.2's Proof.

Under the assumptions of Theorem 1.2, we can apply Lemma 4.6, then as a direct consequence of Theorem 4.1 in [10] we easily get the assertion. □

Proof of Corollaries.

Let $x : M^n \to \mathbb{R}^{n+1}$, with $n = 3$ (resp. $n = 4$), be a locally strongly convex affine hypersphere whose Ricci tensor is parallel with respect to the Levi-Civita of the affine metric. Then, by the classical de Rham-Wu's decomposition theorem [27], (M^n, h) is locally isometric to a Riemannian product of Einstein manifolds.

If $n = 3$, then either (M^3, h) is Einstein and thus M^3 is of constant sectional curvature, or (M^3, h) is locally isometric to a Riemannian product $\mathbb{R} \times \tilde M^2$, where $\tilde M^2$ is a Riemannian manifold with constant sectional curvature. For both of these cases, according to [26] and Theorem 1.2, we obtain Corollary 1.1.

If $n = 4$, then either (M^4, h) is Einstein, or (M^4, h) is locally isometric to a Riemannian product $\mathbb{R} \times \tilde M^3$, or (M^4, h) is locally isometric to a Riemannian product $M_1^2 \times M_2^2$, where $\tilde M^3$, M_1^2 and M_2^2 are Riemannian manifolds with constant sectional curvature. Then, for each of these three cases, applying the results of [12], Theorem 1.1 and Theorem 1.2, we obtain Corollary 1.2. □

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